CHAPTER 25

The Natural Logarithm

The traditional way of defining a logarithm, $\log_a b$, is to define it as that number u such that $a^u = b$. For example, $\log_{10} 100 = 2$ because $10^2 = 100$. However, this definition has a theoretical gap. The flaw is that we have not yet defined a^u when u is an irrational number, for example, $\sqrt{2}$ or π. This gap can be filled in, but that would require an extensive and sophisticated detour.† Instead, we take a different approach that will eventually provide logically unassailable definitions of the logarithmic and exponential functions. A temporary disadvantage is that the motivation for our initial definition will not be obvious.

The Natural Logarithm

We are already familiar with the formula

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C \qquad (r \neq -1)$$

The problem remains of finding out what happens when $r = -1$, that is, of finding the antiderivative of x^{-1}.

The graph of $y = 1/t$, for $t > 0$, is shown in Fig. 25-1. It is one branch of a hyperbola. For $x > 1$, the definite integral

$$\int_1^x \frac{1}{t} dt$$

is the value of the area under the curve $y = 1/t$ and above the t axis, between $t = 1$ and $t = x$.

Definition

$\ln x = \int_1^x \frac{1}{t} dt \quad$ for $x > 0$

The function ln x is called the *natural logarithm*. The reasons for referring to it as a logarithm will be made clear later. By (24.2),

(25.1) $\quad D_x(\ln x) = \frac{1}{x} \quad$ for $x > 0$

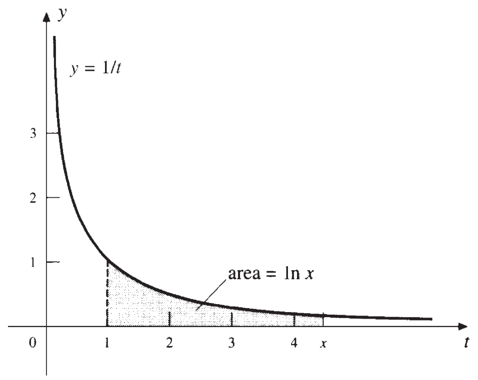

Fig. 25-1

† Some calculus textbooks just ignore the difficulty. They assume that a^u is defined when $a > 0$ and u is *any* real number and that the usual laws for exponents are valid.

Hence, the natural logarithm is the antiderivative of x^{-1}, but only on the interval $(0, +\infty)$. An antiderivative for all $x \neq 0$ will be constructed below in (25.5).

Properties of the Natural Logarithm

(25.2) $\ln 1 = 0$, since $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$.

(25.3) If $x > 1$, then $\ln x > 0$.

This is true by virtue of the fact that $\int_1^x \frac{1}{t}\,dt$ represents an area, or by Problem 15 of Chapter 23.

(25.4) If $0 < x < 1$, then $\ln x < 0$.

$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$ by (23.8). Now, for $0 < x < 1$, if $x \le t \le 1$, then $1/t > 0$ and, therefore, by Problem 15 of Chapter 23, $\int_x^1 \frac{1}{t}\,dt > 0$.

(25.5) (a) $D_x(\ln|x|) = \frac{1}{x}$ for $x \neq 0$

(b) $\int \frac{1}{x}\,dx = \ln|x| + C$ for $x \neq 0$

The argument is simple. For $x > 0$, $|x| = x$, and so $D_x(\ln|x|) = D_x(\ln x) = 1/x$ by (25.1). For $x < 0$, $|x| = -x$, and so

$$D_x(\ln|x|) = D_x(\ln(-x)) = D_u(\ln u)D_x(u) \qquad \text{(Chain Rule, with } u = -x > 0\text{)}$$

$$= \left(\frac{1}{u}\right)(-1) = \frac{1}{-u} = \frac{1}{x}$$

EXAMPLE 25.1: $D_x(\ln|3x+2|) = \frac{1}{3x+2}D_x(3x+2)$ (Chain Rule)

$$= \frac{3}{3x+2}$$

(25.6) $\ln uv = \ln u + \ln v$

Note that

$$D_x(\ln(ax)) = \frac{1}{ax}D_x(ax) \qquad \text{(by the Chain Rule and (25.1))}$$

$$= \frac{1}{ax}(a) = \frac{1}{x} = D_x(\ln x)$$

Hence, $\ln(ax) = \ln x + K$ for some constant K (by Problem 18 of Chapter 13). When $x = 1$, $\ln a = \ln 1 + K = 0 + K = K$. Thus, $\ln(ax) = \ln x + \ln a$. Replacing a and x by u and v yields (25.6).

(25.7) $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$

In (25.6), replace u by $\frac{u}{v}$.

(25.8) $\ln\frac{1}{v} = -\ln v$

In (25.7), replace u by 1 and use (25.2).

(25.9) $\ln(x^r) = r\ln x$ for any rational number r and $x > 0$.

By the Chain Rule, $D_x(\ln(x^r)) = \frac{1}{x^r}(rx^{r-1}) = \frac{r}{x} = D_x(r\ln x)$. So, by Problem 18 of Chapter 13, $\ln(x^r) = r\ln x + K$ for some constant K. When $x = 1$, $\ln 1 = r\ln 1 + K$. Since $\ln 1 = 0$, $K = 0$, yielding (25.9).

EXAMPLE 25.2: $\ln\sqrt[3]{2x-5} = \ln(2x-5)^{1/3} = \frac{1}{3}\ln(2x-5)$.

(25.10) $\ln x$ is an increasing function.

$D_x(\ln x) = \frac{1}{x} > 0$ since $x > 0$. Now use Theorem 13.7.

(25.11) $\ln u = \ln v$ implies $u = v$.

This is a direct consequence of (25.10). For, if $u \neq v$, then either $u < v$ or $v < u$ and, therefore, either $\ln u < \ln v$ or $\ln v < \ln u$.

(25.12) $\frac{1}{2} < \ln 2 < 1$

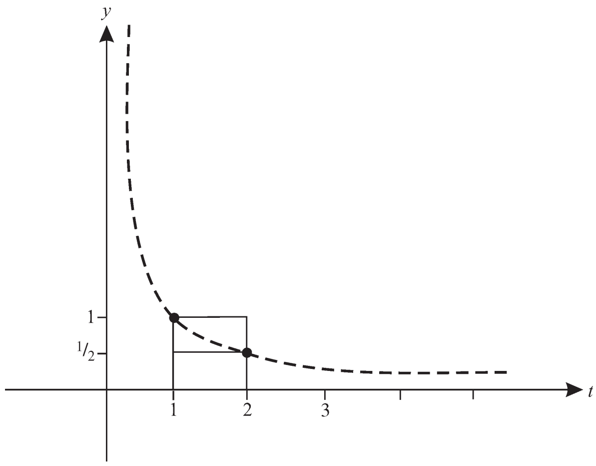

Fig. 25-2

The area under the graph of $y = 1/t$, between $t = 1$ and $t = 2$, and above the t axis, is greater than the area $\frac{1}{2}$ of the rectangle with base [1, 2] and height $\frac{1}{2}$. (See Fig. 25-2.) It is also less than the area 1 of the rectangle with base [1, 2] and height 1. (A more rigorous argument would use Problems 3(c) and 15 of Chapter 23.)

(25.13) $\lim_{x\to+\infty} \ln x = +\infty$

Let k be any positive integer. Then, for $x > 2^{2k}$,

$\ln x > \ln(2^{2k}) = 2k\ln 2 > 2k(\frac{1}{2}) = k$

by (25.10) and (25.9). Thus, as $x \to +\infty$, $\ln x$ eventually exceeds every positive integer.

(25.14) $\lim_{x\to 0^+} \ln x = -\infty$

Let $u = 1/x$. As $x \to 0^+$, $u \to +\infty$. Hence,

$$\lim_{x\to 0^+} \ln x = \lim_{u\to+\infty} \ln\left(\frac{1}{u}\right) = \lim_{u\to+\infty} -\ln u \quad \text{(by (25.8))}$$

$$= -\lim_{u\to+\infty} \ln u = -\infty \quad \text{(by (25.13))}$$

(25.15) Quick Formula II: $\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C$

By the Chain Rule and (25.5) (a), $D_x(\ln|g(x)|) = \frac{1}{g(x)}g'(x)$.

EXAMPLE 25.3:

(a) $\int \frac{2x}{x^2+1}dx = \ln|x^2+1| + C = \ln(x^2+1) + C$

The absolute value sign was dropped because $x^2 + 1 \geq 0$. In the future, we shall do this without explicit mention.

(b) $\int \frac{x^2}{x^3+5}dx = \frac{1}{3}\int \frac{3x^2}{x^3+5}dx = \frac{1}{3}\ln|x^3+5| + C$

SOLVED PROBLEMS

1. Evaluate: (a) $\int \tan x\, dx$; (b) $\int \cot x\, dx$; (c) $\int \sec x\, dx$.

(a) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}dx$

$= -\ln|\cos x| + C$ by Quick Formula II.

$= -\ln\left|\frac{1}{\sec x}\right| + C = -(-\ln|\sec x|) + C = \ln|\sec x| + C$

(25.16) $\int \tan x\, dx = \ln|\sec x| + C$

(b) $\int \cot x\, dx = \int \frac{\cos x}{\sin x}dx = \ln|\sin x| + C$ by Quick Formula II.

(25.17) $\int \cot x\, dx = \ln|\sin x| + C$

(c) $\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx = \ln|\sec x + \tan x| + C$ by Quick Formula II.

(25.18) $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

2. (GC) Estimate the value of ln 2.

A graphing calculator yields the value ln 2 ~ 0.6931471806. Later we shall find another method for calculating ln 2.

3. (GC) Sketch the graph of $y = \ln x$.

A graphing calculator yields the graph shown in Fig. 25-3. Note by (25.10) that ln x is increasing. By (25.13), the graph increases without bound on the right, and, by (25.14), the negative y axis is a vertical asymptote. Since

$$D_x^2(\ln x) = D_x(x^{-1}) = -x^{-2} = -\frac{1}{x^2} < 0$$

the graph is concave downward. By (25.13) and (25.14), and the intermediate value theorem, the range of ln x is the set of all real numbers.

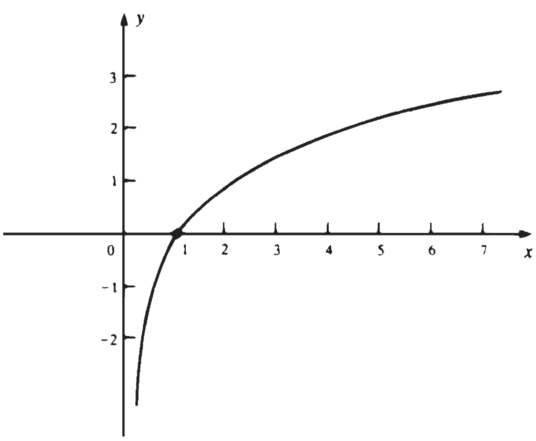

Fig. 25-3

4. Find: (a) $D_x(\ln(x^4+7x))$; (b) $D_x(\ln(\cos 2x))$; (c) $D_x(\cos(\ln 2x))$.

(a) $D_x(\ln(x^4+7x)) = \frac{1}{x^4+7x}(4x^3+7) = \frac{4x^3+7}{x^4+7x}$

(b) $D_x(\ln(\cos 2x)) = \frac{1}{\cos 2x}(-\sin 2x)(2) = -\frac{2\sin 2x}{\cos 2x}$

$= -2\tan 2x$

(c) $D_x(\cos(\ln 2x)) = (-\sin(\ln 2x))\left(\frac{1}{2x}\right)(2) = -\frac{\sin(\ln 2x)}{x}$

5. Find the following antiderivatives. Use Quick Formula II when possible.

(a) $\int \frac{1}{8x-3}dx$; (b) $\int \frac{4x^7}{3x^8-2}dx$; (c) $\int \frac{x-4}{x^2+5}dx$; (d) $\int \frac{x}{x^2-4x+5}dx$

(a) $\int \frac{1}{8x-3}dx = \frac{1}{8}\int \frac{8}{8x-3}dx = \frac{1}{8}\ln|8x-3| + C$

(b) $\int \frac{4x^7}{3x^8-2}dx = \frac{1}{6}\int \frac{24x^7}{3x^8-2}dx = \frac{1}{6}\ln|3x^8-2| + C$

(c) $\int \frac{x-4}{x^2+5}dx = \int \frac{x}{x^2+5}dx - \int \frac{4}{x^2+5}dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+5}dx - 4\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

$$= \frac{1}{2}\ln(x^2+5) - \frac{4\sqrt{5}}{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

(d) Complete the square in the denominator: $\int \frac{x}{x^2-4x+5}dx = \int \frac{x}{(x-2)^2+1}dx.$
Let $u = x-2$, $du = dx$.

$$\int \frac{x}{(x-2)^2+1}dx = \int \frac{u+2}{u^2+1}du = \int \frac{u}{u^2+1}du + \int \frac{2}{u^2+1}du$$

$$= \tfrac{1}{2}\ln(u^2+1) + 2\tan^{-1}u + C = \tfrac{1}{2}\ln(x^2-4x+5) + 2\tan^{-1}(x-2) + C$$

6. **Logarithmic Differentiation**. Find the derivative of $y = \frac{x(1-x^2)^2}{(1+x^2)^{1/2}}$.

First take the natural logarithms of the absolute values of both sides:

$$\ln|y| = \ln\left|\frac{x(1-x^2)^2}{(1+x^2)^{1/2}}\right| = \ln|x(1-x^2)^2| - \ln|(1+x^2)^{1/2}|$$

$$= \ln|x| + \ln|(1-x^2)^2| - \tfrac{1}{2}\ln(1+x^2)$$

$$= \ln|x| + 2\ln|1-x^2| - \tfrac{1}{2}\ln(1+x^2)$$

Now take the derivatives of both sides:

$$\frac{1}{y}y' = \frac{1}{x} + \frac{2}{1-x^2}(-2x) - \frac{1}{2}\frac{1}{1+x^2}(2x) = \frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}$$

$$y' = y\left(\frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}\right) = \frac{x(1-x^2)^2}{(1+x^2)^{1/2}}\left(\frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}\right)$$

7. Show that $1 - \frac{1}{x} \le \ln x \le x - 1$ for $x > 0$. (When $x \neq 1$, the strict inequalities hold.)

When $x > 1$, $1/t$ is a decreasing function on $[1, x]$ and so its minimum on $[1, x]$ is $1/x$ and its maximum is 1. So, by Problems 3(*c*) and 15 of Chapter 23,

$$\frac{1}{x}(x-1) < \ln x = \int_1^x \frac{1}{t}dt < x - 1 \quad \text{and so} \quad 1 - \frac{1}{x} < \ln x < x - 1.$$

For $0 < x < 1$, $-\frac{1}{t}$ is increasing on $[x, 1]$. Then, by Problems 3(*c*) and 15 of Chapter 23,

$$-\frac{1}{x}(1-x) < \ln x = \int_1^x \frac{1}{t}\,dt = \int_x^1 \left(-\frac{1}{t}\right) dt < -1(1-x)$$

Hence, $1-\frac{1}{x} < \ln x < x-1$. When $x = 1$, the three terms are all equal to 0.

SUPPLEMENTARY PROBLEMS

8. Find the derivatives of the following functions.

(a) $y = \ln (x+3)^2 = 2 \ln (x+3)$.

Ans. $y' = \dfrac{2}{x+3}$

(b) $y = (\ln (x+3))^2$

Ans. $y' = 2 \ln (x+3)\dfrac{1}{x+3} = \dfrac{2 \ln (x+3)}{x+3}$

(c) $y = \ln [(x^3+2)(x^2+3)] = \ln (x^3+2) + \ln (x^2+3)$

Ans. $y' = \dfrac{1}{x^3+2}(3x^2) + \dfrac{1}{x^2+3}(2x) = \dfrac{3x^2}{x^3+2} + \dfrac{2x}{x^2+3}$

(d) $y = \ln \dfrac{x^4}{(3x-4)^2} = \ln x^4 - \ln (3x-4)^2 = 4 \ln x - 2 \ln (3x-4)$

Ans. $y' = \dfrac{4}{x} - \dfrac{2}{3x-4}(3) = \dfrac{4}{x} - \dfrac{6}{3x-4}$

(e) $y = \ln \sin 5x$

Ans. $y' = \dfrac{1}{\sin 5x}\cos(5x)(5) = 5 \cot 5x$

(f) $y = \ln (x + \sqrt{1+x^2})$

Ans. $y' = \dfrac{1+\frac{1}{2}(1+x^2)^{-1/2}(2x)}{x+(1+x^2)^{1/2}} = \dfrac{1+x(1+x^2)^{-1/2}}{x+(1+x^2)^{1/2}}\dfrac{(1+x^2)^{1/2}}{(1+x^2)^{1/2}} = \dfrac{1}{\sqrt{1+x^2}}$

(g) $y = \ln \sqrt{3-x^2} = \ln (3-x^2)^{1/2} = \frac{1}{2} \ln (3-x^2)$

Ans. $y' = \dfrac{1}{2}\dfrac{1}{3-x^2}(-2x) = -\dfrac{x}{3-x^2}$

(h) $y = x \ln x - x$

Ans. $y' = \ln x$

(i) $y = \ln (\ln (\tan x))$

Ans. $y' = \dfrac{\tan x + \cot x}{\ln (\tan x)}$

9. Find the following antiderivatives. Use Quick Formula II when possible.

(a) $\int \frac{1}{7x} dx$

Ans. $\frac{1}{7} \ln |x| + C$

(b) $\int \frac{x^8}{x^9 - 1} dx$

Ans. $\frac{1}{9} \ln |x^9 - 1| + C$

(c) $\int \frac{\sqrt{\ln x + 3}}{x} dx$

Ans. Use Quick Formula I. $\frac{2}{3}(\ln x + 3)^{3/2} + C$

(d) $\int \frac{dx}{x \ln x}$

Ans. $\ln |\ln x| + C$

(e) $\int \frac{\sin 3x}{1 - \cos 3x} dx$

Ans. $\frac{1}{3} \ln |1 - \cos 3x| + C$

(f) $\int \frac{2x^4 - x^2}{x^3} dx$

Ans. $x^2 - \ln |x| + C$

(g) $\int \frac{\ln x}{x} dx$

Ans. $\frac{1}{2}(\ln x)^2 + C$

(h) $\int \frac{dx}{\sqrt{x}(1 - \sqrt{x})}$

Ans. $-2 \ln |1 - \sqrt{x}| + C$

10. Use logarithmic differentiation to calculate y'.

(a) $y = x^4 \sqrt{2 - x^2}$

Ans. $y' = x^4 \sqrt{2 - x^2} \left(\frac{4}{x} - \frac{x}{2 - x^2} \right) = 4x^3 \sqrt{2 - x^2} - \frac{x^5}{\sqrt{2 - x^2}}$

(b) $y = \frac{(x - 1)^5 \sqrt[4]{x + 2}}{\sqrt{x^2 + 7}}$

Ans. $y' = y\left(\frac{5}{x - 1} + \frac{1}{4} \frac{1}{x + 2} - \frac{x}{x^2 + 1} \right)$

(c) $y = \dfrac{\sqrt{x^2+3}\cos x}{(3x-5)^3}$

Ans. $y' = y\left(\dfrac{x}{x^2+3} - \tan x - \dfrac{1}{3x-5}\right)$

(d) $y = \sqrt[4]{\dfrac{2x+3}{2x-3}}$

Ans. $y' = -\dfrac{3y}{4x^2-9}$

11. Express in terms of ln 2 and ln 3: (a) $\ln(3^7)$; (b) $\ln\dfrac{2}{27}$.

Ans. (a) 7 ln 3; (b) ln 2–3 ln 3

12. Express in terms of ln 2 and ln 5: (a) ln 50; (b) $\ln\dfrac{1}{4}$; (c) $\ln\sqrt{5}$; (d) $\ln\dfrac{1}{40}$.

Ans. (a) ln 2 + 2 ln 5; (b) – 2 ln 2; (c) $\frac{1}{2}$ ln 5; (d) – (3 ln 2 + ln 5)

13. Find the area under the curve $y = \dfrac{1}{x}$ and above the x axis, between $x = 2$ and $x = 4$.

Ans. ln 2

14. Find the average value of $\dfrac{1}{x}$ on [3, 5].

Ans. $\frac{1}{2}\ln\frac{5}{3}$

15. Use implicit differentiation to find y': (a) $y^3 = \ln(x^3 + y^3)$; (b) $3y - 2x = 1 + \ln xy$.

Ans. (a) $y' = \dfrac{x^2}{y^2(x^3+y^3-1)}$; (b) $y' = \dfrac{y^2x+1}{x^3y-1}$

16. Evaluate $\lim\limits_{h\to 0}\dfrac{1}{h}\ln\dfrac{2+h}{2}$.

Ans. $\frac{1}{2}$

17. Check the formula $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.

18. (GC) Approximate $\ln 2 = \int_1^2 \frac{1}{t}\,dt$ to six decimal places by (a) the trapezoidal rule; (b) the midpoint rule; (c) Simpson's rule, in each case with $n = 10$.

Ans. (a) 0.693771; (b) 0.692835; (c) 0.693147

19. (GC) Use Newton's method to approximate the root of $x^2 + \ln x = 2$ to four decimal places.

Ans. 1.3141