CHAPTER 39

Plane Vectors

Scalars and Vectors

Quantities such as time, temperature, and speed, which have magnitude only, are called *scalars*. Quantities such as force, velocity, and acceleration, which have both magnitude and direction, are called *vectors*. Vectors are represented geometrically by directed line segments (arrows). The direction of the arrow (the angle that it makes with some fixed directed line of the plane) is the direction of the vector, and the length of the arrow represents the magnitude of the vector.

Scalars will be denoted by letters a, b, c, . . . in ordinary type; vectors will be denoted in bold type by letters **a**, **b**, **c**, . . ., or by an expression of the form **OP** (where it is assumed that the vector goes from O to P. (See Fig. 39-1(a).) The magnitude (length) of a vector **a** or **OP** will be denoted |**a**| or |**OP**|.

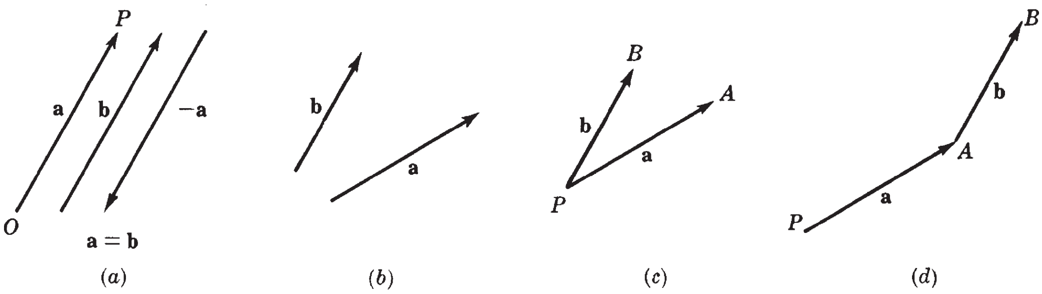

Fig. 39-1

Two vectors **a** and **b** are said to be equal (and we write **a** = **b**) if they have the same direction and magnitude. A vector whose magnitude is that of **a**, but whose direction is opposite that of **a**, is called the *negative* of **a** and is denoted **–a**. (See Fig. 39-1(*a*).)

If **a** is a vector and k is a positive scalar, then k**a** is defined to be a vector whose direction is that of **a** and whose magnitude is k times that of **a**. If k is a negative scalar, then k**a** has direction opposite that of **a** and has magnitude $|k|$ times that of **a**.

We also assume a *zero vector* **0** with magnitude 0 and no direction. We define **–0** = **0**, 0**a** = **0**, and k**0** = **0**.

Unless indicated otherwise, a given vector has no fixed position in the plane and so may be moved under parallel displacement at will. In particular, if **a** and **b** are two vectors (Fig. 39-1(*b*)), they may be placed so as to have a common initial or beginning point P (Fig. 39-1(*c*)) or so that the initial point of **b** coincides with the terminal or endpoint of **a** (Fig. 39-1 (*d*)).

Sum and Difference of Two Vectors

If **a** and **b** are the vectors of Fig. 39-1(*b*), their *sum* **a** + **b** is to be found in either of two equivalent ways:

1. By placing the vectors as in Fig. 39-1(*c*) and completing the parallelogram *PAQB* of Fig. 39-2(*a*). The vector **PQ** is the required sum.
2. By placing the vectors as in Fig. 39-1(*d*) and completing the triangle *PAB* of Fig. 39-2(*b*). Here, the vector **PB** is the required sum.

It follows from Fig. 39-2(*b*) that three vectors may be displaced to form a triangle, provided that one of them is either the sum or the negative of the sum of the other two.

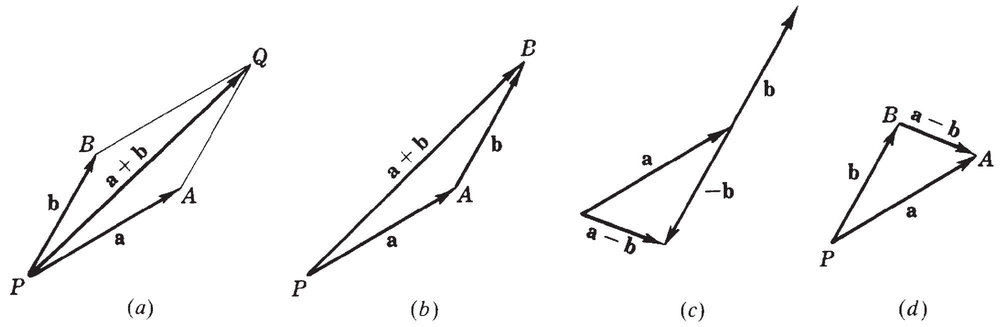

Fig. 39-2

If **a** and **b** are the vectors of Fig. 39-1(*b*), their difference **a** − **b** is to be found in either of two equivalent ways:

1. From the relation **a** − **b** = **a** + (− **b**) as in Fig. 39-2(*c*).
2. By placing the vectors as in Fig. 39-1(*c*) and completing the triangle. In Fig. 39-2(*d*), the vector

   BA = **a** − **b**.

If **a**, **b**, and **c** are vectors, the following laws are valid.

PROPERTY (39.1) (*Commutative Law*) **a** + **b** = **b** + **a**
PROPERTY (39.2) (*Associative Law*) **a** + (**b** + **c**) = (**a** + **b**) + **c**
PROPERTY (39.3) (*Distributive Law*) $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

See Problems 1 to 4.

Components of a Vector

In Fig. 39-3(*a*), let **a** = **PQ** be a given vector, and let *PM* and *PN* be any two other directed lines through *P*. Construct the parallelogram *PAQB*. Then

$$\mathbf{a} = \mathbf{PA} + \mathbf{PB}$$

and **a** is said to be *resolved* in the directions *PM* and *PN*. We shall call **PA** and **PB** the vector components of **a** in the pair of directions *PM* and *PN*.

Consider next the vector **a** in a rectangular coordinate system (Fig. 39-3(*b*)), having equal units of measure on the two axes. Denote by **i** the vector from (0, 0) to (1, 0), and by **j** the vector from (0, 0) to (0, 1). The direction of **i** is that of the positive *x* axis, the direction of **j** is that of the positive *y* axis, and both are *unit vectors*, that is, vectors of magnitude 1.

From the initial point *P* and the terminal point *Q* of **a**, drop perpendiculars to the *x* axis, meeting it at *M* and *N*, respectively, and to the *y* axis, meeting it at *S* and *T*, respectively. Now, $MN = a_1\mathbf{i}$, with a_1 positive, and $ST = a_2\mathbf{j}$, with a_2 negative. Then: $\mathbf{MN} = \mathbf{RQ} = a_1\mathbf{i}$, $\mathbf{ST} = \mathbf{PR} = a_2\mathbf{j}$, and

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} \tag{39.1}$$

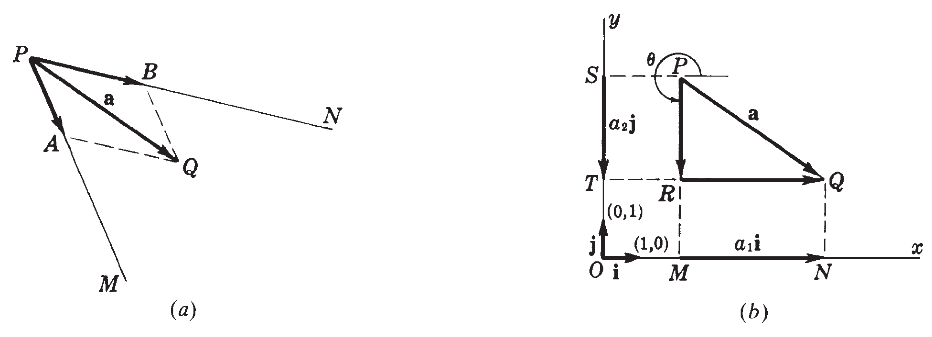

Fig. 39-3

Let us call $a_1\mathbf{i}$ and $a_2\mathbf{j}$ the *vector components* of **a**.† The scalars a_1 and a_2 will be called the *scalar components* (or the *x component* and *y component*, or simply the *components*) of **a**. Note that $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$.

Let the direction of **a** be given by the angle θ, with $0 \le \theta < 2\pi$, measured counterclockwise from the positive x axis to the vector. Then

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2} \tag{39.2}$$

and

$$\tan\theta = \frac{a_2}{a_1} \tag{39.3}$$

with the quadrant of θ being determined by

$$a_1 = |\mathbf{a}|\cos\theta, \quad a_2 = |\mathbf{a}|\sin\theta$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{j} + b_2\mathbf{j}$, then the following hold.

PROPERTY (39.4)	$\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1$ and $a_2 = b_2$
PROPERTY (39.5)	$k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j}$
PROPERTY (39.6)	$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$
PROPERTY (39.7)	$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}$

Scalar Product (or Dot Product)

The *scalar product* (or *dot product*) of vectors **a** and **b** is defined by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \tag{39.4}$$

where θ is the smaller angle between the two vectors when they are drawn with a common initial point (see Fig. 39-4). We also define: $\mathbf{a}\cdot\mathbf{0} = \mathbf{0}\cdot\mathbf{a} = 0$.

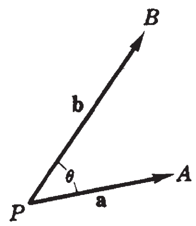

Fig. 39-4

From the definitions, we can derive the following properties of the scalar product.

PROPERTY (39.8) (*Commutative Law*)	$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
PROPERTY (39.9)	$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$ and $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$
PROPERTY (39.10)	$\mathbf{a}\cdot\mathbf{b} = 0$ if and only if ($\mathbf{a} = 0$ or $\mathbf{b} = 0$ or **a** is perpendicular to **b**)
PROPERTY (39.11)	$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = 0$
PROPERTY (39.12)	$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j})\cdot(b_1\mathbf{i} + b_2\mathbf{j}) = a_1b_1 + a_2b_2$
PROPERTY (39.13) (*Distributive Law*)	$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
PROPERTY (39.14)	$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d}) = \mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}$

†A pair of directions (such as *OM* and *OT*) need not be mentioned, since they are determined by the coordinate system.

Scalar and Vector Projections

In equation (39.1), the scalar a_1 is called the *scalar projection* of **a** on any vector whose direction is that of the positive x axis, while the vector $a_1\mathbf{i}$ is called the *vector projection* of **a** on any vector whose direction is that of the positive x axis. In general, for any nonzero vector **b** and any vector **a**, we define we define $\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}$ to be the scalar projection of **a** on **b**, and $\left(\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|}$ to be the vector projection of **a** on **b**. (See Problem 7.) Note that, when **b** has the direction of the positive x axis, $\frac{\mathbf{b}}{|\mathbf{b}|}=\mathbf{i}$.

PROPERTY (39.15) $\mathbf{a}\cdot\mathbf{b}$ is the product of the length of **a** and the scalar projection of **b** on **a**. Likewise, $\mathbf{a}\cdot\mathbf{b}$ is the product of the length of **b** and the scalar projection of **a** on **b**. (See Fig. 39-5.)

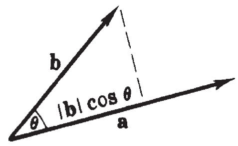

Fig. 39-5

Differentiation of Vector Functions

Let the curve of Fig. 39-6 be given by the parametric equations $x=f(u)$ and $y=g(u)$. The vector

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}=f(u)\mathbf{i}+g(u)\mathbf{j}$$

joining the origin to the point $P(x, y)$ of the curve is called the *position vector* or the *radius vector* of P. It is a function of u. (From now on, the letter **r** will be used exclusively to denote position vectors. Thus, $\mathbf{a}=3\mathbf{i}+4\mathbf{j}$ is meant to be a “free” vector, whereas $\mathbf{r}=3\mathbf{i}+4\mathbf{j}$ is meant to be the vector joining the origin to $P(3, 4)$.)

The derivative $\frac{d\mathbf{r}}{du}$ of the function **r** with respect to u is defined to be $\lim_{\Delta u\to 0}\frac{\mathbf{r}(u+\Delta u)-\mathbf{r}(u)}{\Delta u}$.

Straightforward computation yields:

$$\frac{d\mathbf{r}}{du}=\frac{dx}{du}\mathbf{i}+\frac{dy}{du}\mathbf{j} \tag{39.5}$$

Let s denote the arc length measured from a fixed point P_0 of the curve so that s increases with u. If τ is the angle that $d\mathbf{r}/du$ makes with the positive x axis, then

$$\tan\tau=\left(\frac{dy}{du}\right)\bigg/\left(\frac{dx}{du}\right)=\frac{dy}{dx}=\text{the slope of the curve at } P$$

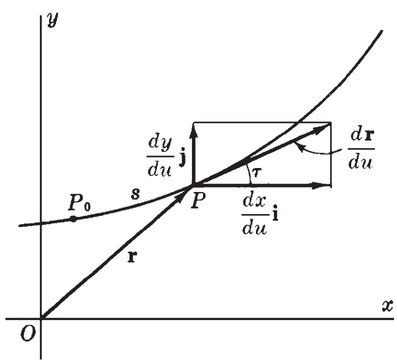

Fig. 39-6

Moreover, $\frac{d\mathbf{r}}{du}$ is a vector of magnitude

$$\left|\frac{d\mathbf{r}}{du}\right| = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} = \frac{ds}{du} \tag{39.6}$$

whose direction is that of the tangent line to the curve at P. It is customary to show this vector with P as its initial point.

If now the scalar variable u is taken to be the arc length s, then equation (39.5) becomes

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \tag{39.7}$$

The direction of $\mathbf{t}$ is τ, while its magnitude is $\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}$, which is equal to 1. Thus, $\mathbf{t} = d\mathbf{r}/ds$ is the *unit tangent vector* to the curve at P.

Since $\mathbf{t}$ is a unit vector, $\mathbf{t}$ and $d\mathbf{t}/ds$ are perpendicular. (See Problem 10.) Denote by $\mathbf{n}$ a unit vector at P having the direction of $d\mathbf{t}/ds$. As P moves along the curve shown in Fig. 39-7, the magnitude of $\mathbf{t}$ remains constant; hence $d\mathbf{t}/ds$ measures the rate of change of the direction of $\mathbf{t}$. Thus, the magnitude of $d\mathbf{t}/ds$ at P is the absolute value of the curvature at P, that is, $|d\mathbf{t}/ds| = |K|$, and

$$\frac{d\mathbf{t}}{ds} = |K|\,\mathbf{n} \tag{39.8}$$

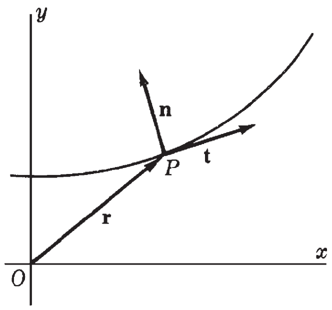

Fig. 39-7

SOLVED PROBLEMS

1. Prove $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

From Fig. 39-8, $\mathbf{a} + \mathbf{b} = \mathbf{PQ} = \mathbf{b} + \mathbf{a}$.

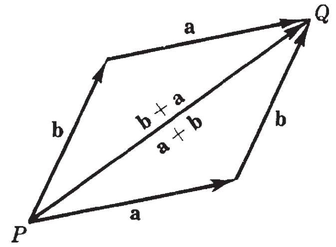

Fig. 39-8

2. Prove $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

From Fig. 39-9, $\mathbf{PC} = \mathbf{PB} + \mathbf{BC} = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$. Also, $\mathbf{PC} = \mathbf{PA} + \mathbf{AC} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

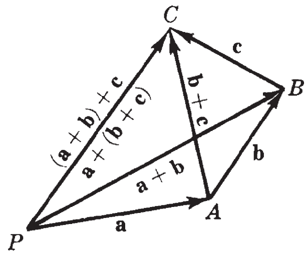

Fig. 39-9

3. Let **a**, **b**, and **c** be three vectors issuing from P such that their endpoints A, B, and C lie on a line, as shown in Fig. 39-10. If C divides BA in the ratio $x:y$, where $x + y = 1$, show that $\mathbf{c} = x\mathbf{a} + y\mathbf{b}$.

Just note that

$$\mathbf{c} = \mathbf{PB} + \mathbf{BC} = \mathbf{b} + x(\mathbf{a} - \mathbf{b}) = x\mathbf{a} + (1 - x)\mathbf{b} = x\mathbf{a} + y\mathbf{b}$$

As an example, if C bisects BA, then $\mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{BC} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$.

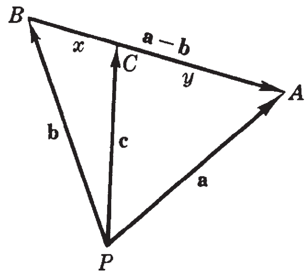

Fig. 39-10

4. Prove: The diagonals of a parallelogram bisect each other.

Let the diagonals intersect at Q, as in Fig. 39-11. Since $\mathbf{PB} = \mathbf{PQ} + \mathbf{QB} = \mathbf{PQ} - \mathbf{BQ}$, there are positive numbers x and y such that $\mathbf{b} = x(\mathbf{a} + \mathbf{b}) - y(\mathbf{a} - \mathbf{b}) = (x - y)\mathbf{a} + (x + y)\mathbf{b}$. Then $x + y = 1$ and $x - y = 0$. Hence, $x = y = \frac{1}{2}$, and Q is the midpoint of each diagonal.

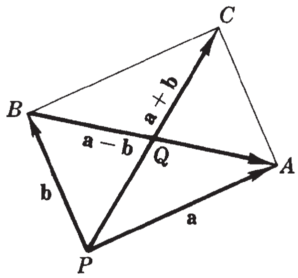

Fig. 39-11

5. For the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$, find the magnitude and direction of (a) **a** and **b**; (b) $\mathbf{a} + \mathbf{b}$; (c) $\mathbf{b} - \mathbf{a}$.

(a) For $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2} = \sqrt{3^2 + 4^2} = 5$; $\tan\theta = a_2/a_1 = \frac{4}{3}$ and $\cos\theta = a_1/|\mathbf{a}| = \frac{3}{5}$; then θ is a first quadrant angle and is $53°8'$.

For $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$: $|\mathbf{b}| = \sqrt{4 + 1} = \sqrt{5}$; $\tan\theta = -\frac{1}{2}$ and $\cos\theta = 2/\sqrt{5}$; $\theta = 360° - 26°34' = 333°26'$.

(b) $\mathbf{a} + \mathbf{b} = (3\mathbf{i} + 4\mathbf{j}) + (2\mathbf{i} - \mathbf{j}) = 5\mathbf{i} + 3\mathbf{j}$. Then $|\mathbf{a} + \mathbf{b}| = \sqrt{5^2 + 3^2} = \sqrt{34}$. Since $\tan\theta = \frac{3}{5}$ and $\cos\theta = 5/\sqrt{34}$, $\theta = 30°58'$.

(c) $\mathbf{b} - \mathbf{a} = (2\mathbf{i} - \mathbf{j}) - (3\mathbf{i} + 4\mathbf{j}) = -\mathbf{i} - 5\mathbf{j}$. Then $|\mathbf{b} - \mathbf{a}| = \sqrt{26}$. Since $\tan\theta = 5$ and $\cos\theta = -1/\sqrt{26}$, $\theta = 258°41'$.

6. Prove: The median to the base of an isosceles triangle is perpendicular to the base. (See Fig. 39-12, where $|\mathbf{a}| = |\mathbf{b}|$.)

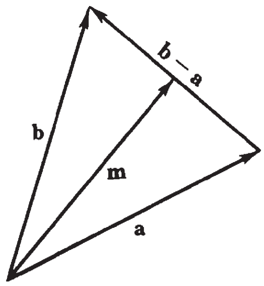

Fig. 39-12

From Problem 3, since $\mathbf{m}$ bisects the base, $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$. Then

$$\begin{aligned}\mathbf{m} \cdot (\mathbf{b} - \mathbf{a}) &= \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \tfrac{1}{2}(\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a}) = \tfrac{1}{2}(\mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a}) = 0\end{aligned}$$

Thus, the median is perpendicular to the base.

7. If $\mathbf{b}$ is a nonzero vector, resolve a vector $\mathbf{a}$ into components $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively parallel and perpendicular to $\mathbf{b}$.

In Fig. 39-13, we have $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}_1 = c\mathbf{b}$, and $\mathbf{a}_2 \cdot \mathbf{b} = 0$. Hence, $\mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = \mathbf{a} - c\mathbf{b}$. Moreover, $\mathbf{a}_2 \cdot \mathbf{b} = (\mathbf{a} - c\mathbf{b}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} - c|\mathbf{b}|^2 = 0$, whence $c = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}$. Thus,

$$\mathbf{a}_1 = c\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \quad \text{and} \quad \mathbf{a}_2 = \mathbf{a} - c\mathbf{b} = \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

The scalar $\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|}$ is the scalar projection of $\mathbf{a}$ on $\mathbf{b}$. The vector $\left(\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|}\right)\dfrac{\mathbf{b}}{|\mathbf{b}|}$ is the vector projection of $\mathbf{a}$ on $\mathbf{b}$.

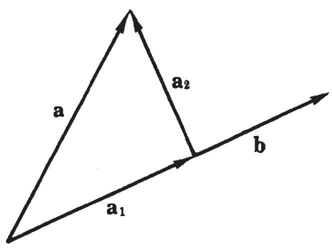

Fig. 39-13

8. Resolve $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ into components $\mathbf{a}_1$ and $\mathbf{a}_2$, parallel and perpendicular, respectively, to $\mathbf{b} = 3\mathbf{i} + \mathbf{j}$.

From Problem 7, $c = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} = \dfrac{12 + 3}{10} = \dfrac{3}{2}$. Then

$$\mathbf{a}_1 = c\mathbf{b} = \tfrac{9}{2}\mathbf{i} + \tfrac{3}{2}\mathbf{j} \text{ and } \mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = -\tfrac{1}{2}\mathbf{i} + \tfrac{3}{2}\mathbf{j}$$

9. If $\mathbf{a} = f_1(u)\mathbf{i} + f_2(u)\mathbf{j}$ and $\mathbf{b} = g_1(u)\mathbf{i} + g_2(u)\mathbf{j}$, show that $\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{du}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{du}$.

By Property 39.12, $\mathbf{a}\cdot\mathbf{b} = (f_1(u)\mathbf{i} + f_2(u)\mathbf{j})\cdot(g_1(u)\mathbf{i} + g_2(u)\mathbf{j}) = f_1g_1 + f_2g_2$. Then

$$\begin{aligned}\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) &= \frac{df_1}{du}g_1 + f_1\frac{dg_1}{du} + \frac{df_2}{du}g_2 + f_2\frac{dg_2}{du}\\ &= \left(\frac{df_1}{du}g_1 + \frac{df_2}{du}g_2\right) + \left(f_1\frac{dg_1}{du} + f_2\frac{dg_2}{du}\right)\\ &= \left(\frac{df_1}{du}\mathbf{i} + \frac{df_2}{du}\mathbf{j}\right)\cdot(g_1\mathbf{i} + g_2\mathbf{j}) + (f_1(u)\mathbf{i} + f_2(u)\mathbf{j})\cdot\left(\frac{dg_1}{du}\mathbf{i} + \frac{dg_2}{du}\mathbf{j}\right)\\ &= \frac{d\mathbf{a}}{du}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{du}\end{aligned}$$

10. If $\mathbf{a} = f_1(u)\mathbf{i} + f_2(u)\mathbf{j}$ is of constant nonzero magnitude, show that $\mathbf{a}\cdot\frac{d\mathbf{a}}{du} = 0$ and, therefore, when $\frac{d\mathbf{a}}{du}$ is not zero, $\mathbf{a}$ and $\frac{d\mathbf{a}}{du}$ are perpendicular.

Let $|\mathbf{a}| = c$. Thus, $\mathbf{a}\cdot\mathbf{a} = c^2$. By Problem 9,

$$\frac{d}{du}(\mathbf{a}\cdot\mathbf{a}) = \frac{d\mathbf{a}}{du}\cdot\mathbf{a} + \mathbf{a}\cdot\frac{d\mathbf{a}}{du} = 2\mathbf{a}\cdot\frac{d\mathbf{a}}{du} = 0$$

Then $\mathbf{a}\cdot\frac{d\mathbf{a}}{du} = 0$.

11. Given $\mathbf{r} = (\cos^2\theta)\mathbf{i} + (\sin^2\theta)\mathbf{j}$, for $0 \le \theta \le \pi/2$, find $\mathbf{t}$.

Since $\frac{d}{d\theta}\cos^2\theta = -2\cos\theta\sin\theta = -\sin 2\theta$ and $\frac{d}{d\theta}\sin^2\theta = 2\sin\theta\cos\theta = \sin 2\theta$, equation (39.5) yields

$$\frac{d\mathbf{r}}{d\theta} = -(\sin 2\theta)\mathbf{i} + (\sin 2\theta)\mathbf{j}$$

Therefore, by equation (39.6),

$$\frac{ds}{d\theta} = \left|\frac{d\mathbf{r}}{d\theta}\right| = \sqrt{\frac{d\mathbf{r}}{d\theta}\cdot\frac{d\mathbf{r}}{d\theta}} = \sqrt{2}\sin 2\theta$$

by Property 39.12. So,

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

12. Given $x = a\cos^3\theta$, $y = a\sin^3\theta$, with $0 \le \theta \le \pi/2$, find $\mathbf{t}$ and $\mathbf{n}$ when $\theta = \pi/4$.

We have $\mathbf{r} = a(\cos^3\theta)\mathbf{i} + a(\sin^3\theta)\mathbf{j}$. Then

$$\frac{d\mathbf{r}}{d\theta} = -3a(\cos^2\theta)(\sin\theta)\mathbf{i} + 3a(\sin^2\theta)(\cos\theta)\mathbf{j} \qquad \text{and} \qquad \frac{ds}{d\theta} = \left|\frac{d\mathbf{r}}{d\theta}\right| = 3a\sin\theta\cos\theta$$

Hence,

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds} = -(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \qquad \text{and} \qquad \frac{d\mathbf{t}}{ds} = ((\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j})\frac{d\theta}{ds}$$

$$= \frac{1}{3a\cos\theta}\mathbf{i} + \frac{1}{3a\sin\theta}\mathbf{j}$$

At $\theta = \pi/4$,

$$\mathbf{t} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j},\ \frac{d\mathbf{t}}{ds} = \frac{\sqrt{2}}{3a}\mathbf{i} + \frac{\sqrt{2}}{3a}\mathbf{j},\ |K| = \left|\frac{d\mathbf{t}}{ds}\right| = \frac{2}{3a} \quad \text{and} \quad \mathbf{n} = \frac{1}{|K|}\frac{d\mathbf{t}}{ds} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

13. Show that the vector $\mathbf{a} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by + c = 0$.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two distinct points on the line. Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$. Subtracting the first from the second yields

$$a(x_2 - x_1) + b(y_2 - y_1) = 0 \tag{1}$$

Now

$$a(x_2 - x_1) + b(y_2 - y_1) = (a\mathbf{i} + b\mathbf{j}) \cdot [(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}]$$
$$= \mathbf{a} \cdot \mathbf{P_1P_2}$$

By (1), the left side is zero. Thus, **a** is perpendicular (normal) to the line.

14. Use vector methods to find:

(a) The equation of the line through $P_1(2, 3)$ and perpendicular to the line $x + 2y + 5 = 0$.
(b) The equation of the line through, $P_1(2, 3)$ and $P_2(5, -1)$.

Take $P(x, y)$ to be any other point on the required line.

(a) By Problem 13, the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ is normal to the line $x + 2y + 5 = 0$. Then $\mathbf{P_1P} = (x - 2)\mathbf{i} + (y - 3)\mathbf{j}$ is parallel to **a** if $(x - 2)\mathbf{i} + (y - 3)\mathbf{j} = k(\mathbf{i} + 2\mathbf{j})$ for some scalar k. Equating components, we get $x - 2 = k$ and $y - 3 = 2k$. Eliminating k, we obtain the required equation $y - 3 = 2(x - 2)$, or, equivalently, $2x - y - 1 = 0$.
(b) We have $\mathbf{P_1P} = (x - 2)\mathbf{i} + (y - 3)\mathbf{j}$ and $\mathbf{P_1P_2} = 3\mathbf{i} - 4\mathbf{j}$. Now $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ is perpendicular to $\mathbf{P_1P_2}$ and, hence, to $\mathbf{P_1P}$. Thus, $0 = \mathbf{a} \cdot \mathbf{P_1P} = (4\mathbf{i} + 3\mathbf{j}) \cdot [(x - 2)\mathbf{i} + (y - 3)\mathbf{j}]$ and, equivalently, $4x + 3y - 17 = 0$.

15. Use vector methods to find the distance of the point $P_1(2, 3)$ from the line $3x + 4y - 12 = 0$.

At any convenient point on the line, say $A(4, 0)$, construct the vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ perpendicular to the line. The required distance is $d = |\mathbf{AP_1}|\ cos\ \theta$ in Fig. 39-14. Now, $\mathbf{a} \cdot \mathbf{AP_1} = |\mathbf{a}|\ |\mathbf{AP_1}| \cos\theta = |\mathbf{a}|\ d$. Hence,

$$d = \frac{\mathbf{a} \cdot \mathbf{AP_1}}{|\mathbf{a}|} = \frac{(3\mathbf{i} + 4\mathbf{j}) \cdot (-2\mathbf{i} + 3\mathbf{j})}{5} = \frac{-6 + 12}{5} = \frac{6}{5}$$

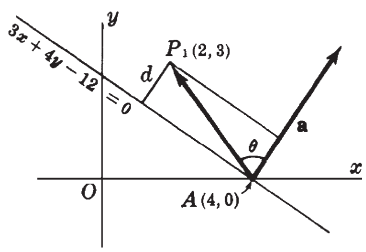

Fig. 39-14

16. The *work* done by a force expressed as a vector **b** in moving an object along a vector **a** is defined as the product of the magnitude of **b** in the direction of **a** and the distance moved. Find the work done in moving an object along the vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ if the force applied is $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$.

The work done is

$$(\text{magnitude of } \mathbf{b} \text{ in the direction of } \mathbf{a}) \cdot (\text{distance moved}) = (|\mathbf{b}|\cos\theta)\ |\mathbf{a}| = \mathbf{b} \cdot \mathbf{a} = (2\mathbf{i} + \mathbf{j}) \cdot (3\mathbf{i} + 4\mathbf{j}) = 10$$

SUPPLEMENTARY PROBLEMS

17. Given the vectors **a**, **b**, **c** in Fig. 39-15, construct (a) 2**a**; (b) –3**b**; (c) **a** + 2**b**; (d) **a** + **b** – **c**; (e) **a** – 2**b** + 3**c**.

18. Prove: The line joining the midpoints of two sides of a triangle is parallel to and one-half the length of the third side. (See Fig. 39-16.)

19. If **a**, **b**, **c**, **d** are consecutive sides of a quadrilateral (see Fig. 39-17), show that **a** + **b** + **c** + **d** = 0. (*Hint:* Let *P* and *Q* be two nonconsecutive vertices.) Express **PQ** in two ways.

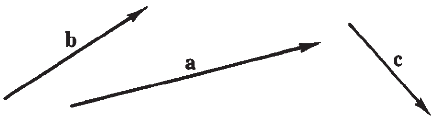

Fig. 39-15

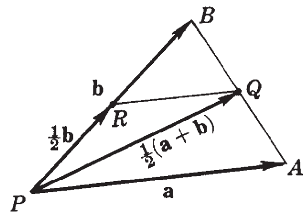

Fig. 39-16

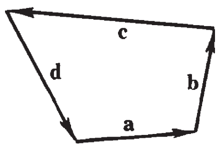

Fig. 39-17

20. Prove: If the midpoints of the consecutive sides of any quadrilateral are joined, the resulting quadrilateral is a parellelogram. (See Fig. 39-18.)

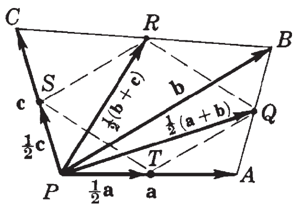

Fig. 39-18

21. Using Fig. 39-19, in which |**a**| = |**b**| is the radius of a circle, prove that the angle inscribed in a semicircle is a right angle.

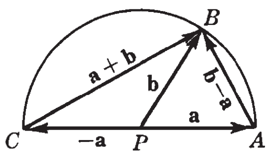

Fig. 39-19

22. Find the length of each of the following vectors and the angle it makes with the positive *x* axis: (a) $\mathbf{i}+\mathbf{j}$; (b) $-\mathbf{i}+\mathbf{j}$; (c) $\mathbf{i}+\sqrt{3}\mathbf{j}$; (d) $\mathbf{i}-\sqrt{3}\mathbf{j}$.

Ans. (a) $\sqrt{2}$, $\theta=\frac{1}{4}\pi$; (b) $\sqrt{2}$, $\theta=3\pi/4$; (c) 2, $\theta=\pi/3$; (d) 2, $\theta=5\pi/3$

23. Prove: If **u** is obtained by rotating the unit vector **i** counterclockwise about the origin through the angle θ, then $\mathbf{u}=\mathbf{i}\cos\theta+\mathbf{j}\sin\theta$.

24. Use the law of cosines for triangles to obtain $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = \frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2)$.

25. Write each of the following vectors in the form $a\mathbf{i} + b\mathbf{j}$.

(a) The vector joining the origin to $P(2, -3)$; (b) The vector joining $P_1(2, 3)$ to $P_2(4, 2)$;
(c) The vector joining $P_2(4, 2)$ to P_1 $(2, 3)$; (d) The unit vector in the direction of $3\mathbf{i} + 4\mathbf{j}$;
(e) The vector having magnitude 6 and direction 120°

Ans. (a) $2\mathbf{i} - 3\mathbf{j}$; (b) $2\mathbf{i} - \mathbf{j}$; (c) $-2\mathbf{i} + \mathbf{j}$; (d) $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$; (e) $-3\mathbf{i} + 3\sqrt{3}\mathbf{j}$

26. Using vector methods, derive the formula for the distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

27. Given $O(0, 0)$, $A(3, 1)$, and $B(1, 5)$ as vertices of the parallelogram $OAPB$, find the coordinates of P.

Ans. (4, 6)

28. (a) Find k so that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + k\mathbf{j}$ are perpendicular. (b) Write a vector perpendicular to $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$.

29. Prove Properties (39.8) to (39.15).

30. Find the vector projection and scalar projection of $\mathbf{b}$ on $\mathbf{a}$, given: (a) $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = -3\mathbf{i} + \mathbf{j}$; (b) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 10\mathbf{i} + 2\mathbf{j}$.

Ans. (a) $-\mathbf{i} + 2\mathbf{j}, -\sqrt{5}$; (b) $4\mathbf{i} + 6\mathbf{j}, 2\sqrt{13}$

31. Prove: Three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ will, after parallel displacement, form a triangle provided (a) one of them is the sum of the other two or (b) $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

32. Show that $\mathbf{a} = 3\mathbf{i} - 6\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$, and $\mathbf{c} = -7\mathbf{i} + 4\mathbf{j}$ are the sides of a right triangle. Verify that the midpoint of the hypotenuse is equidistant from the vertices.

33. Find the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds$, given: (a) $\mathbf{r} = 4\mathbf{i} \cos\theta + 4\mathbf{j} \sin\theta$; (b) $\mathbf{r} = e^{\theta}\mathbf{i} + e^{-\theta}\mathbf{j}$; (c) $\mathbf{r} = \theta\mathbf{i} + \theta^2\mathbf{j}$.

Ans. (a) $-\mathbf{i} \sin\theta + \mathbf{j} \cos\theta$; (b) $\dfrac{e^{\theta}\mathbf{i} - e^{-\theta}\mathbf{j}}{\sqrt{e^{2\theta} + e^{-2\theta}}}$; (c) $\dfrac{\mathbf{i} + 2\theta\mathbf{j}}{\sqrt{1 + 4\theta^2}}$

34. (a) Find $\mathbf{n}$ for the curve of Problem 33(a); (b) Find $\mathbf{n}$ for the curve of Problem 33(c); (c) Find $\mathbf{t}$ and $\mathbf{n}$ given $x = \cos\theta + \theta \sin\theta$, $y = \sin\theta - \theta\cos\theta$.

Ans. (a) $\mathbf{i} \cos\theta - \mathbf{j} \sin\theta$; (b) $\dfrac{-2\theta}{\sqrt{1 + 4\theta^2}}\mathbf{i} + \dfrac{1}{\sqrt{1 + 4\theta^2}}\mathbf{j}$; (c) $\mathbf{t} = \mathbf{i} \cos\theta + \mathbf{j} \sin\theta$, $\mathbf{n} = -\mathbf{i} \sin\theta + \mathbf{j} \cos\theta$