CHAPTER 31

Techniques of Integration I: Integration by Parts

If u and v are functions, the product rule yields

$$D_x(uv) = uv' + vu'$$

which can be rewritten in terms of antiderivatives as follows:

$$uv = \int uv'\,dx + \int vu'\,dx$$

Now, $\int uv'\,dx$ can be written as $\int u\,dv$, and $\int vu'\,dx$ can be written as $\int v\,du$.[†] Thus, $uv = \int u\,dv + \int v\,du$ and, therefore,

$$\int u\,dv = uv - \int v\,du \qquad \text{(integration by parts)}$$

The purpose of integration by parts is to replace a "difficult" integration $\int u\,dv$ by an "easy" integration $\int v\,du$.

EXAMPLE 31.1: Find $\int x \ln x\,dx$.

In order to use the integration by parts formula, we must divide the integrand $x \ln x\,dx$ into two "parts" u and dv so that we can easily find v by an integration and also easily find $\int v\,du$. In this example, let $u = \ln x$ and $dv = x\,dx$. Then we can set $v = \frac{1}{2}x^2$ and note that $du = \frac{1}{x}dx$. So, the integration by parts formula yields:

$$\int x \ln x\,dx = \int u\,dv = uv - \int v\,du = (\ln x)(\tfrac{1}{2}x^2) - \int \tfrac{1}{2}x^2\left(\frac{1}{x}dx\right)$$

$$= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\int x\,dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C$$

$$= \tfrac{1}{4}x^2(2\ln x - 1) + C$$

Integration by parts can be made easier to apply by setting up a rectangle such as the following one for Example 1.

$u = \ln x$	$dv = x\,dx$
$du = \frac{1}{x}dx$	$v = \frac{1}{2}x^2$

[†] $\int uv'\,dx = \int u\,dv$, where, after the integration on the right, the variable v is replaced by the corresponding function of x. In fact, by the Chain Rule, $D_x\left(\int u\,dv\right) = D_v\left(\int u\,dv\right) \cdot D_x v = u \cdot v'$. Hence, $\int u\,dv = \int uv'\,dx$. Similarly, $\int v\,du = \int vu'\,dx$.

In the first row, we place u and dv. In the second row, we place the results of computing du and v. The desired result of the integration parts formula $uv - \int v\,du$ can be obtained by first multiplying the upper-left corner u by the lower-right corner v, and then subtracting the integral of the product $v\,du$ of the two entries v and du in the second row.

EXAMPLE 31.2: Find $\int xe^x\,dx$.

Let $u = x$ and $dv = e^x\,dx$. We can picture this in the box below.

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

Then,
$$\int xe^x dx = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C$$
$$= e^x(x-1) + C$$

EXAMPLE 31.3: Find $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$. Then we get the box

$u = e^x$	$dv = \cos x\,dx$
$du = e^x\,dx$	$v = \sin x$

So,
$$\int e^x \cos x\,dx = uv - \int v\,du = e^x \sin x - \int e^x \sin x\,dx \tag{1}$$

Now we have the problem of finding $\int e^x \sin x\,dx$, which seems to be just as hard as the original integral $\int e^x \cos x\,dx$. However, let us try to find $\int e^x \sin x\,dx$ by another integration by parts. This time, let $u = e^x$ and $dv = \sin x\,dx$.

$u = e^x$	$dv = \sin x\,dx$
$du = e^x\,dx$	$v = -\cos x$

Then,
$$\int e^x \sin x\,dx = -e^x \cos x - \int -e^x \cos x\,dx$$
$$= -e^x \cos x + \int e^x \cos x\,dx$$

Substituting in formula (1) above, we get:

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\,dx\right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

Adding $\int e^x \cos x\,dx$ to both sides yields $2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$. So,

$$\int e^x \cos x\,dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x)$$

We must add an arbitrary constant:

$$\int e^x \cos x\,dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x) + C$$

Notice that this example required an iterated application of integration by parts.

SOLVED PROBLEMS

1. Find $\int x^3 e^{x^2}\,dx$.

Let $u = x^2$ and $dv = xe^{x^2}\,dx$. Note that v can be evaluated by using the substitution $w = x^2$. (We get $v = \frac{1}{2}\int e^w dw = \frac{1}{2}e^w = \frac{1}{2}e^{x^2}$.)

$u = x^2$	$dv = xe^{x^2}\,dx$
$du = 2x\,dx$	$v = \frac{1}{2}e^{x^2}$

Hence,

$$\int x^3 e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \int xe^{x^2}\,dx$$
$$= \tfrac{1}{2}x^2 e^{x^2} - \tfrac{1}{2}e^{x^2} + C$$
$$= \tfrac{1}{2}e^{x^2}(x^2 - 1) + C$$

2. Find $\int \ln(x^2+2)\,dx$.

Let $u = \ln(x^2+2)$ and $dv = dx$.

$u = \ln(x^2+2)$	$dv = dx$
$du = \dfrac{2x}{x^2+2}dx$	$v = x$

So,

$$\int \ln(x^2+2)\,dx = x\ln(x^2+2) - 2\int \frac{x^2}{x^2+2}dx$$
$$= x\ln(x^2+2) - 2\int\left(1 - \frac{2}{x^2+2}\right)dx$$
$$= x\ln(x^2+2) - 2x + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$
$$= x(\ln(x^2+2) - 2) + 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

3. Find $\int \ln x\,dx$.

Let $u = \ln x$ and $dv = dx$.

$u = \ln x$	$dv = dx$
$du = \dfrac{1}{x}dx$	$v = x$

So,

$$\int \ln x\,dx = x\ln x - \int 1\,dx = x\ln x - x + C$$
$$= x(\ln x - 1) + C$$

4. Find $\int x\sin x\,dx$.

We have three choices: (a) $u = x\sin x$, $dv = dx$; (b) $u = \sin x$, $dv = x\,dx$; (c) $u = x$, $dv = \sin x\,dx$.

(a) Let $u = x\sin x$, $dv = dx$. Then $du = (\sin x + x\cos x)\,dx$, $v = x$, and

$$\int x\sin x\,dx = x\cdot x\sin x - \int x(\sin x + x\cos x)\,dx$$

The resulting integral is not as simple as the original, and this choice is discarded.

(b) Let $u = \sin x$, $dv = x\,dx$. Then $du = \cos x\,dx$, $v = \frac{1}{2}x^2$, and

$$\int x \sin x\,dx = \tfrac{1}{2}x^2 \sin x - \int \tfrac{1}{2}x^2 \cos x\,dx$$

The resulting integral is not as simple as the original, and this choice too is discarded.

(c) Let $u = x$, $dv = \sin x\,dx$. Then $du = dx$, $v = -\cos x$, and

$$\int x \sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

5. Find $\int x^2 \ln x\,dx$.

Let $u = \ln x$, $dv = x^2\,dx$. Then $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{dx}{x} = \frac{x^3}{3}\ln x - \tfrac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \tfrac{1}{9}x^3 + C$$

6. Find $\int \sin^{-1} x\,dx$.

Let $u = \sin^{-1} x$, $dv = dx$.

$u = \sin^{-1} x$	$dv = dx$
$du = \frac{1}{\sqrt{1-x^2}}dx$	$v = x$

So,

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1} x + \tfrac{1}{2}\int (1-x^2)^{-1/2}(-2x)dx$$

$$= x\sin^{-1} x + \tfrac{1}{2}(2(1-x^2)^{1/2}) + C \qquad \text{(by Quick Formula I)}$$

$$= x\sin^{-1} x + (1-x^2)^{1/2} + C = x\sin^{-1} x + \sqrt{1-x^2} + C$$

7. Find $\int \tan^{-1} x\,dx$.

Let $u = \tan^{-1}x$, $dv = dx$.

$u = \tan^{-1} x$	$dv = dx$
$du = \frac{1}{1+x^2}dx$	$v = x$

So,

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}dx = x\tan^{-1} x - \tfrac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C \qquad \text{(by Quick Formula II)}$$

8. Find $\int \sec^3 x\,dx$.

Let $u = \sec x$, $dv = \sec^2 x\,dx$.

$u = \sec x$	$dv = \sec^2 x\,dx$
$du = \sec x \tan x\,dx$	$v = \tan x$

Thus, $\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|$

Then, $2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$

Hence, $\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$

9. Find $\int x^2 \sin x\,dx$.

Let $u = x^2$, $dv = \sin x\,dx$. Thus, $du = 2x\,dx$ and $v = -\cos x$. Then

$$\int x^2 \sin x\,dx = -x^2 \cos x - \int -2x \cos x\,dx$$

$$= -x^2 \cos x + 2\int x \cos x\,dx$$

Now apply integration by parts to $\int x \cos x\,dx$, with $u = x$ and $dv = \cos x\,dx$, getting

$$\int x \cos x = x \sin x - \int \sin x\,dx = x \sin x + \cos x$$

Hence, $\int x^2 \sin x\,dx = -x^2 \cos x + 2(x \sin x + \cos x) + C$

10. Find $\int x^3 e^{2x} dx$.

Let $u = x^3$, $dv = e^{2x}\,dx$. Then $du = 3x^2\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x} dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\int x^2 e^{2x} dx$$

For the resulting integral, let $u = x^2$ and $dv = e^{2x}\,dx$. Then $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x} dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\left(\tfrac{1}{2}x^2 e^{2x} - \int x e^{2x} dx\right) = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\int x e^{2x} dx$$

For the resulting integral, let $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x} dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\left(\tfrac{1}{2}x e^{2x} - \tfrac{1}{2}\int e^{2x} dx\right) = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{4}x e^{2x} - \tfrac{3}{8}e^{2x} + C$$

11. Derive the following *reduction formula* for $\int \sin^m x\,dx$.

$$\int \sin^m x\,dx = -\frac{\sin^{m-1} x \cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x\,dx$$

Let $u = \sin^{m-1} x$ and $dv = \sin x\,dx$.

$u = \sin^{m-1} x$	$dv = \sin x\,dx$
$du = (m-1)\sin^{m-2} x\,dx$	$v = -\cos x$

Then
$$\int \sin^m x\,dx = -\cos x \sin^{m-1} x + (m-1)\int \sin^{m-2} x \cos^2 x\,dx$$
$$= -\cos x \sin^{m-1} x + (m-1)\int \sin^{m-2} x(1-\sin^2 x)\,dx$$
$$= -\cos x \sin^{m-1} x + (m-1)\int \sin^{m-2} x\,dx - (m-1)\int \sin^m x\,dx$$

Hence,
$$m\int \sin^m x\,dx = -\cos x \sin^{m-1} x + (m-1)\int \sin^{m-2} x\,dx$$
and division by m yields the required formula.

12. Apply the reduction formula of Problem 11 to find $\int \sin^2 x\,dx$.
When $m = 2$, we get

$$\int \sin^2 x\,dx = -\frac{\sin x \cos x}{2} + \tfrac{1}{2}\int \sin^0 x\,dx$$
$$= -\frac{\sin x \cos x}{2} + \tfrac{1}{2}\int 1\,dx$$
$$= -\frac{\sin x \cos x}{2} + \frac{x}{2} + C = \frac{x - \sin x \cos x}{2} + C$$

13. Apply the reduction formula of Problem 11 to find $\int \sin^3 x\,dx$.
When $m = 3$, we get
$$\int \sin^3 x\,dx = -\frac{\sin^2 x \cos x}{3} + \tfrac{2}{3}\int \sin x\,dx$$
$$= -\frac{\sin^2 x \cos x}{3} - \tfrac{2}{3}\cos x + C$$
$$= -\frac{\cos x}{3}(2 + \sin^2 x) + C$$

SUPPLEMENTARY PROBLEMS

In Problems 14–21, use integration by parts to verify the specified formulas.

14. $\int x \cos x\,dx = x \sin x + \cos x + C$

15. $\int x \sec^2 3x\,dx = \frac{1}{3}x \tan 3x - \frac{1}{9}\ln|\sec x| + C$

16. $\int \cos^{-1} 2x\,dx = x\cos^{-1} 2x - \frac{1}{2}\sqrt{1-4x^2} + C$

17. $\int x \tan^{-1} x\,dx = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + C$

18. $\int x^2 e^{-3x}\,dx = -\frac{1}{3}e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C$

19. $\int x^3 \sin x\,dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

20. $\int x \sin^{-1}(x^2)\,dx = \frac{1}{2}x^2 \sin^{-1}(x^2) + \frac{1}{2}\sqrt{1-x^4} + C$

21. $\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x + 1}{x} + C$

22. Show that $\int_0^{2\pi} x \sin nx\,dx = -\frac{2\pi}{n}$ for any positive integer n.

23. Prove the following reduction formula: $\int \sec^n x\,dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$.

24. Apply Problem 23 to find $\int \sec^4 x\,dx$.

Ans. $\frac{1}{3}\tan x(\sec^2 x + 2) + C$

25. Prove the reduction formula:

$$\int \frac{x^2}{(a^2+x^2)^n}\,dx = \frac{1}{2n-2}\left(-\frac{x}{(a^2+x^2)^{n-1}} + \int \frac{dx}{(a^2+x^2)^{n-1}}\right)$$

26. Apply Problem 25 to find $\int \frac{x^2}{(a^2+x^2)^2}\,dx$.

Ans. $\frac{1}{2}\left(-\frac{x}{a^2+x^2} + \frac{1}{a}\tan^{-1}\frac{x}{a}\right) + C$

27. Prove $\int x^n \ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1)] + C$ for $n \neq -1$.

28. Prove the reduction formula: $\int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax} dx$.

29. Use Problem 28 and Example 2 to show that: $\int x^2 e^x dx = e^x(x^2 - 2x + 2) + C$.