CHAPTER 56

Double Integration Applied to Volume Under a Surface and the Area of a Curved Surface

Let $z = f(x, y)$ or $z = f(\rho, \theta)$ define a surface.

The volume V under the surface, that is, the volume of a vertical column whose upper base is in the surface and whose lower base is in the xy plane, is given by the double integral

$$V = \iint_R z\, dA \tag{56.1}$$

where R is the region forming the lower base.

The area S of the portion R^* of the surface lying above the region R is given by the double integral

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \tag{56.2}$$

If the surface is given by $x = f(y, z)$ and the region R lies in the yz plane, then

$$S = \iint_R \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dA \tag{56.3}$$

If the surface is given by $y = f(x, z)$ and the region R lies in the xz plane, then

$$S = \iint_R \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dA \tag{56.4}$$

SOLVED PROBLEMS

1. Find the volume in the first octant between the planes $z = 0$ and $z = x + y + 2$, and inside the cylinder $x^2 + y^2 = 16$.

From Fig. 56-1, it is evident that $z = x + y + 2$ is to be integrated over a quadrant of the circle $x^2 + y^2 = 16$ in the xy plane. Hence,

$$V = \iint_R z\,dA = \int_0^4 \int_0^{\sqrt{16-x^2}} (x + y + 2)\,dy\,dx = \int_0^4 \left(x\sqrt{16 - x^2} + 8 - \frac{1}{2}x^2 + 2\sqrt{16 - x^2}\right)dx$$

$$= \left[-\frac{1}{3}(16 - x^2)^{3/2} + 8x - \frac{x^3}{6} + x\sqrt{16 - x^2} + 16\sin^{-1}\frac{1}{4}x\right]_0^4 = \left(\frac{128}{3} + 8\pi\right) \text{ cubic units}$$

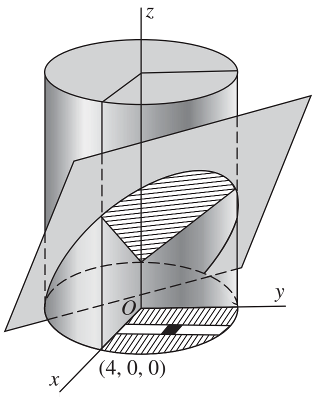

Fig. 56-1

2. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

From Fig. 56-2, it is evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy plane. Hence,

$$V = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(4-y)dx\,dy = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}(4-y)dx\,dy = 16\pi \text{ cubic units}$$

3. Find the volume bounded above by the paraboloid $x^2 + 4y^2 = z$, below by the plane $z = 0$, and laterally by the cylinders $y^2 = x$ and $x^2 = y$. (See Fig. 56-3.)

The required volume is obtained by integrating $z = x^2 + 4y^2$ over the region R common to the parabolas $y^2 = x$ and $x^2 = y$ in the xy plane. Hence,

$$V = \int_0^1\int_{x^2}^{\sqrt{x}}(x^2+4y^2)dy\,dx = \int_0^1\left[x^2y + \frac{4}{3}y^3\right]_{x^2}^{\sqrt{x}} dx = \tfrac{3}{7} \text{ cubic units}$$

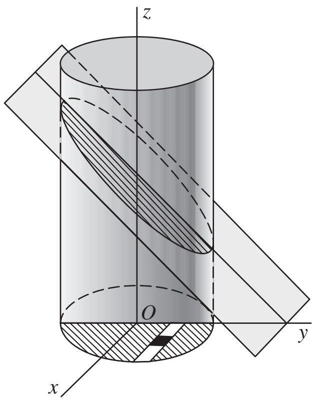

Fig. 56-2

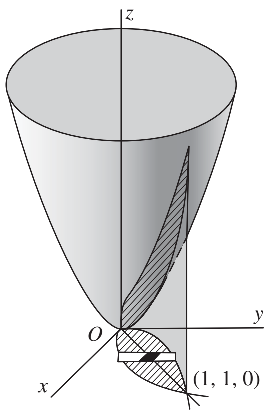

Fig. 56-3

4. Find the volume of one of the wedges cut from the cylinder $4x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = my$. (See Fig. 56-4.)

The volume is obtained by integrating $z = my$ over half the ellipse $4x^2 + y^2 = a^2$. Hence,

$$V = 2\int_0^{a/2}\int_0^{\sqrt{a^2-4x^2}} my\,dy\,dx = m\int_0^{a/2}[y^2]_0^{\sqrt{a^2-4x^2}}dx = \frac{ma^3}{3} \text{ cubic units}$$

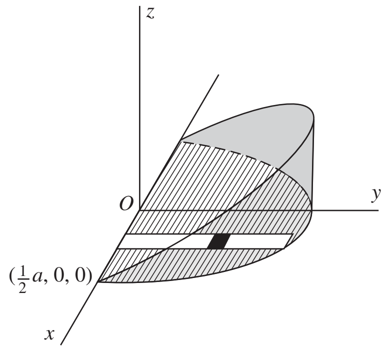

Fig. 56-4

5. Find the volume bounded by the paraboloid $x^2 + y^2 = 4z$, the cylinder $x^2 + y^2 = 8y$, and the plane $z = 0$. (See Fig. 56-5.)

The required volume is obtained by integrating $z = \frac{1}{4}(x^2 + y^2)$ over the circle $x^2 + y^2 = 8y$. Using cylindrical coordinates (see Chapter 57), the volume is obtained by integrating $z = \frac{1}{4}\rho^2$ over the circle $\rho = 8 \sin \theta$. Then,

$$V = \iint_R z\, dA = \int_0^{\pi}\int_0^{8\sin\theta} z\rho\, d\rho\, d\theta = \frac{1}{4}\int_0^{\pi}\int_0^{8\sin\theta} \rho^3 d\rho\, d\theta$$

$$= \frac{1}{16}\int_0^{\pi} [\rho^4]_0^{8\sin\theta} d\theta = 256\int_0^{\pi} \sin^4\theta\, d\theta = 96\pi \text{ cubic units}$$

6. Find the volume removed when a hole of radius a is bored through a sphere of radius $2a$, the axis of the hole being a diameter of the sphere. (See Fig. 56-6.)

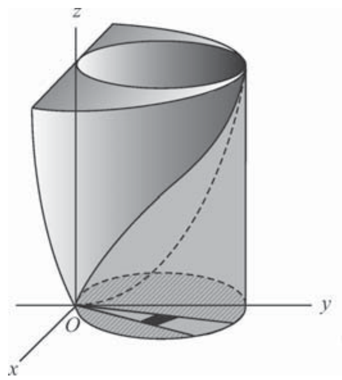

Fig. 56-5

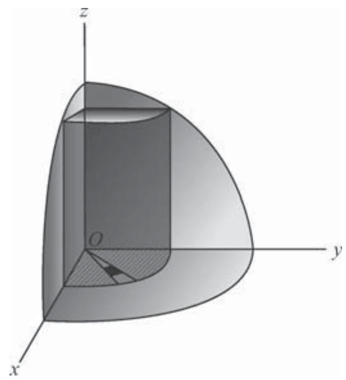

Fig. 56-6

From the figure, it is obvious that the required volume is eight times the volume in the first octant bounded by the cylinder $\rho^2 = a^2$, the sphere $\rho^2 + z^2 = 4a^2$, and the plane $z = 0$. The latter volume is obtained by integrating $z = \sqrt{4a^2 - \rho^2}$ over a quadrant of the circle $\rho = a$. Hence,

$$V = 8\int_0^{\pi/2}\int_0^{a} \sqrt{4a^2 - \rho^2}\rho\, d\rho\, d\theta = \frac{8}{3}\int_0^{\pi/2}(8a^3 - 3\sqrt{3}a^3)d\theta = \tfrac{4}{3}(8 - 3\sqrt{3})a^3\pi \text{ cubic units}$$

7. Derive formula (56.2).

Consider a region R^* of area S on the surface $z = f(x, y)$. Through the boundary of R^* pass a vertical cylinder (see Fig. 56-7) cutting the xy plane in the region R. Now divide R into n subregions $R_1, \ldots, R_n$ of areas $\Delta A_1, \ldots,$

ΔA_n, and denote by ΔS_i, the area of the projection of ΔA_i on R^*. In that ith subregion of R^*, choose a point P_i and draw there the tangent plane to the surface. Let the area of the projection of R_i on this tangent plane be denoted by ΔT_i. We shall use ΔT_i as an approximation of the corresponding surface area ΔS_i.

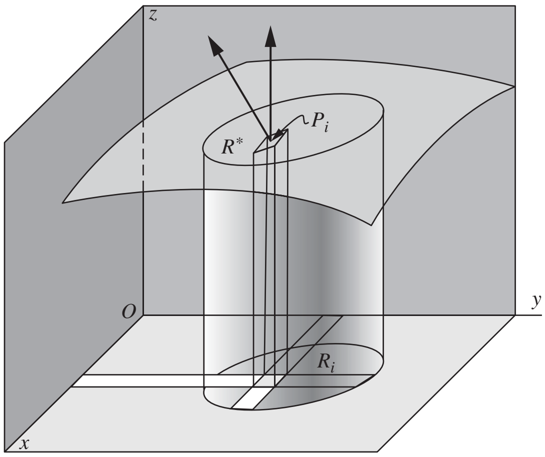

Fig. 56-7

Now the angle between the xy plane and the tangent plane at P_i is the angle γ_i between the z axis with direction numbers [0, 0, 1] and the normal, $\left[-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right] = \left[-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right]$, to the surface at P_i. Thus,

$$\cos\gamma_i = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

Then (see Fig. 56-8)

$$\Delta T_i \cos\gamma_i = \Delta A_i \quad \text{and} \quad \Delta T_i = \sec\gamma_i \Delta A_i$$

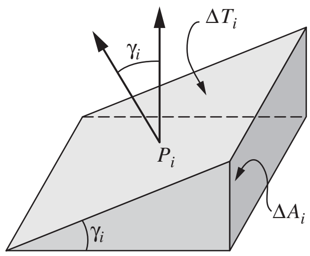

Fig. 56-8

Hence, an approximation of S is $\sum_{i=1}^{n} \Delta T_i = \sum_{i=1}^{n} \sec\gamma_i \Delta A_i$, and

$$S = \lim_{n\to+\infty} \sum_{i=1}^{n} \sec\gamma_i \Delta A_i = \iint_R \sec\gamma \, dA = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

8. Find the area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above the xy plane and inside the cylinder $x^2 + y^2 = 4y$.

Solution 1*:* Refer to Fig. 56-9. The projection of the required area on the xy plane is the region R enclosed by the circle $x^2 + y^2 = 4y$. For the cone,

$$\frac{\partial z}{\partial x} = \frac{1}{3}\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{3}\frac{y}{z}. \qquad \text{So} \qquad 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{9z^2 + x^2 + y^2}{9z^2} = \frac{12z^2}{9z^2} = \frac{4}{3}$$

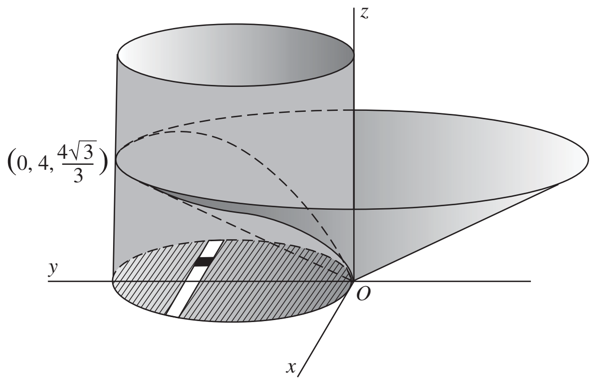

Fig. 56-9

Then $$S = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \int_0^4\int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \frac{2}{\sqrt{3}}\,dx\,dy = 2\frac{2}{\sqrt{3}}\int_0^4\int_0^{\sqrt{4y-y^2}} dx\,dy$$

$$= \frac{4}{\sqrt{3}}\int_0^4 \sqrt{4y-y^2}\,dy = \frac{8\sqrt{3}}{3}\pi \text{ square units}$$

Solution 2*:* Refer to Fig. 56-10. The projection of one-half the required area on the *yz* plane is the region *R* bounded by the line $y = \sqrt{3}z$ and the parabola $y = \frac{3}{4}z^2$, the latter having been obtained by eliminating *x* from the equations of the two surfaces. For the cone,

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = \frac{3z}{x}. \quad \text{So} \quad 1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2+y^2+9z^2}{x^2} = \frac{12z^2}{x^2} = \frac{12z^2}{3z^2-y^2}.$$

Then $$S = 2\int_0^4\int_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}} \frac{2\sqrt{3}z}{\sqrt{3z^2-y^2}}\,dz\,dy = \frac{4\sqrt{3}}{3}\int_0^4 [\sqrt{3z^2-y^2}\,]_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}}\,dy = \frac{4\sqrt{3}}{3}\int_0^4 \sqrt{4y-y^2}\,dy.$$

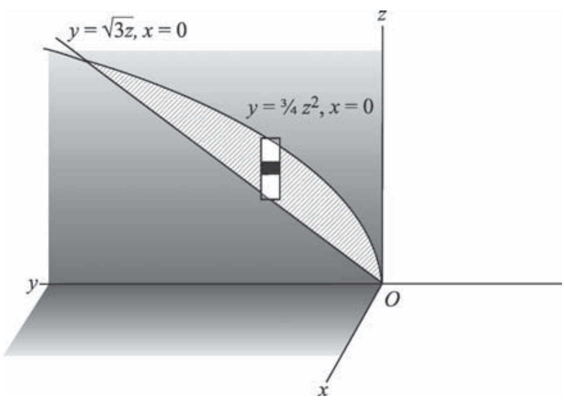

Fig. 56-10

Solution 3*:* Using polar coordinates in solution 1, we must integrate $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \frac{2}{\sqrt{3}}$ over the region *R* enclosed by the circle $\rho = 4\sin\theta$. Then,

$$S = \iint_R \frac{2}{\sqrt{3}}\,dA = \int_0^\pi\int_0^{4\sin\theta} \frac{2}{\sqrt{3}}\rho\,d\rho\,d\theta = \frac{1}{\sqrt{3}}\int_0^\pi [\rho^2]_0^{4\sin\theta}\,d\theta$$

$$= \frac{16}{\sqrt{3}}\int_0^\pi \sin^2\theta\,d\theta = \frac{8\sqrt{3}}{3}\pi \text{ square units}$$

9. Find the area of the portion of the cylinder $x^2 + z^2 = 16$ lying inside the cylinder $x^2 + y^2 = 16$.

Fig. 56-11 shows one-eighth of the required area, its projection on the xy plane being a quadrant of the circle $x^2 + y^2 = 16$. For the cylinder $x^2 + z^2 = 16$,

$$\frac{\partial z}{\partial x} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = 0. \quad \text{So} \quad 1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{x^2+z^2}{z^2} = \frac{16}{16-x^2}.$$

Then

$$S = 8\int_0^4 \int_0^{\sqrt{16-x^2}} \frac{4}{\sqrt{16-x^2}}\, dy\, dx = 32\int_0^4 dx = 128 \text{ square units}$$

10. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 16$ outside the paraboloid $x^2 + y^2 + z = 16$.

Fig. 56-12 shows one-fourth of the required area, its projection on the yz plane being the region R bounded by the circle $y^2 + z^2 = 16$, the y and z axes, and the line $z = 1$. For the sphere,

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = -\frac{z}{x}. \quad \text{So} \quad 1+\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2+y^2+z^2}{x^2} = \frac{16}{16-y^2-z^2}.$$

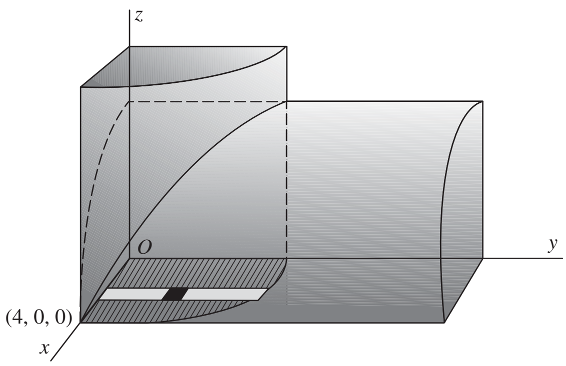

Fig. 56-11

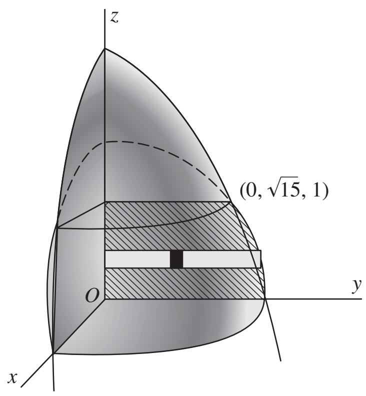

Fig. 56-12

Then

$$S = 4\iint_R \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dA = 4\int_0^1 \int_0^{\sqrt{16-z^2}} \frac{4}{\sqrt{16-y^2-z^2}}\, dy\, dz$$

$$= 16\int_0^1 \left[\sin^{-1}\left(\frac{y}{\sqrt{16-z^2}}\right)\right]_0^{\sqrt{16-z^2}} dz = 16\int_0^1 \frac{\pi}{2}\, dz = 8\pi \text{ square units}$$

11. Find the area of the portion of the cylinder $x^2 + y^2 = 6y$ lying inside the sphere $x^2 + y^2 + z^2 = 36$.

Fig. 56-13 shows one-fourth of the required area. Its projection on the yz plane is the region R bounded by the z and y axes and the parabola $z^2 + 6y = 36$, the latter having been obtained by eliminating x from the equations of the two surfaces. For the cylinder,

$$\frac{\partial x}{\partial y} = \frac{3-y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = 0. \quad \text{So} \quad 1+\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2+9-6y+y^2}{x^2} = \frac{9}{6y-y^2}$$

Then

$$S = 4\int_0^6 \int_0^{\sqrt{36-6y}} \frac{3}{\sqrt{6y-y^2}}\, dz\, dy = 12\int_0^6 \frac{\sqrt{6}}{\sqrt{y}}\, dy = 144 \text{ square units}$$

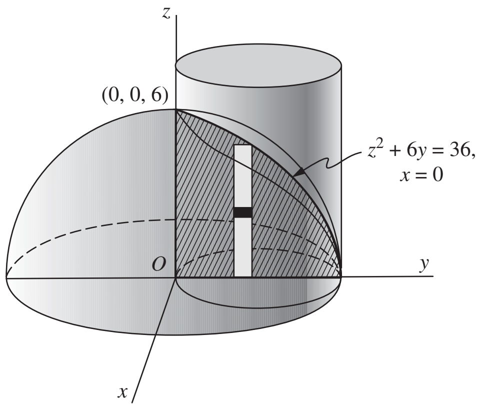

Fig. 56-13

SUPPLEMENTARY PROBLEMS

12. Find the volume cut from $9x^2 + 4y^2 + 36z = 36$ by the plane $z = 0$.

Ans. 3π cubic units

13. Find the volume under $z = 3x$ and above the first-quadrant area bounded by $x = 0$, $y = 0$, $x = 4$, and $x^2 + y^2 = 25$.

Ans. 98 cubic units

14. Find the volume in the first octant bounded by $x^2 + z = 9$, $3x + 4y = 24$, $x = 0$, $y = 0$, and $z = 0$.

Ans. 1485/16 cubic units

15. Find the volume in the first octant bounded by $xy = 4z$, $y = x$, and $x = 4$.

Ans. 8 cubic units

16. Find the volume in the first octant bounded by $x^2 + y^2 = 25$ and $z = y$.

Ans. 125/3 cubic units

17. Find the volume common to the cylinders $x^2 + y^2 = 16$ and $x^2 + z^2 = 16$.

Ans. 1024/3 cubic units

18. Find the volume in the first octant inside $y^2 + z^2 = 9$ and outside $y^2 = 3x$.

Ans. $27\pi/16$ cubic units

19. Find the volume in the first octant bounded by $x^2 + z^2 = 16$ and $x - y = 0$.

Ans. 64/3 cubic units

20. Find the volume in front of $x = 0$ and common to $y^2 + z^2 = 4$ and $y^2 + z^2 + 2x = 16$.

Ans. 28π cubic units

21. Find the volume inside $\rho = 2$ and outside the cone $z^2 = \rho^2$.

Ans. $32\pi/3$ cubic units

22. Find the volume inside $y^2 + z^2 = 2$ and outside $x^2 - y^2 - z^2 = 2$.

Ans. $8\pi(4 - \sqrt{2})/3$ cubic units

23. Find the volume common to $\rho^2 + z^2 = a^2$ and $\rho = a \sin \theta$.

Ans. $2(3\pi - 4)a^2/9$ cubic units

24. Find the volume inside $x^2 + y^2 = 9$, bounded below by $x^2 + y^2 + 4z = 16$ and above by $z = 4$.

Ans. $81\pi/8$ cubic units

25. Find the volume cut from the paraboloid $4x^2 + y^2 = 4z$ by the plane $z - y = 2$.

Ans. 9π cubic units

26. Find the volume generated by revolving the cardiod $\rho = 2(1 - \cos \theta)$ about the polar axis.

Ans. $V = 2\pi \iint y\rho \, d\rho \, d\theta = 64\pi/3$ cubic units

27. Find the volume generated by revolving a petal of $\rho = \sin 2\theta$ about either axis.

Ans. $32\pi/105$ cubic units

28. Find the area of the portion of the cone $x^2 + y^2 = z^2$ inside the vertical prism whose base is the triangle bounded by the lines $y = x$, $x = 0$, and $y = 1$ in the xy plane.

Ans. $\frac{1}{2}\sqrt{2}$ square units

29. Find the area of the portion of the plane $x + y + z = 6$ inside the cylinder $x^2 + y^2 = 4$.

Ans. $4\sqrt{3}\pi$ square units

30. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 36$ inside the cylinder $x^2 + y^2 = 6y$.

Ans. $72(\pi - 2)$ square units

31. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 4z$ inside the paraboloid $x^2 + y^2 = z$.

Ans. 4π square units

32. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 25$ between the planes $z = 2$ and $z = 4$.

Ans. 20π square units

33. Find the area of the portion of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$.

Ans. $2\pi(2\sqrt{2} - 1)/3$ square units

34. Find the area of the surface of the cone $x^2 + y^2 - 9z^2 = 0$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 6y$.

Ans. $3\sqrt{10}\pi$ square units

35. Find the area of that part of the sphere $x^2 + y^2 + z^2 = 25$ that is within the elliptic cylinder $2x^2 + y^2 = 25$.

Ans. 50π square units

36. Find the area of the surface of $x^2 + y^2 - az = 0$ which lies directly above the lemniscate $4\rho^2 = a^2 \cos 2\theta$.

Ans. $S = \frac{4}{a}\iint \sqrt{4\rho^2 + a^2}\,\rho\, d\rho\, d\theta = \frac{a^2}{3}\left(\frac{5}{3} - \frac{\pi}{4}\right)$ square units

37. Find the area of the surface of $x^2 + y^2 + z^2 = 4$ which lies directly above the cardioid $\rho = 1 - \cos\theta$.

Ans. $8[\pi - \sqrt{2} - \ln(\sqrt{2} + 1)]$ square units