# CHAPTER 17

# Differentiation of Trigonometric Functions

## Continuity of cos *x* and sin *x*

It is clear that cos *x* and sin *x* are continuous functions, that is, for any $\theta$,

$$\lim_{h\to 0} \cos(\theta + h) = \cos\theta \qquad \text{and} \qquad \lim_{h\to 0} \sin(\theta + h) = \sin\theta$$

To see this, observe that, in Fig. 17-1, as $h$ approaches 0, point $C$ approaches point $B$. Hence, the $x$ coordinate of $C$ (namely, $\cos(\theta + h)$) approaches the $x$ coordinate of $B$ (namely, $\cos\theta$), and the $y$ coordinate of $C$ (namely, $\sin(\theta + h)$) approaches the $y$ coordinate of $B$ (namely, $\sin\theta$).

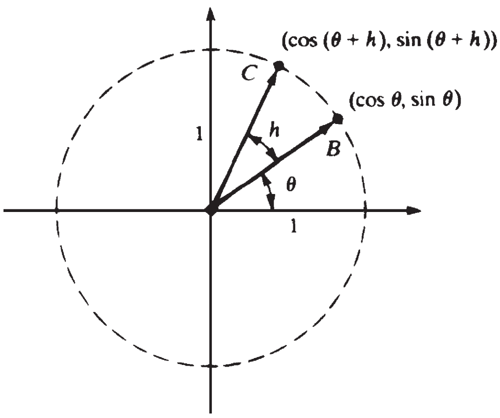


Fig. 17-1

To find the derivative of sin *x* and cos *x*, we shall need the following limits.

**(17.1)** $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

**(17.2)** $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$

For a proof of (17.1), see Problem 1. From (17.1), (17.2) is derived as follows:

$$\frac{1-\cos\theta}{\theta} = \frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}.$$

Hence,

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}=1\cdot\frac{\sin 0}{1+\cos 0}=1\cdot\frac{0}{1+1}=1\cdot 0=0$$

**(17.3)** $D_x(\sin x)=\cos x$

**(17.4)** $D_x(\cos x)=-\sin x$

For a proof of (17.3), see Problem 2. From (17.3) we can derive (17.4), with the help of the chain rule and (16.8), as follows:

$$D_x(\cos x)=D_x\left(\sin\left(\frac{\pi}{2}-x\right)\right)=\cos\left(\frac{\pi}{2}-x\right)\cdot(-1)=-\sin x$$

## Graph of sin *x*

Since sin $(x+2\pi)=\sin x$, we need only construct the graph for $0\le x\le 2\pi$. Setting $D_x(\sin x)=\cos x=0$ and noting that $\cos x=0$ in $[0, 2\pi]$ when and only when $x=\pi/2$ or $x=3\pi/2$, we find the critical numbers $\pi/2$ and $3\pi/2$. Since $D_x^2(\sin x)=D_x(\cos x)=-\sin x$, and $-\sin(\pi/2)=-1<0$ and $-\sin(3\pi/2)=1>0$, the second derivative test implies that there is a relative maximum at $(\pi/2, 1)$ and a relative minimum at $(3\pi/2, -1)$. Since $D_x(\sin x)=\cos x$ is positive in the first and fourth quadrants, sin $x$ is increasing for $0<x<\pi/2$ and for $3\pi/2<x<2\pi$. Since $D_x^2(\sin x)=-\sin x$ is positive in the third and fourth quadrants, the graph is concave upward for $\pi<x<2\pi$. Thus, there will be an inflection point at $(\pi, 0)$, as well as at $(0, 0)$ and $(2\pi, 0)$. Part of the graph is shown in Fig. 17-2.

## Graph of cos *x*

Note that $\sin(\pi/2+x)=\sin(\pi/2)\cos x+\cos(\pi/2)\sin x=1\cdot\cos x+0\cdot\sin x=\cos x$. Thus, the graph of cos $x$ can be drawn by moving the graph of sin $x$ by $\pi/2$ units to the left, as shown in Fig. 17-3.

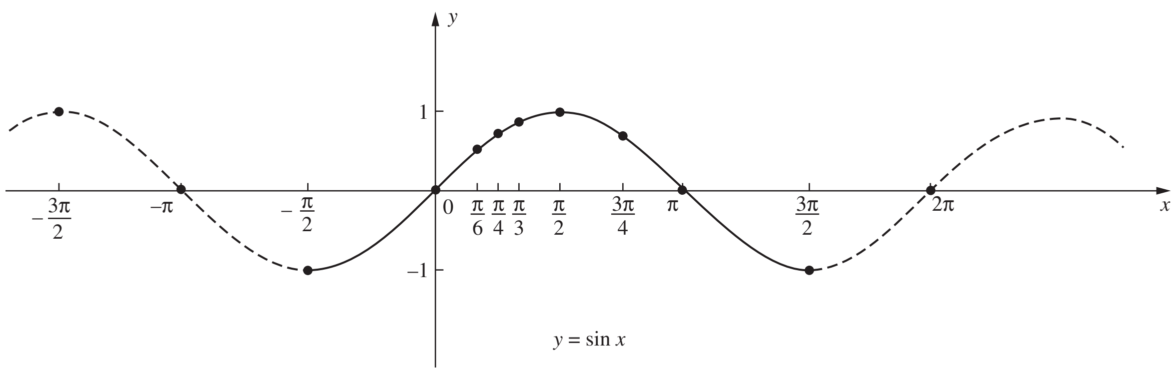


Fig. 17-2

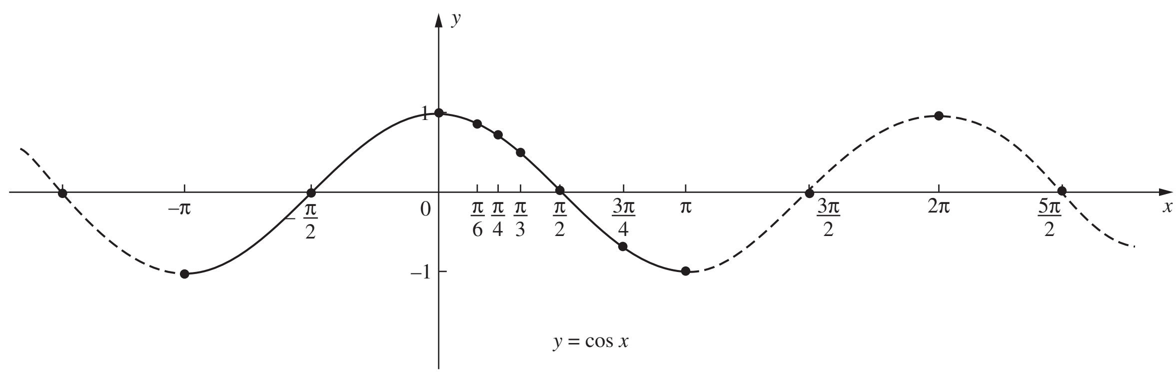


Fig. 17-3

The graphs of $y = \sin x$ and $y = \cos x$ consist of repeated waves, with each wave extending over an interval of length $2\pi$. The length (*period*) and height (*amplitude*) of the waves can be changed by multiplying the argument and the value, respectively, by constants.

**EXAMPLE 17.1:** Let $y = \cos 3x$. The graph is sketched in Fig. 17-4. Because $\cos 3(x + 2\pi/3) = \cos(3x + 2\pi) = \cos 3x$, the function is of period $p = 2\pi/3$. Hence, the length of each wave is $2\pi/3$. The number of waves over an interval of length $2\pi$ (corresponding to one complete rotation of the ray determining the angle $x$) is 3. This number is called the *frequency* $f$ of $\cos 3x$. In general, $pf$ = (length of each wave) × (number of waves in an interval of $2\pi$) = $2\pi$. Hence, $f = 2\pi/p$.

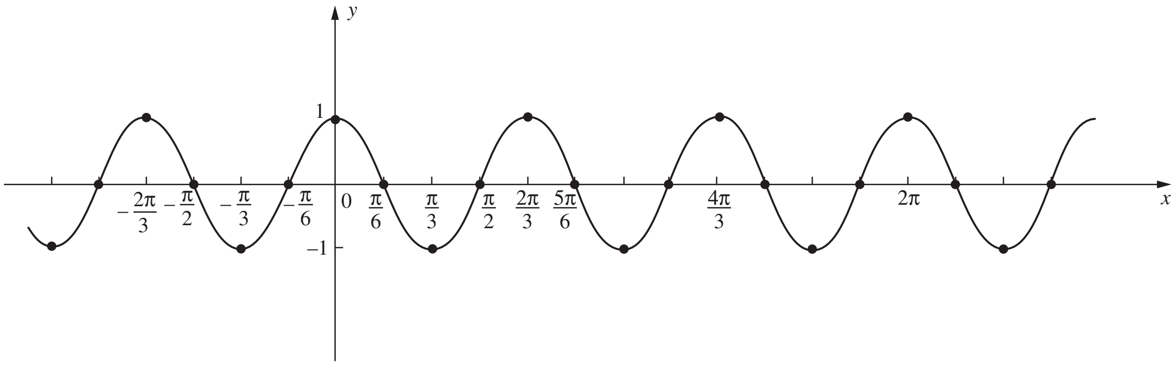


Fig. 17-4

For any $b > 0$, the functions $\sin bx$ and $\cos bx$ have frequency $b$ and period $2\pi/b$.

**EXAMPLE 17.2:** $y = 2 \sin x$. The graph of this function (see Fig. 17-5) is obtained from that of $y = \sin x$ by doubling the $y$ values. The period and frequency are the same as those of $y = \sin x$, that is, $p = 2\pi$ and $f = 1$. The amplitude, the maximum height of each wave, is 2.

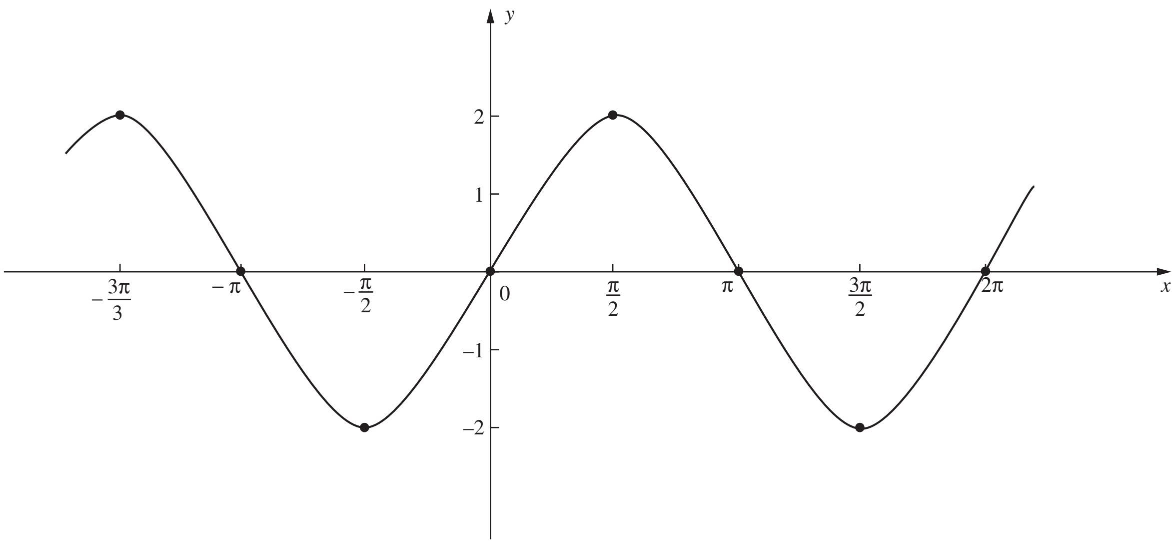


Fig. 17-5

**EXAMPLE 17.3:** In general, if $b > 0$, then $y = A \sin bx$ and $y = A \cos bx$ have period $2\pi/b$, frequency $b$, and amplitude $|A|$. Figure 17-6 shows the graph of $y = 1.5 \sin 4x$.

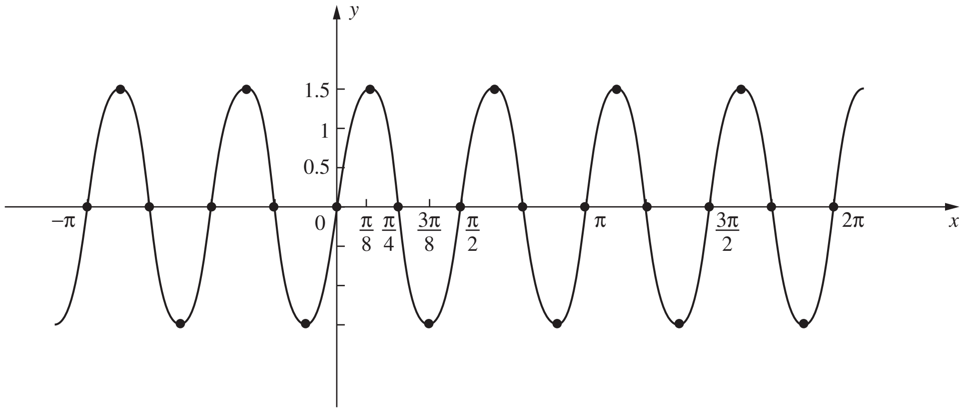


Fig. 17-6

## Other Trigonometric Functions

Tangent $\tan x = \dfrac{\sin x}{\cos x}$

Cotangent $\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$

Secant $\sec x = \dfrac{1}{\cos x}$

Cosecant $\csc x = \dfrac{1}{\sin x}$

## Derivatives

**(17.5)** $D_x(\tan x) = \sec^2 x$

**(17.6)** $D_x(\cot x) = -\csc^2 x$

**(17.7)** $D_x(\sec x) = \tan x \sec x$

**(17.8)** $D_x(\csc x) = -\cot x \csc x$

For the proofs, see Problem 3.

## Other Relationships

**(17.9)** $\tan^2 x + 1 = \sec^2 x$

$$\tan^2 x + 1 = \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

**(17.10)** $\tan(x+\pi) = \tan x$ and $\cot(x+\pi) = \cot x$

Thus, tan $x$ and cot $x$ have period $\pi$. See Problem 4.

**(17.11)** $\tan(-x) = -\tan x$ and $\cot(-x) = -\cot x$

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x, \text{ and similarly for } \cot x$$

## Graph of $y = \tan x$

Since $\tan x$ has period $\pi$, it suffices to determine the graph in $(-\pi/2, \pi/2)$. Since $\tan(-x) = -\tan x$, we need only draw the graph in $(0, \pi/2)$ and then reflect in the origin. Since $\tan x = (\sin x)/(\cos x)$, there will be vertical asymptotes at $x = \pi/2$ and $x = -\pi/2$. By (17.5), $D_x(\tan x) > 0$ and, therefore, $\tan x$ is increasing.

$$D_x^2(\tan x) = D_x(\sec^2 x) = 2\sec x(\tan x \sec x) = 2\tan x \sec^2 x.$$

Thus, the graph is concave upward when $\tan x > 0$, that is, for $0 < x < \pi/2$, and there is an inflection point at (0, 0). Some special values of $\tan x$ are given in Table 17-1, and the graph is shown in Fig. 17-7.

**TABLE 17-1**

| $x$ | $\tan x$ |
|---|---|
| 0 | 0 |
| $\frac{\pi}{6}$ | $\frac{\sqrt{3}}{3} \sim 0.58$ |
| $\frac{\pi}{4}$ | 1 |
| $\frac{\pi}{3}$ | $\sqrt{3} \sim 1.73$ |

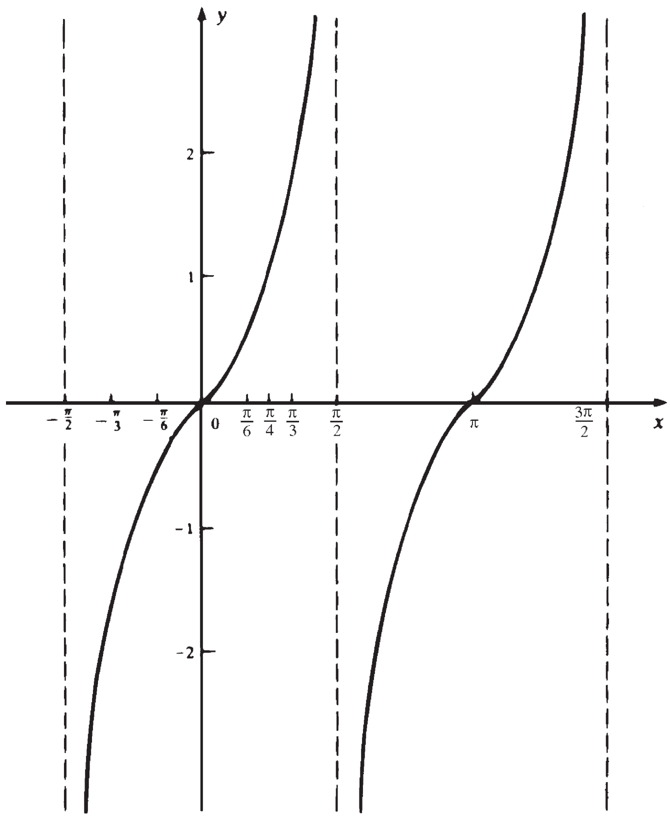


Fig. 17-7

For an acute angle $\theta$ of a right triangle,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{hypotenuse}} \div \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{opposite}}{\text{adjacent}}$$

## Graph of $y = \sec x$

Since $\sec x = 1/(\cos x)$, the graph will have a vertical asymptote $x = x_0$ for all $x_0$ for which $\cos x_0 = 0$, that is, for $x = (2n + 1)\pi/2$, where $n$ is any integer. Like $\cos x$, $\sec x$ has a period of $2\pi$, and we can confine our attention to $(-\pi, \pi)$. Note that $|\sec x| \geq 1$, since $|\cos x| \leq 1$. Setting $D_x(\sec x) = \tan x \sec x = 0$, we find critical numbers at $x = 0$ and $x = \pi$, and the first derivative test tells us that there is a relative minimum at $x = 0$ and a relative maximum at $x = \pi$.

Since

$$D_x^2(\sec x) = D_x(\tan x \sec x) = \tan x(\tan x \sec x) + \sec x(\sec^2 x) = \sec x(\tan^2 x + \sec^2 x)$$

there are no inflection points and the curve is concave upward for $-\pi/2 < x < \pi/2$. The graph is shown in Fig. 17-8.

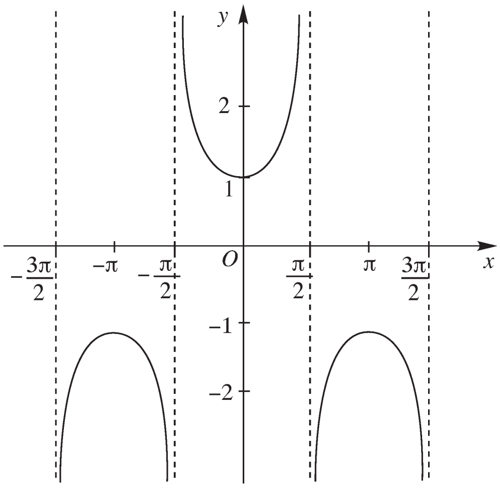


Fig. 17-8

## Angles Between Curves

By the *angle of inclination* of a nonvertical line $\mathscr{L}$, we mean the smaller counterclockwise angle $\alpha$ from the positive $x$ axis to the line. (See Fig. 17-9.) If $m$ is the slope of $\mathscr{L}$, then $m = \tan \alpha$. (To see this, look at Fig. 17-10, where the line $\mathscr{L}'$ is assumed to be parallel to $\mathscr{L}$ and, therefore, has the same slope $m$. Then $m = (\sin \alpha - 0)/(\cos \alpha - 0) = (\sin \alpha)/(\cos \alpha) = \tan \alpha$.)

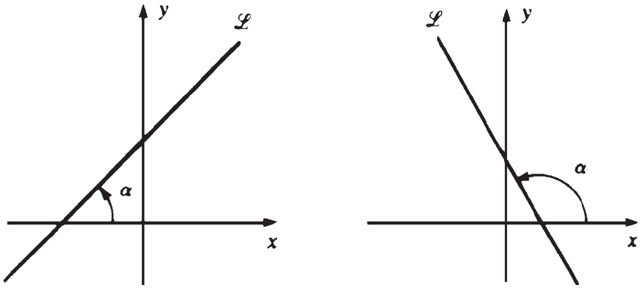


Fig. 17-9

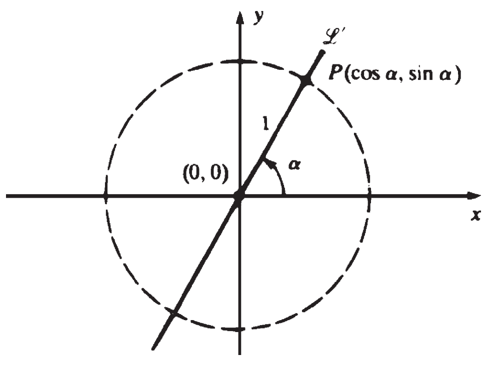


Fig. 17-10

By the *angle between two curves at a point of intersection P*, we mean the smaller of the two angles between the tangent lines to the curves at $P$. (See Problems 17 and 18.)

## SOLVED PROBLEMS

**1.** Prove (17.1): $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$.

Since $\frac{\sin(-\theta)}{-\theta}=\frac{\sin\theta}{\theta}$, we need consider only $\theta > 0$. In Fig. 17-11, let $\theta=\angle AOB$ be a small positive central angle of a circle of radius $OA = OB = 1$. Let $C$ be the foot of the perpendicular from $B$ onto $OA$. Note that $OC = \cos\theta$ and $CB = \sin\theta$. Let $D$ be the intersection of $OB$ and an arc of a circle with center at $O$ and radius $OC$. So,

$$\text{Area of sector } COD \le \text{area of } \Delta COB \le \text{area of sector } AOB$$

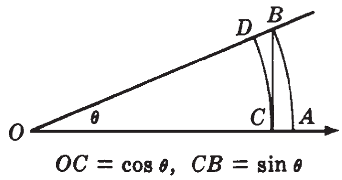


Fig. 17-11

Observe that area of sector $COD=\frac{1}{2}\theta\cos^2\theta$ and that area of sector $AOB=\frac{1}{2}\theta$. (If $W$ is the area of a sector determined by a central angle $\theta$ of a circle of radius $r$, then $W$/(area of circle) = $\theta/2\pi$. Thus, $W/\pi r^2=\theta/2\pi$ and, therefore, $W=\frac{1}{2}\theta r^2$.)

Hence,

$$\tfrac{1}{2}\theta\cos^2\theta \le \tfrac{1}{2}\sin\theta\cos\theta \le \tfrac{1}{2}\theta$$

Division by $\frac{1}{2}\theta\cos\theta > 0$ yields

$$\cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}$$

As $\theta$ approaches $0^+$, $\cos\theta \to 1$, $1/(\cos\theta) \to 1$. Hence,

$$1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1 \quad \text{Thus} \quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

**2.** Prove (17.3): $D_x(\sin x) = \cos x$.

Here we shall use (17.1) and (17.2).
Let $y = \sin x$. Then $y + \Delta y = \sin(x + \Delta x)$ and

$$\Delta y = \sin(x+\Delta x) - \sin x = \cos x \sin\Delta x + \sin x\cos\Delta x - \sin x$$
$$= \cos x\sin\Delta x + \sin x(\cos\Delta x - 1)$$
$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\cos x\frac{\sin\Delta x}{\Delta x}+\sin x\frac{\cos\Delta x-1}{\Delta x}\right)$$
$$=(\cos x)\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}+(\sin x)\lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x}$$
$$=(\cos x)(1)+(\sin x)(0)=\cos x$$

**3.** Prove: (a) $D_x(\tan x) = \text{see}^2 x$ (17.5); (b) $D_x(\sec x) = \tan x \sec x$ (17.7).

(a) $$\frac{d}{dx}(\tan x)=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}$$
$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

(b) Differentiating both sides of (17.9), $\tan^2 x + 1 = \sec^2 x$, by means of the chain rule, we get

$$2\tan x \sec^2 x = 2\sec x\, D_x(\sec x).$$

Hence, $D_x(\sec x) = \tan x \sec x$.

**4.** Prove (17.10): $\tan(x + \pi) = \tan x$.

$$\sin(x+\pi) = \sin x\cos\pi + \cos x \sin\pi = -\sin x$$
$$\cos(x+\pi) = \cos x\cos\pi - \sin x \sin\pi = -\cos x$$

Hence,

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x} = \tan x$$

**5.** Derive $\tan(u-v) = \dfrac{\tan u - \tan v}{1+\tan u \tan v}$

$$\tan(u-v) = \frac{\sin(u-v)}{\cos(u-v)} = \frac{\sin u\cos v - \cos u \sin v}{\cos u \cos v + \sin u \sin v}$$

$$= \frac{\dfrac{\sin u}{\cos u} - \dfrac{\sin v}{\cos v}}{1 + \dfrac{\sin u}{\cos u}\dfrac{\sin v}{\cos v}} \quad \text{(divide numerator and denominator by } \cos u \cos v\text{)}$$

$$= \frac{\tan u - \tan v}{1+\tan u \tan v}$$

**6.** Calculate the derivatives of the following functions: (a) $2\cos 7x$; (b) $\sin^3(2x)$; (c) $\tan(5x)$; (d) $\sec(1/x)$.

(a) $D_x(2\cos 7x) = 2(-\sin 7x)(7) = -14\sin 7x$
(b) $D_x(\sin^3(2x)) = 3(\sin^2(2x))(\cos(2x))(2) = 6\sin^2(2x)\cos(2x)$
(c) $D_x(\tan(5x)) = (\sec^2(5x))(5) = 5\sec^2(5x)$
(d) $D_x(\sec(1/x)) = \tan(1/x)\sec(1/x)(-1/x^2) = -(1/x^2)\tan(1/x)\sec(1/x)$

**7.** Find all solutions of the equation $\cos x = \frac{1}{2}$.

Solving $(\frac{1}{2})^2 + y^2 = 1$, we see that the only points on the unit circle with abscissa $\frac{1}{2}$ are $(\frac{1}{2}, \sqrt{3}/2)$ and $(\frac{1}{2}, -\sqrt{3}/2)$. The corresponding central angles are $\pi/3$ and $5\pi/3$. So, these are the solutions in $[0, 2\pi]$. Since $\cos x$ has period $2\pi$, the solutions are $\pi/3 + 2\pi n$ and $5\pi/3 + 2\pi n$, where $n$ is any integer.

**8.** Calculate the limits (a) $\lim_{x\to 0}\dfrac{\sin 5x}{2x}$; (b) $\lim_{x\to 0}\dfrac{\sin 3x}{\sin 7x}$; (c) $\lim_{x\to 0}\dfrac{\tan x}{x}$

(a) $\lim_{x\to 0}\dfrac{\sin 5x}{2x} = \lim_{x\to 0}\dfrac{5}{2}\dfrac{\sin 5x}{5x} = \dfrac{5}{2}\lim_{u\to 0}\dfrac{\sin u}{u} = \frac{5}{2}(1) = \frac{5}{2}$

(b) $\lim_{x\to 0}\dfrac{\sin 3x}{\sin 7x} = \lim_{x\to 0}\dfrac{\sin 3x}{3x}\cdot\dfrac{7x}{\sin 7x}\cdot\dfrac{3}{7} = \dfrac{3}{7}\lim_{u\to 0}\dfrac{\sin u}{u}\lim_{u\to 0}\dfrac{u}{\sin u}$

$= \frac{3}{7}(1)(1) = \frac{3}{7}$

(c) $\lim_{x\to 0}\dfrac{\tan x}{x} = \lim_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x} = \lim_{x\to 0}\dfrac{\sin x}{x}\cdot\lim_{u\to 0}\dfrac{1}{\cos x}$

$= (1)(\frac{1}{1}) = 1$

**9.** Let $y = x \sin x$. Find $y'''$.

$$y' = x\cos x + \sin x$$
$$y'' = x(-\sin x) + \cos x + \cos x = -x\sin x + 2\cos x$$
$$y''' = -x\cos x - \sin x - 2\sin x = -x\cos x - 3\sin x$$

**10.** Let $y = \tan^2(3x - 2)$. Find $y''$.

$$y' = 2\tan(3x-2)\sec^2(3x-2)\cdot 3 = 6\tan(3x-2)\sec^2(3x-2)$$
$$y'' = 6[\tan(3x-2)\cdot 2\sec(3x-2)\cdot\sec(3x-2)\tan(3x-2)\cdot 3 + \sec^2(3x-2)\sec^2(3x-2)\cdot 3]$$
$$= 36\tan^2(3x-2)\sec^2(3x-2) + 18\sec^4(3x-2)$$

**11.** Assume $y = \sin(x + y)$. Find $y'$.

$$y' = \cos(x+y)\cdot(1+y') = \cos(x+y) + \cos(x+y)\cdot(y')$$

Solving for $y'$,

$$y' = \frac{\cos(x+y)}{1-\cos(x+y)}$$

**12.** Assume $\sin y + \cos x = 1$. Find $y'$.

$$\cos y\cdot y' - \sin x = 0. \quad \text{So} \quad y' = \frac{\sin x}{\cos y}$$
$$y'' = \frac{\cos y\cos x - \sin x(-\sin y)\cdot y'}{\cos^2 y} = \frac{\cos x\cos y + \sin x\sin y\cdot y'}{\cos^2 y}$$
$$= \frac{\cos x\cos y + \sin x\sin y(\sin x)/(\cos y)}{\cos^2 y} = \frac{\cos x\cos^2 y + \sin^2 x\sin y}{\cos^3 y}$$

**13.** A pilot is sighting a location on the ground directly ahead. If the plane is flying 2 miles above the ground at 240 mi/h, how fast must the sighting instrument be turning when the angle between the path of the plane and the line of sight is 30°? See Fig. 17-12.

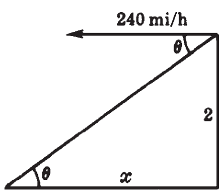


Fig. 17-12

$$\frac{dx}{dt} = -240\,\text{mi/h} \quad \text{and} \quad x = 2\cot\theta$$

From the last equation, $\frac{dx}{dt} = -2\csc^2\theta\frac{d\theta}{dt}$. Thus, $-240 = -2(4)\frac{d\theta}{dt}$ when $\theta = 30°$

$$\frac{d\theta}{dt} = 30\text{ rad/h} = \frac{3}{2\pi}\text{deg/s}$$

**14.** Sketch the graph of $f(x) = \sin x + \cos x$.

$f(x)$ has a period of $2\pi$. Hence, we need only consider the interval $[0, 2\pi]$. $f'(x) = \cos x - \sin x$, and $f''(x) = -(\sin x + \cos x)$. The critical numbers occur where $\cos x = \sin x$ or $\tan x = 1$, $x = \pi/4$ or $x = 5\pi/4$. $f''(\pi/4) = -(\sqrt{2}/2 + \sqrt{2}/2) = -\sqrt{2} < 0$. So, there is a relative maximum at $x = \pi/4$, $y = \sqrt{2}$. $f''(5\pi/4) = -(-\sqrt{2}/2 - \sqrt{2}/2) = \sqrt{2} > 0$. Thus, there is a relative minimum at $x = 5\pi/4, y = -\sqrt{2}$. The inflection points occur where $f''(x) = -(\sin x + \cos x) = 0$, $\sin x = -\cos x$, $\tan x = -1$, $x = 3\pi/4$ or $x = 7\pi/4$, $y = 0$. See Fig. 17-13.

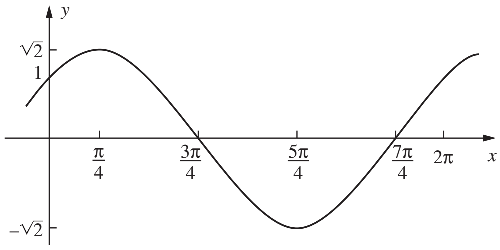


Fig. 17-13

**15.** Sketch the graph of $f(x) = \cos x - \cos^2 x$.

$$f'(x) = -\sin x - 2(\cos x)(-\sin x) = (\sin x)(2\cos x - 1)$$

and

$$f''(x) = (\sin x)(-2\sin x) + (2\cos x - 1)(\cos x)$$
$$= 2(\cos^2 x - \sin^2 x) - \cos x = 4\cos^2 x - \cos x - 2$$

Since $f$ has period $2\pi$, we need only consider $[-\pi, \pi]$, and since $f$ is even, we have to look at only $[0, \pi]$. The critical numbers are the solutions in $[0, \pi]$ of $\sin x = 0$ or $2 \cos x - 1 = 0$. The first equation has solutions 0 and $\pi$, and the second is equivalent to $\cos x = \frac{1}{2}$, which has the solution $\pi/3$. $f''(0) = 1 > 0$; so there is a relative minimum at (0, 0). $f''(\pi) = 3 > 0$; so there is a relative minimum at $(\pi, -2)$. $f''(\pi/3) = -\frac{3}{2} < 0$; hence there is a relative maximum at $(\pi/3, \frac{1}{4})$. There are inflection points between 0 and $\pi/3$ and between $\pi/3$ and $\pi$; they can be found by using the quadratic formula to solve $4 \cos^2 x - \cos x - 2 = 0$ for $\cos x$ and then using a cosine table or a calculator to approximate $x$. See Fig. 17-14.

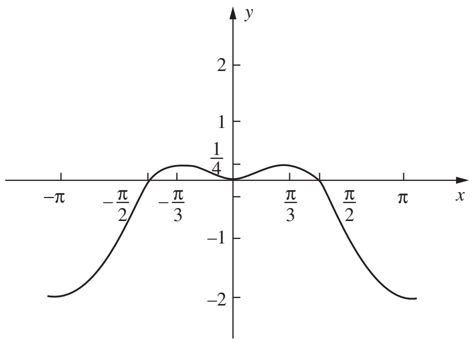


Fig. 17-14

**16.** Find the absolute extrema of $f(x) = \sin x + x$ on $[0, 2\pi]$.

$f'(x) = \cos x + 1$. Setting $f'(x) = 0$, we get $\cos x = -1$ and, therefore, the only critical number in $[0, 2\pi]$ is $x = \pi$. We list $\pi$ and the two endpoints 0 and $2\pi$ and compute the values of $f(x)$.

| $x$ | $f(x)$ |
|---|---|
| $\pi$ | $\pi$ |
| 0 | 0 |
| $2\pi$ | $2\pi$ |

Hence, the absolute maximum $2\pi$ is achieved at $x = 2\pi$, and the absolute minimum 0 at $x = 0$.

**17.** Find the angle at which the lines $\mathscr{L}_1$: $y = x + 1$ and $\mathscr{L}_2$: $y = -3x + 5$ intersect.

Let $\alpha_1$ and $\alpha_2$ be the angles of inclination of $\mathscr{L}_1$ and $\mathscr{L}_2$ (see Fig. 17-15), and let $m_1$ and $m_2$ be the respective slopes. Then tan $\alpha_1 = m_1 = 1$ and tan $\alpha_2 = m_2 = -3$. $\alpha_2 - \alpha_1$ is the angle of intersection. Now, by Problem 5,

$$\tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-3-1}{1+(-3)(1)}$$

$$= \frac{-4}{-2} = 2$$

From a graphing calculator, $\alpha_2 - \alpha_1 \sim 63.4°$.

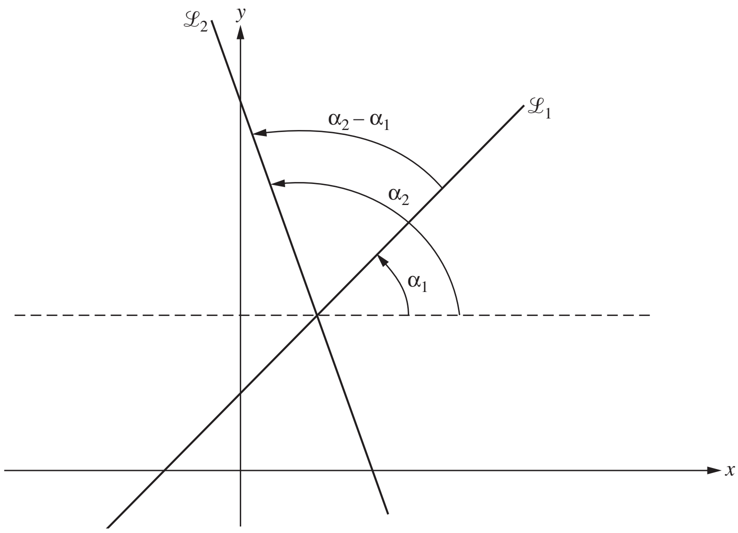


Fig. 17-15

**18.** Find the angle $\alpha$ between the parabolas $y = x^2$ and $x = y^2$ at (1, 1).

Since $D_x(x^2) = 2x$ and $D_x(\sqrt{x}) = 1/(2\sqrt{x})$, the slopes at (1, 1) are 2 and $\frac{1}{2}$. Hence, $\tan\alpha = \frac{2-(\frac{1}{2})}{1+2(\frac{1}{2})} = \frac{\frac{3}{2}}{2} = \frac{3}{4}$. Thus, using a graphing calculator, we approximate $\alpha$ by 36.9°.

## SUPPLEMENTARY PROBLEMS

**19.** Show that $\cot(x + \pi) = \cot x$, $\sec(x + 2\pi) = \sec x$, and $\csc(x + 2\pi) = \csc x$.

**20.** Find the period $p$, frequency $f$, and amplitude $A$ of 5 sin($x$/3) and sketch its graph.

*Ans.* $p = 6\pi, f = \frac{1}{3}, A = 5$

**21.** Find all solutions of $\cos x = 0$.

*Ans.* $x = (2n+1)\frac{\pi}{2}$ for all integers $n$

**22.** Find all solutions of $\tan x = 1$.

*Ans.* $x = (4n+1)\frac{\pi}{4}$ for all integers $n$

**23.** Sketch the graph of $f(x)=\dfrac{\sin x}{2-\cos x}$.

*Ans.* See Fig. 17–16.

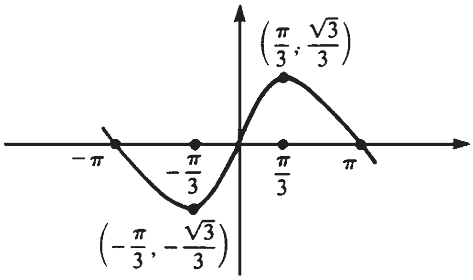


Fig. 17-16

**24.** Derive the formula $\tan(u+v)=\dfrac{\tan u+\tan v}{1-\tan u\tan v}$.

**25.** Find $y'$.

| | |
|---|---|
| (a) $y = \sin 3x + \cos 2x$ | *Ans.* $y' = 3\cos 3x - 2\sin 2x$ |
| (b) $y = \tan(x^2)$ | *Ans.* $y' = 2x\sec^2(x^2)$ |
| (c) $y = \tan^2 x$ | *Ans.* $y' = 2\tan x\sec^2 x$ |
| (d) $y = \cot(1 - 2x^2)$ | *Ans.* $y' = 4x\csc^2(1 - 2x^2)$ |
| (e) $y = x^2 \sin x$ | *Ans.* $y' = x^2\cos x + 2x\sin x$ |
| (f) $y = \dfrac{\cos x}{x}$ | *Ans.* $y' = \dfrac{-x\sin x - \cos x}{x^2}$ |

**26.** Evaluate: (a) $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$; (b) $\lim\limits_{x\to 0}\dfrac{\sin^3(2x)}{x\sin^2(3x)}$

*Ans.* (a) $\dfrac{a}{b}$; (b) $\frac{8}{9}$

**27.** If $x = A\sin kt + B\cos kt$, show that $\dfrac{d^2x}{dt^2}=-k^2x$.

**28.** (a) If $y = 3\sin(2x + 3)$, show that $y'' + 4y = 0$. (b) If $y = \sin x + 2\cos x$, show that $y''' + y'' + y' + y = 0$.

**29.** (i) Discuss and sketch the following on the interval $0 \le x < 2\pi$. (ii) (GC) Check your answers to (i) on a graphing calculator.

(a) $y=\frac{1}{2}\sin 2x$
(b) $y = \cos^2 x - \cos x$
(c) $y = x - 2\sin x$
(d) $y = \sin x(1 + \cos x)$
(e) $y = 4\cos^3 x - 3\cos x$

*Ans.* (a) maximum at $x = \pi/4, 5\pi/4$; minimum at $x = 3\pi/4, 7\pi/4$; inflection point at $x = 0, \pi/2, \pi, 3\pi/2$
(b) maximum at $x = 0, \pi$; minimum at $x = \pi/3, 5\pi/3$; inflection point at $x = 32°32', 126°23', 233°37', 327°28'$
(c) maximum at $x = 5\pi/3$; minimum at $x = \pi/3$; inflection point at $x = 0, \pi$
(d) maximum at $x = \pi/3$; minimum at $x = 5\pi/3$; inflection point at $x = 0, \pi, 104°29', 255°31'$
(e) maximum at $x = 0, 2\pi/3, 4\pi/3$; minimum at $x = \pi/3, \pi, 5\pi/3$; inflection point at $x = \pi/2, 3\pi/2, \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$

**30.** If the angle of elevation of the sun is 45° and is decreasing by $\frac{1}{4}$ radians per hour, how fast is the shadow cast on the ground by a pole 50 ft tall lengthening?

*Ans.* 25 ft/h

**31.** Use implicit differentiation to find $y'$: (a) $\tan y = x^2$; (b) $\cos(xy) = 2y$.

*Ans.* (a) $y'' = 2x\cos^2 y$; (b) $y' = -\dfrac{y\sin(xy)}{2 + x\sin(xy)}$