Chapter 6

Cardinal Numbers

6.1 INTRODUCTION

It is natural to ask whether or not two sets have the same number of elements. For finite sets the answer can be found by simply counting the number of elements. For example, each of the sets

$$\{a, b, c, d\}, \qquad \{2, 3, 5, 7\}, \qquad \{x, y, z, t\}$$

has four elements. Thus these sets have the same number of elements. However, it is not always necessary to know the number of elements in two finite sets before we know that they have the same number of elements. For example, if each chair in a room is occupied by exactly one person and there is no one standing, then clearly there are "just as many" people as there are chairs in the room.

The above simple notion, that two sets have "the same number of elements" if their elements can be "paired-off", can also apply to infinite sets. In fact, it has the following startling results:

(*a*) Infinite sets need not have the "same number of elements"; some are "more infinite" than others.
(*b*) There are "just as many" even integers as there are integers, and "just as many" rational numbers **Q** as positive integers **P**.
(*c*) There are "more" points on the real line **R** than there are positive integers **P**; and there are "more" curves in the plane $\mathbf{R}^2$ than there are points in the plane.

This chapter will investigate and prove the above results. First we will formally define when two sets, finite or infinite, have the same number of elements or, in other words, the same cardinality. Lastly, we define addition and multiplication for these "cardinal numbers", and show that many of their properties reflect corresponding properties of sets.

We remark that, at one time, all infinite sets were considered to have the same number of elements. The German mathematician Georg Cantor (1845–1918) gave the above alternative definition which revolutionized the entire theory of sets.

6.2 ONE-TO-ONE CORRESPONDENCE, EQUIPOTENT SETS

Recall that a one-to-one correspondence between sets A and B is a function $f: A \to B$ which is bijective, that is, which is one-to-one and onto. In such a case, each element $a \in A$ is paired with a unique element $b \in B$ given by $b = f(a)$. We sometimes write

$$a \leftrightarrow b$$

to denote such a pairing.

Remark: Frequently, a child counts the objects of a set by forming a one-to-one correspondence between the objects and his fingers. An adult counts the objects of a set by forming a one-to-one correspondence between the objects and the set

$$\{1, 2, 3, \ldots, n\}$$

In fact, if one is asked the question:

"How many days are there until next Saturday?"

the response is often to actually pair the remaining days with one's fingers.

The following definition applies.

Definition 6.1: Sets A and B are said to have the *same cardinality* or the *same number of elements*, or to be *equipotent*, written

$$A \approx B$$

if there is a function $f: A \to B$ which is bijective, that is, both one-to-one and onto.

Recall that such a function f is said to define a *one-to-one correspondence* between A and B.

Since the identity function is bijective, and the composition and inverse of bijective functions are bijective, we immediately obtain the following theorem:

Theorem 6.1: The relation $\approx$ of being equipotent is an equivalence relation in any collection of sets. That is:

(i) $A \approx A$ for any set A.
(ii) If $A \approx B$, then $B \approx A$.
(iii) If $A \approx B$ and $B \approx C$, then $A \approx C$.

EXAMPLE 6.1

(*a*) Let A and B be sets with exactly three elements, say,

$$A = \{2, 3, 5\}, \qquad \text{and } B = \{\text{Marc, Erik, Audrey}\}$$

Then clearly we can find a one-to-one correspondence between A and B. For example, we can label the elements of A as the first element, the second element, and the third element, and label B similarly. Then the rule which pairs the first elements of A and B, pairs the second elements of A and B, and pairs the third elements of A and B, that is, the function $f: A \to B$ defined by

$$f(2) = \text{Marc}, \qquad f(3) = \text{Erik}, \qquad f(5) = \text{Audrey}$$

is one-to-one and onto. Thus A and B are equipotent.

The same idea may be used to show that any two finite sets with the same number of elements are equipotent.

(*b*) Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Then A and B are not equipotent. For suppose there were a rule for pairing the elements of A and B. If there were four or more pairs, then an element of B would be used twice, and if there were three or fewer pairs then some element of A would not be used. In other words, since A has more elements than B, any function $f: A \to B$ must assign at least two elements of A to the same element of B, and hence f would not be one-to-one.

In a similar way, we can see that any two finite sets with different numbers of elements are not equipotent.

(*c*) Let $\mathbf{I} = [0, 1]$, the closed unit interval, and let S be any other closed interval, say $S = [a, b]$ where $a < b$. The function $f: \mathbf{I} \to S$ defined by

$$f(x) = (b - a)x + a$$

is one-to-one and onto. Thus $\mathbf{I}$ and S have the same cardinality. Therefore, by Theorem 6.1, any two closed intervals have the same cardinality.

(*d*) Consider the set $\mathbf{P} = \{1, 2, 3, \ldots\}$ of positive integers and the set $E = \{2, 4, 6, \ldots\}$ of even positive integers. The following defines a one-to-one correspondence between $\mathbf{P}$ and E:

$$\begin{array}{rcccccccc} \mathbf{P} = \{1, & 2, & 3, & 4, & 5, & . & . & .\} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ E = \{2, & 4, & 6, & 8, & 10, & . & . & .\} \end{array}$$

In other words, the function $f: \mathbf{P} \to E$ defined by $f(n) = 2n$ is one-to-one and onto. Thus $\mathbf{P}$ and E have the same cardinality.

More generally, if $K = \{0, k, 2k, 3k, \ldots\}$ is the set of multiples of a positive integer k, then $f: \mathbf{P} \to K$ defined by $f(n) = kn$ is a one-to-one correspondence between $\mathbf{P}$ and K. Therefore $\mathbf{P}$ and K have the same cardinality.

Parts (*a*) and (*b*) of the above Example 6.1 show that finite sets are equipotent if and only if they contain the same number of elements. Thus, for finite sets, Definition 6.1 corresponds to the usual meaning of two sets containing the same number of elements.

On the other hand, Example 6.1(*d*) shows that the infinite set **P** has the same cardinality as a proper subset of itself. This property is characteristic of infinite sets. In fact, we state this observation formally.

Definition 6.2: A set S is *infinite* if it has the same cardinality as a proper subset of itself. Otherwise S is *finite*.

Familiar examples of infinite sets are the counting numbers (positive integers) **P**, the natural numbers (nonnegative integers) **N**, the integers **Z**, the rational numbers **Q**, and the real numbers **R**.

There might be a temptation to think that all infinite sets have the same cardinality; but we will show later that this is definitely not true.

We conclude this section with the following example, which tells us that any two sets have the same cardinality, respectively, to two disjoint sets.

EXAMPLE 6.2 Consider any two sets A and B. Let $A' = A \times \{1\}$ and $B' = B \times \{2\}$. Then

$$A \approx A' \qquad \text{and} \qquad B \approx B'$$

For example, the functions

$$f(a) = (a, 1),\ a \in A \qquad \text{and} \qquad g(b) = (b, 2),\ b \in B$$

are each bijective. Although A and B need not be disjoint, the sets A' and B' are disjoint, i.e.,

$$A' \cap B' = \varnothing$$

Specifically, each ordered pair in A' has 1 as a second component, whereas each ordered pair in B' has 2 as a second component.

6.3 DENUMERABLE AND COUNTABLE SETS

The reader is familiar with the set $\mathbf{P} = \{1, 2, 3, \ldots\}$ of counting numbers or positive integers. The following definitions apply.

Definition 6.3: A set D is said to be *denumerable* or *countably infinite* if D has the same cardinality as **P**.

Definition 6.4: A set is *countable* if it is finite or denumerable, and a set is *nondenumerable* if it is not countable.

Thus a set S is nondenumerable if S is infinite and S does not have the same cardinality as **P**.

EXAMPLE 6.3

(*a*) Any infinite sequence

$$a_1, a_2, a_3, \ldots$$

of distinct elements is countably infinite, for a sequence is essentially a function $f(n) = a_n$ whose domain is **P**. So if the a_n are distinct, the function is one-to-one and onto. Thus each of the following sets is countably infinite:

$$\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$$
$$\{1, -2, 3, -4, \ldots (-1)^{n-1}n, \ldots\}$$
$$\{(1, 1), (4, 8), (9, 27), \ldots, (n^2, n^3), \ldots\}$$

(*b*) Consider the product set $\mathbf{P} \times \mathbf{P}$ as exhibited in Fig. 6-1. The set $\mathbf{P} \times \mathbf{P}$ can be written as an infinite sequence as follows:

$$\{(1,1),\ (2,1),\ (1,2),\ (1,3),\ (2,2), \ldots\}$$

This sequence is determined by "following the arrows" in Fig. 6-1. Thus $\mathbf{P} \times \mathbf{P}$ is countably infinite for the reasons stated in (*a*).

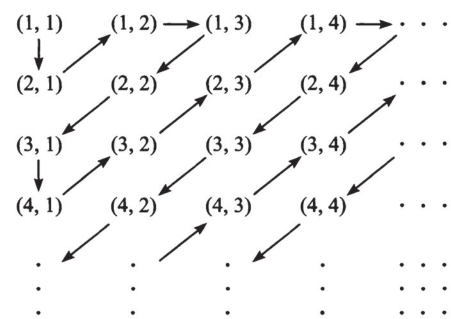


Fig. 6-1

(*c*) Recall that $\mathbf{N} = \{0, 1, 2, \ldots\} = \mathbf{P} \cup \{0\}$ is the set of natural numbers or nonnegative integers. Now each positive integer $a \in \mathbf{P}$ can be written uniquely in the form

$$a = 2^r(2s + 1)$$

where $r, s \in \mathbf{N}$. Consider the function $f\colon \mathbf{P} \to \mathbf{N} \times \mathbf{N}$ defined by

$$f(a) = (r, s)$$

where r and s are as above. Then f is one-to-one and onto. Thus $\mathbf{N} \times \mathbf{N}$ is denumerable (countably infinite) or, in other words, $\mathbf{N} \times \mathbf{N}$ has the same cardinality as $\mathbf{P}$. Note that $\mathbf{P} \times \mathbf{P}$ is a subset of $\mathbf{N} \times \mathbf{N}$.

The following theorems apply.

Theorem 6.2: Every infinite set contains a subset which is denumerable.

Theorem 6.3: A subset of a denumerable set is finite or denumerable.

Corollary 6.4: A subset of a countable set is countable.

Theorem 6.5: Let $A_1, A_2, A_3, \ldots$ be a sequence of pairwise disjoint denumerable sets. Then the union

$$A_1 \cup A_2 \cup A_3 \cup \cdots = \cup(A_i : i \in \mathbf{P}\}$$

is denumerable.

Corollary 6.6: A countable union of countable sets is countable.

Observe that Corollary 6.6 tells us that if each of the sets $A_1, A_2, A_3, \ldots$ is countable then the union

$$A_1 \cup A_2 \cup A_3 \cup \cdots$$

is also countable.

The next theorem gives a very important, and not entirely obvious, example of a denumerable (countably infinite) set.

Theorem 6.7: The set $\mathbf{Q}$ of rational numbers is denumerable.

Proof: Note that $\mathbf{Q} = \mathbf{Q}^+ \cup \{0\} \cup \mathbf{Q}^-$ where $\mathbf{Q}^+$ and $\mathbf{Q}^-$ denote, respectively, the sets of positive and negative rational numbers. Let $f: \mathbf{Q}^+ \to \mathbf{P} \times \mathbf{P}$ be defined by

$$f(p/q) = (p, q)$$

where p/q is any element of $\mathbf{Q}^+$ expressed as the ratio of two relatively prime positive integers. Then f is one-to-one and so $\mathbf{Q}^+$ has the same cardinality as a subset of $\mathbf{P} \times \mathbf{P}$. By Example 6.3($b$), $\mathbf{P} \times \mathbf{P}$ is denumerable; hence, by Theorem 6.3, the infinite set $\mathbf{Q}^+$ is denumerable. Similarly $\mathbf{Q}^-$ is denumerable. Thus the set $\mathbf{Q}$ of rational numbers, the union of $\mathbf{Q}^+$, $\{0\}$, and $\mathbf{Q}^-$, is also denumerable.

Remark: Theorem 6.7 tells us that there are just as many rational numbers as there are positive integers, that is, that $\mathbf{Q}$ has the same cardinality as $\mathbf{P}$.

6.4 REAL NUMBERS R AND THE POWER OF THE CONTINUUM

Not every infinite set is countable. The next theorem (proved in Problem 6.15) gives a specific and extremely important example of such a set.

Theorem 6.8: The unit interval $\mathbf{I} = [0, 1]$ is nondenumerable.

Observe that this theorem also tells us that infinite sets need not have the same cardinality.

The following definition applies.

Definition 6.5: A set A is said to have the *power of the continuum* if A has the same cardinality as the unit interval $\mathbf{I} = [0, 1]$.

Besides the unit interval $\mathbf{I}$, all the other intervals also have the power of the continuum. There are several such kinds of intervals. Specifically, if a and b are real numbers with $a < b$, then we define:

closed interval:	$[a, b] = \{x \in \mathbf{R} : a \le x \le b\}$
open interval:	$(a, b) = \{x \in \mathbf{R} : a < x < b\}$
half-open intervals:	$[a, b) = \{x \in \mathbf{R} : a \le x < b\}$
	$(a, b] = \{x \in \mathbf{R} : a < x \le b\}$

Example 6.1(c) shows that any closed interval $[a, b]$ has the power of the continuum. Problem 6.3 shows that any open or half-open interval also has the power of the continuum.

Real Numbers R

Lastly, we note that the set $\mathbf{R}$ of real numbers also has the power of the continuum. Specifically, consider the function $f: \mathbf{R} \to D$ where $D = (-1, 1)$ and f is defined by

$$f(x) = \frac{x}{1 + |x|}$$

Figure 6-2 is the graph of this function. Clearly the values of f belong to $(-1, 1)$ since $|x| < 1 + |x|$. It is not difficult to show that f is both one-to-one and onto. Thus the set $\mathbf{R}$ of real numbers has the same cardinality as the open interval $D = (-1, 1)$, and hence $\mathbf{R}$ has the power of the continuum.

Remark: Some texts define a set A to have the power of the continuum if it has the same cardinality as $\mathbf{R}$ rather than the unit interval $\mathbf{I}$. By the above remark, both definitions are equivalent. The use here of $\mathbf{I}$ rather than $\mathbf{R}$ is motivated by Theorem 6.8.

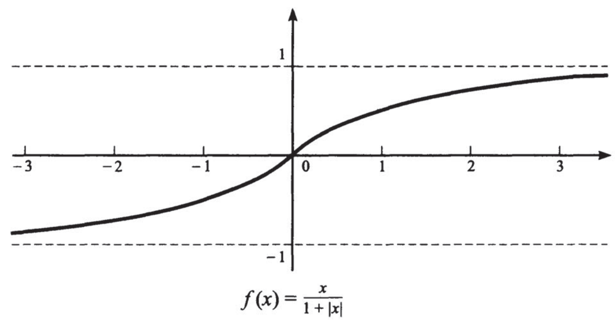


Fig. 6-2

6.5 CARDINAL NUMBERS

Frequently, we want to know the "size" of a given set without necessarily comparing it to another set. For finite sets, there is no difficulty. For example, the set $A = \{a, b, c\}$ has 3 elements. Any other set with 3 elements is equipotent to A. On the other hand, for infinite sets it is not sufficient to just say that the set has infinitely many elements since not all infinite sets are equipotent. To solve this problem, we introduce the concept of a cardinal number.

Each set A is assigned a symbol in such a way that two sets A and B are assigned the same symbol if and only if they are equipotent. This symbol is called the *cardinality* or *cardinal number* of A, and it is denoted by

$$|A|, \qquad n(A), \qquad \text{or} \qquad \text{card}(A)$$

We will use $|A|$. Thus:

$$\boxed{|A| = |B| \qquad \text{if and only if} \qquad A \approx B}$$

One may also view a cardinal number as the equivalence class of all sets which are equipotent.

Finite Cardinal Numbers

The obvious symbols are used for the cardinal numbers of finite sets. That is, 0 is assigned to the empty set $\varnothing$, and n is assigned to the set $\{1, 2, \ldots, n\}$. Thus:

$$\boxed{|A| = n \qquad \text{if and only if} \qquad A \approx \{1, 2, \ldots, n\}}$$

Alternatively, the symbols $0, 1, 2, 3, \ldots$ are assigned, respectively, to the sets

$$\varnothing,\ \{\varnothing\},\ [\varnothing,\ \{\varnothing\}], [\varnothing,\ \{\varnothing\},\ [\varnothing,\ \{\varnothing\}]], \ldots$$

Although the natural number n and the cardinal number n are technically different things, there is no conflict using the same symbol in these two roles. The cardinal numbers of finite sets are called *finite cardinal numbers*.

Transfinite Cardinal Numbers, $\aleph_0$ and c

Cardinal numbers of infinite sets are called *infinite* or *transfinite cardinal numbers.*

The cardinal number of the infinite set **P** of positive integers is

$$\aleph_0$$

which is read aleph-nought. This notation was introduced by Cantor. (The symbol $\aleph$ is the first letter aleph of the Hebrew alphabet.) Thus:

$$|A| = \aleph_0 \qquad \text{if and only if} \qquad A \approx \mathbf{P}$$

In particular, we have $|\mathbf{Z}| = \aleph_0$ and $|\mathbf{Q}| = \aleph_0$. (The significance of 0 in $\aleph_0$ is discussed in Chapter 8.)

The cardinal number of the unit interval $\mathbf{I} = [0, 1]$ is denoted by:

$$\mathbf{c}$$

and it is called the *power of the continuum*. Thus:

$$|A| = \mathbf{c} \qquad \text{if and only if} \qquad A \approx \mathbf{I}$$

In particular, we have $|\mathbf{R}| = \mathbf{c}$, and the cardinal number of any interval is **c**.

The following statements follow directly from the above definitions:

(*a*) A is denumerable or countably infinite means $|A| = \aleph_0$.
(*b*) A is countable means $|A|$ is finite or $|A| = \aleph_0$.
(*c*) A has the power of the continuum means $|A| = \mathbf{c}$.

6.6 ORDERING OF CARDINAL NUMBERS

One frequently wants to compare the size of two sets. This is done by means of an inequality relation which is defined for cardinal numbers as follows.

Definition 6.6: Let A and B be sets. We say that

$$|A| \leq |B|$$

if A has the same cardinality as a subset of B or, equivalently, if there exists a one-to-one (injective) function $f: A \to B$.

As expected, $|A| \leq |B|$ is read:

"The cardinal number of A is less than or equal to the cardinal number of B."

As usual with the symbol $\leq$, we have the following addition notation:

$$\begin{array}{lll} |A| < |B| & \text{means} & |A| \leq |B| \quad \text{but } |A| \neq |B| \\ |A| \geq |B| & \text{means} & |B| \leq |A| \\ |A| > |B| & \text{means} & |B| < |A| \end{array}$$

Again, as usual, the symbols $<$, $\geq$, $>$ are read "less than", "greater than or equal to", and "greater than", respectively.

We emphasize that the above relations between cardinal numbers are well defined, that is, the relations are independent of the particular sets involved. Namely, if $A \approx A'$ and $B \approx B'$, then

$$|A| \leq |B| \text{ if and only if } |A'| \leq |B'| \quad \text{and} \quad |A| < |B| \text{ if and only if } |A'| < |B'|$$

EXAMPLE 6.4

(*a*) Let A be a proper subset of a finite set B. Clearly, $|A| \leq |B|$. Since A is a proper subset of B, where A and B are finite, we know that $|A| \neq |B|$. Thus $|A| < |B|$. In other words, for finite cardinals m and n, we have $m < n$ as cardinal numbers if and only if $m < n$ as nonnegative integers. Accordingly, the inequality relation $\leq$ for cardinal numbers is an extension of the inequality relation $\leq$ for nonnegative integers.

(*b*) Let n be a finite cardinal. Then $n < \aleph_0$ since any finite set A is equipotent to a subset of **P** and $|A| \neq |\mathbf{P}|$. Thus we may write

$$0 < 1 < 2 < \cdots < \aleph_0$$

(*c*) Consider the set **P** of positive integers and the unit interval **I**, that is, consider the sets

$$\mathbf{P} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbf{I} = \{x \in \mathbf{R} : 0 \leq x \leq 1\}$$

The function $f\colon \mathbf{P} \to \mathbf{I}$ defined by $f(n) = 1/n$ is one-to-one. Therefore, $|\mathbf{P}| \leq |\mathbf{I}|$. On the other hand, by Theorem 6.7, $|\mathbf{P}| \neq |\mathbf{I}|$. Therefore, $\aleph_0 = |\mathbf{P}| < |\mathbf{I}| = \mathbf{c}$. Accordingly, we may now write

$$0 < 1 < 2 < \cdots < \aleph_0 < \mathbf{c}$$

(*d*) Let A be any infinite set. By Theorem 6.2, A contains a subset which is denumerable. Accordingly, for any infinite set A, we always have $\aleph_0 \leq |A|$.

Cantor's Theorem

The only transfinite cardinal numbers we have seen are $\aleph_0$ and **c**. It is natural to ask if there are any others. The answer is yes. In fact, Cantor's theorem, which follows, tells us that the cardinal number of the power set $\mathscr{P}(A)$ of any set A is larger than the cardinal number of the set A itself; namely:

Theorem 6.9 (Cantor): For any set A, we have $|A| < |\mathscr{P}(A)|$.

This important theorem is proved in Problem 6.18.

Notation: If $\alpha = |A|$, then we let $2^\alpha = |\mathscr{P}(A)|$. This no doubt comes from the fact that if a finite set A has n elements then $\mathscr{P}(A)$ has 2^n elements.

Accordingly, Cantor's theorem may be restated as follows.

Theorem 6.9 (Cantor): For any cardinal number α, we have $\alpha < 2^\alpha$.

Schroeder–Bernstein Theorem, Law of Trichotomy

Note first that the relation $\leq$ for cardinal numbers is reflexive and transitive. That is:

(i) For any set A, we have $|A| = |A|$.

(ii) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

The second property (transitivity) comes from the fact that if $f\colon A \to B$ and $g\colon B \to C$ are both one-to-one, then the composition $g \circ f\colon A \to C$ is also one-to-one.

Since we have used the familiar $\leq$ notation, we would hope that the relation $\leq$ for cardinal numbers possesses other commonly used properties of the relation $\leq$ for the real numbers **R** and the integers **Z**. One such property follows:

If a and b are real numbers such that $a \leq b$ and $b \leq a$, then $a = b$.

This property certainly holds for finite cardinal numbers. If A is a proper subset of a finite set B, then $|A| < |B|$. Therefore, for finite sets A and B, the only way that we can have $|A| \leq |B|$ and $|B \leq |A|$ is that A and B have the same number of elements, that is, that $|A| = |B|$.

On the other hand, it is possible for a proper subset of an infinite set to have as many elements as the entire set. For example, consider the infinite sets

$$E = \{2, 4, 6, \ldots\} \quad \text{and} \quad \mathbf{P} = \{1, 2, 3, \ldots\}$$

As illustrated in Example 6.1(d), the subset E does have the same cardinality as $\mathbf{P}$. Accordingly, the above property for infinite cardinal numbers is not obvious. But it is still indeed true in view of the celebrated Schroeder–Bernstein theorem which follows.

Theorem 6.10 (Schroeder–Bernstein): If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

In other words, if α and β are cardinal numbers such that $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$. This important theorem, proved in Problem 6.19, can be stated in the following equivalent form.

Theorem 6.11: Let X, Y, X_1 be sets such that $X \supseteq Y \supseteq X_1$ and $X \approx X_1$. Then $X \approx Y$.

Another familiar property of the relation $\leq$ for the real numbers $\mathbf{R}$, called the law of trichotomy, is the following:

> If a and b are real numbers, then exactly one of the following is true:
> $$a < b, \qquad a = b, \qquad a > b$$

It is clear that the above property holds for finite cardinal numbers. Again, it is not obvious that it holds for infinite cardinal numbers. The fact that it does is the content of the next theorem.

Theorem 6.12 (Law of Trichotomy): For any two sets A and B, exactly one of the following is true:

$$|A| < |B|, \qquad |A| = |B|, \qquad |A| > |B|$$

In other words, if α and β are cardinal numbers, then either $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$. The proof of this theorem uses transfinite induction which is discussed in Chapter 9; hence the proof will be postponed until then.

Continuum Hypothesis

By Cantor's theorem, $\aleph_0 < 2^{\aleph_0}$ and, as noted previously, $\aleph_0 < \mathbf{c}$. The next theorem (proved in Problem 6.20) tells us the relationship between $2^{\aleph_0}$ and $\mathbf{c}$.

Theorem 6.13: $2^{\aleph_0} = \mathbf{c}$.

It is natural to ask if there exists a cardinal number β which lies "between" $\aleph_0$ and $\mathbf{c}$. Originally, Cantor supported the conjecture, which is known as the continuum hypothesis, that the answer to the above question is in the negative. Specifically:

Continuum Hypothesis: There exists no cardinal number β such that

$$\aleph_0 < \beta < \mathbf{c}$$

In 1963 it was shown by Paul Cohen that the continuum hypothesis is independent of our axioms of set theory in somewhat the same sense that Euclid's fifth postulate on parallel lines is independent of the other axioms of geometry.

6.7 CARDINAL ARITHMETIC

The collection of all cardinal numbers can be considered to be a superset of the finite cardinal numbers (nonnegative integers)

$$0, 1, 2, 3, \ldots$$

This section shows how certain arithmetic operations on the finite cardinals can be extended to all the cardinal numbers.

Cardinal Addition and Multiplication

Addition and multiplication of the counting numbers **N** are sometimes treated from the point of view of set theory. The interpretation of $2 + 3 = 5$, for example, is given by the picture in Fig. 6-3. Namely, the union of two disjoint sets, one having two elements and the other having three elements, is a set with five elements. This idea leads to a completely general definition of addition of cardinal numbers.

(x x) + (x x x) = (x x   x x x)

Fig. 6-3

Definition 6.7: Let α and β be cardinal numbers and let A and B be disjoint sets with $\alpha = |A|$ and $\beta = |B|$. Then the *sum* of α and β is denoted and defined by

$$\alpha + \beta = |(A \cup B)|$$

Two comments are appropriate with this definition. First of all, the addition of cardinal numbers is well-defined. That is, if A' and B' are also disjoint sets with cardinality α and β respectively, then

$$|(A' \cup B')| = |(A \cup B)|$$

Second, if A and B are any two sets, then $A \times (1)$ and $B \times \{2\}$ are disjoint. Accordingly, there is no difficulty in finding disjoint sets with given cardinalities.

EXAMPLE 6.5

(*a*) Let m and n be finite cardinal numbers. Then $m + n$ corresponds to the usual addition in **N**.

(*b*) Let n be a finite cardinal number. Then $n + \aleph_0 = \aleph_0$ since

$$n + \aleph_0 = |\{1, 2, \ldots, n\} \cup \{n + 1, n + 2, \ldots\}| = \aleph_0$$

(*c*) $\aleph_0 + \aleph_0 = \aleph_0$ since

$$\aleph_0 + \aleph_0 = |\{2, 4, 6, \ldots\} \cup \{1, 3, 5, \ldots\}| = \aleph_0$$

(*d*) $\mathbf{c} + \mathbf{c} = \mathbf{c}$ since

$$\mathbf{c} + \mathbf{c} = |[0, \tfrac{1}{2}] \cup (\tfrac{1}{2}, 1]| = \mathbf{c}$$

The definition of cardinal multiplication follows.

Definition 6.8: Let α and β be cardinal numbers and let A and B be sets with $\alpha = |A|$ and $\beta = |B|$. Then the *product* of α and β is denoted and defined by

$$\alpha\beta = |A \times B|$$

As with addition, multiplication of cardinal numbers is well-defined. (Observe that, in the definition of cardinal multiplication, A and B need not be disjoint.)

EXAMPLE 6.6

(*a*) Let m and n be finite cardinal numbers. Then mn corresponds to the usual multiplication in **N**.

(*b*) Since $\mathbf{N} \times \mathbf{N}$ is countably infinite, $\aleph_0\aleph_0 = \aleph_0$.

(*c*) Theorem 6.15 below tells us that the cartesian plane $\mathbf{R}^2$ has the same cardinality as **R**. That is, $\mathbf{cc} = \mathbf{c}$.

Table 6-1 lists properties of the addition and multiplication of cardinal numbers and gives the corresponding properties of sets under union and cartesian product. We state this result formally.

Theorem 6.14: The addition and multiplication of cardinal numbers satisfy the properties in Table 6-1.

Table 6-1

Cardinal numbers	Sets
(1) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$	(1) $(A \cup B) \cup C = A \cup (B \cup C)$
(2) $\alpha + \beta = \beta + \alpha$	(2) $A \cup B = B \cup A$
(3) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$	(3) $(A \times B) \times C \approx A \times (B \times C)$
(4) $\alpha\beta = \beta\alpha$	(4) $A \times B \approx B \times A$
(5) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$	(5) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(6) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$	(6) If $A \subseteq B$, then $(A \cup C) \subseteq (B \cup C)$
(7) If $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$	(7) If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$

We emphasize that not every property of addition and multiplication of finite cardinals holds for cardinal numbers in general. For example, cancellation holds for finite cardinal numbers, that is,

(i) If $a + b = a + c$, then $b = c$.
(ii) If $ab = ac$ and $a \neq 0$, then $b = c$.

On the other hand, using Example 6.5 and Example 6.6, we have

(i) $\aleph_0 + \aleph_0 = \aleph_0 = \aleph_0 + 1$, but $\aleph_0 \neq 1$.
(ii) $\aleph_0\aleph_0 = \aleph_0 = \aleph_0 1$, but $\aleph_0 \neq 1$.

Accordingly, the cancellation law is not true for the operations of addition and multiplication of infinite cardinal numbers.

On the other hand, the addition and multiplication of infinite cardinal numbers turn out to be very simple. We state the following theorem whose proof lies beyond the scope of this text.

Theorem 6.15: Let α and β be nonzero cardinal numbers such that β is infinite and $\alpha \leq \beta$. Then

$$\alpha + \beta = \alpha\beta = \beta$$

That is, given two nonzero cardinal numbers, at least one of which is infinite, their sum or product is simply the larger of the two. Examples 6.5 and 6.6 verify some instances of the theorem.

Exponents and Cardinal Numbers

First we note that if A and B are sets, then

$$A^B$$

denotes the set of all functions from B (the exponent) into A. This notation comes from the fact that if A and B are finite sets, say, $|A| = m$ and $|B| = n$, then there are m^n functions from B into A. This is illustrated in the next example, where $|A| = 2$ and $|B| = 3$.

EXAMPLE 6.7 Let $A = \{1, 2\}$ and $B = \{x, y, z\}$. Then A^B consists of exactly eight functions, which follow:

$$\begin{array}{llll} \{(x,1),(y,1),(z,1)\}, & \{(x,1),(y,1),(z,2)\}, & \{(x,1),(y,2),(z,1)\}, & \{(x,1),(y,2),(z,2)\}, \\ \{(x,2),(y,1),(z,1)\}, & \{(x,2),(y,1),(z,2)\}, & \{(x,2),(y,2),(z,1)\}, & \{(x,2),(y,2),(z,2)\} \end{array}$$

That is, there are 2 choices for x, 2 choices for y, and 2 choices for z, and hence there are $2^3 = 8$ functions altogether.

Exponents are introduced into the arithmetic of cardinal numbers in the next definition and, as illustrated above, this definition agrees with the case when A and B are finite sets.

Definition 6.9: Let α and β be cardinal numbers and let A and B be sets with $\alpha = |A|$ and $\beta = |B|$. Then *α to the power β* is denoted and defined by

$$\alpha^\beta = |A^B|$$

Remark: Previously, if $\alpha = |A|$, then we used the exponent notation $2^\alpha = |\mathscr{P}(A)|$ where $\mathscr{P}(A)$ is the power set (collection of all subsets) of a set A. We note that there is a one-to-one correspondence between the subsets X of A and functions $f\colon A \to \{0, 1\}$ as follows:

$$f(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{if } a \notin X \end{cases}$$

Thus there is no contradiction between the two notations.

The following familiar rules for working with exponents continue to hold.

Theorem 6.16: Let α, β, γ be cardinal numbers. Then:

(1) $(\alpha\beta)^\gamma = \alpha^\gamma\beta^\gamma$.
(2) $\alpha^\beta\alpha^\gamma = \alpha^{\beta+\gamma}$.
(3) $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$.
(4) If $\alpha \le \beta$, then $\alpha^\gamma \le \beta^\gamma$.

EXAMPLE 6.8 Using the rules for exponentiation we can make the following calculations:

(*a*) $\mathbf{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0} = \mathbf{c}$.

(*b*) $\mathbf{c}^{\mathbf{c}} = (2^{\aleph_0})^{\mathbf{c}} = 2^{\aleph_0\mathbf{c}} = 2^{\mathbf{c}}$.

Solved Problems

EQUIPOTENT SETS, DENUMERABLE SETS, CONTINUUM

6.1. Consider the following concentric circles:

$$C_1 = \{(x, y) : x^2 + y^2 = a^2\}, \qquad C_2 = \{(x, y) : x^2 + y^2 = b^2\}$$

where, say, $0 < a < b$. Establish, geometrically, a one-to-one correspondence between C_1 and C_2.

Let $x \in C_2$. Consider the function $f\colon C_2 \to C_1$ where $f(x)$ is the point of intersection of the radius from the center of C_2, (and C_1) to x and C_1, as shown in Fig. 6-4. Note that f is both one-to-one and onto. Thus f defines a one-to-one correspondence between C_1 and C_2.

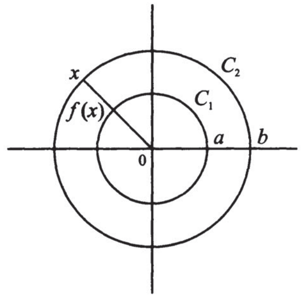


Fig. 6-4

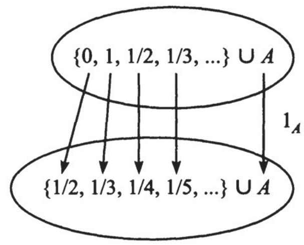


Fig. 6-5

6.2. Prove: (*a*) $[0,1] \approx (0,1)$; (*b*) $[0,1] \approx [0,1)$; (*c*) $[0,1] \approx (0,1]$.

(*a*) Note that

$$[0,1] = \{0, 1, 1/2, 1/3, \ldots\} \cup A$$
$$(0,1) = \{1/2, 1/3, 1/4, \ldots\} \cup A$$

where

$$A = [0,1] \setminus \{0, 1, 1/2, 1/3, \ldots\} = (0,1) \setminus \{1/2, 1/3, \ldots\}$$

Consider the function f: $[0,1] \to (0,1)$ defined by the diagram in Fig. 6-5. That is,

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/(n+2) & \text{if } x = 1/n, n \in \mathbf{P} \\ x & \text{if } x \neq 0, 1/n, n \in \mathbf{P} \end{cases}$$

The function f is one-to-one and onto. Consequently, $[0,1] \approx (0,1)$.

(*b*) The function f: $[0,1] \to [0,1)$ defined by

$$f(x) = \begin{cases} 1/(n+1) & \text{if } x = 1/n, n \in \mathbf{P} \\ x & \text{if } x \neq 1/n, n \in \mathbf{P} \end{cases}$$

is one-to-one and onto. [It is similar to the function in part (*a*).] Hence $[0,1] \approx [0,1)$.

(*c*) Let f: $[0,1) \to (0,1]$ be the function defined by $f(x) = 1 - x$. Then f is one-to-one and onto and, therefore, $[0,1) \approx (0,1]$. By part (*b*) and Theorem 6.1, we have $[0,1] \approx (0,1]$.

6.3. Prove that each of the following intervals (where $a < b$) has the power of the continuum, i.e., has cardinality **c**:

$$(1)\ [a,b], \qquad (2)\ (a,b), \qquad (3)\ [a,b), \qquad (4)\ (a,b]$$

The formula $f(x) = a + (b - a)x$ defines a bijective mapping between each pair of sets:

(1) $[0,1)$ and $[a,b]$ (3) $[0,1)$ and $[a,b)$
(2) $(0,1)$ and (a,b) (4) $(0,1]$ and $(a,b]$

Thus, by Theorem 6.1 and Problem 6.2, every interval has the same cardinality as the unit interval $\mathbf{I} = [0,1]$, that is, has the power of the continuum.

6.4. Prove Theorem 6.1: The relation $A \approx B$ in sets is an equivalence relation. Specifically:

(1) $A \approx A$ for any set A.

(2) If $A \approx B$, then $B \approx A$.

(3) If $A \approx B$ and $B \approx C$, then $A \approx C$.

(1) The identity function $1_A : A \to A$ is bijective (one-to-one and onto); hence $A \approx A$.

(2) Suppose $A \approx B$. Then there exists a bijective function $f: A \to B$. Hence f has an inverse function $f^{-1}: B \to A$ which is also bijective. Hence $B \approx A$. Therefore, if $A \approx B$ then $B \approx A$.

(3) Suppose $A \approx B$ and $B \approx C$. Then there exist bijective functions $f: A \to B$ and $g: B \to C$. Then the composition function $g \circ f: A \to C$ is also bijective. Hence $A \approx B$. Therefore, if $A \approx B$ and $B \approx C$, then $A \approx C$.

6.5. Prove Theorem 6.2: Every infinite set A contains a subset D which is denumerable.

Let $f: \mathscr{P}(A) \to A$ be a choice function. Consider the following sequence:

$$\begin{aligned} a_1 &= f(A) \\ a_2 &= f(A \backslash \{a_1\}) \\ a_3 &= f(A \backslash \{a_1, a_2\}) \\ &\cdots\cdots\cdots \\ a_n &= f(A \backslash \{a_1, a_2, \ldots, a_{n-1}\}) \\ &\cdots\cdots\cdots \end{aligned}$$

Since A is infinite, $A \backslash \{a_1, a_2, \ldots, a_{n-1}\}$ is not empty for every $n \in \mathbf{P}$. Furthermore, since f is a choice function,

$$a_n \not\approx a_i \quad \text{for} \quad i < n$$

Thus the a_n are distinct and, therefore, $D = \{a_1, a_2, \ldots\}$ is a denumerable subset of A.

Essentially, the choice function f "chooses" an element $a_1 \in A$, then chooses an element a_2 from the elements which "remain" in A, and so on. Since A is infinite, the set of elements which "remain" in A is nonempty.

6.6. Prove: (*a*) For any sets A and B, $A \times B \approx B \times A$.

(*b*) For any sets A, B, C,

$$(A \times B) \times C \approx A \times B \times C \approx A \times (B \times C)$$

(*c*) If $A \approx C$ and $B \approx D$, then $A \times B \approx C \times D$.

(*a*) Let $f: A \times B \to B \times A$ be defined by

$$f((a, b)) = (b, a)$$

Clearly f is bijective. Hence $A \times B \approx B \times A$.

(*b*) Let $f: (A \times B) \times C \to A \times B \times C$ be defined by

$$f((a, b), c) = (a, b, c)$$

Then f is bijective. Hence $(A \times B) \times C \approx A \times B \times C$. Similarly, $A \times (B \times C) \approx A \times B \times C$. Thus

$$(A \times B) \times C \approx A \times B \times C \approx A \times (B \times C)$$

(c) Let $f: A \to C$ and $g: B \to D$ be one-to-one correspondences. Define $h: A \times B \to C \times D$ by

$$h(a, b) = (f(a), g(b))$$

One can easily check that h is one-to-one and onto. Hence $A \times B \approx C \times D$.

6.7. Prove: Let X be any set and let $C(X)$ be the family of characteristic functions of X, that is, the family of functions $f: X \to \{0, 1\}$. Then $\mathscr{P}(X) \approx C(X)$ where $\mathscr{P}(X)$ is the power set of X, i.e., the collection of subsets of X.

Let A be any subset of X, i.e., let $A \in \mathscr{P}(X)$. Let $f: \mathscr{P}(X) \to C(X)$ be defined by

$$f(A) = \chi_A$$

that is, f maps each subset A of X into the characteristic function χ_A of A (relative to X). [Recall $\chi_A : X \to \{0, 1\}$ is defined by $f(x) = 1$ if and only if $x \in A$.] Then f is both one-to-one and onto. Hence $\mathscr{P}(X) \approx C(X)$.

6.8. Suppose A is an infinite set and F is a finite subset of A. Show that $A \backslash F \approx A$. In other words, removing a finite number of elements from an infinite set does not change its cardinality.

Suppose $F = \{a_1, a_2, \ldots, a_n\}$. Choose a denumerable subset $D = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$ of A so that the first n elements of D are the elements of F. Let $g: A \to A \backslash F$ be defined by

$$g(a) = a \text{ if } a \notin D \qquad \text{and} \qquad g(a_k) = a_{k+n} \text{ if } a \in D$$

Then g is one-to-one correspondence between A and $A \backslash F$. Thus $A \approx A \backslash F$.

6.9. Prove Theorem 6.3: A subset of a denumerable set is either finite or denumerable.

Consider any denumerable set, say,

$$A = \{a_1, a_2, a_3, \ldots\} \tag{1}$$

Let B be a subset of A. If $B = \varnothing$, then B is finite. Suppose $B \neq 0$. Let b_1 be the first element in the sequence in (1) such that $b_1 \in B$; let b_2 be the first element which follows b_1 in the sequence in (1) such that $b_2 \in B$; and so on. Then $B = \{b_1, b_2, \ldots\}$. If the sequence $b_1, b_2, \ldots$ ends, then B is finite. Otherwise B is denumerable.

6.10. Prove: A countable union of finite sets is countable.

Let $\mathscr{C} = \{S_i : i \in \mathbf{P}\}$ be a countable collection of finite sets, and let $C = \cup_i S_i$. If C is empty, then C is countable. Suppose $C \neq \varnothing$. Define $A_1 = S_1$, $A_2 = S_2 \backslash S_1$, $A_3 = S_3 \backslash S_2$, and so on. Then the sets A_i are finite and pairwise disjoint. Say,

$$A_1 = \{a_{11}, a_{12}, \ldots, a_{1n_1}\}, \qquad A_2 = \{a_{21}, a_{22}, \ldots a_{2n_2}\}, \ldots$$

Then the union $B = \cup_i A_i$ can be written as a sequence as follows:

$$B = \{a_{11}, a_{12}, \ldots, a_{1n_1}, a_{21}, a_{22}, \ldots, a_{2n_2}, \ldots\}$$

That is, first we write down the elements of A_1, then the elements of A_2, and so on. Formally, define $f: D \to \mathbf{P}$ as follows:

$$f(a_{ij}) = n_1 + n_2 + \cdots + n_{i-1+j}$$

Then f is bijective. Hence B is countable. However, B is also the union of the sets in $\mathscr{C}$; that is, $B = C$. Therefore, C is countable, as claimed.

6.11. Prove Theorem 6.5: Let $A_1, A_2, A_3, \ldots$ be a sequence of pairwise disjoint denumerable sets. Then the union $S = \cup_i A_i$ is denumerable.

Suppose

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}, \qquad A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}, \ \ldots$$

Define $D_n = \{a_{ij} : i + j = n, n > 1\}$. For example,

$$D_2 = \{a_{11}\}, \qquad D_3 = \{a_{12,}, a_{21}\}, \qquad D_4 = \{a_{13}, a_{22}, a_{31}\}, \ \ldots$$

Note that each D_n is finite. In fact, D_n has $n-1$ elements. By Problem 6.10, $T = \bigcup(D_j : j > 1)$ is countable. On the other hand, the union of the finite D's is the same as the union of the A's, that is, $T = S$. Thus S is countable.

6.12. Show that $\mathbf{R} \approx \mathbf{R}^+$. (The sets of positive and negative real numbers are denoted, respectively, by $\mathbf{R}^+$ and $\mathbf{R}^-$.)

The function $f(x) = x/(1 + |x|)$ is a one-to-one correspondence between $\mathbf{R}^-$ and the open interval $(-1, 0)$. Hence the function h defined by

$$h(x) = \begin{cases} \dfrac{x}{1 + |x|} + 1 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$$

is a one-to-one correspondence between $\mathbf{R}$ and $\mathbf{R}^+$. Hence $\mathbf{R} \approx \mathbf{R}^+$.

6.13. Suppose A is any uncountable set and B is a denumerable subset of A. Show that $A \backslash B \approx A$. In other words, removing a denumerable set from an uncountable set does not change its cardinality.

Suppose $B = \{b_1, b_2, b_3, \ldots\}$. The set $A \backslash B$ is infinite (indeed uncountable) and contains a denumerable subset, say, $D = \{d_1, d_2, d_3, \ldots\}$. Let $A^* = A \backslash (B \cup D)$. Then A and $A \backslash B$ are the following disjoint unions,

$$A = A^* \cup D \cup B = A^* \cup \{d_1, d_2, d_3, \ldots\} \cup \{b_1, b_2, b_3, \ldots\}$$
$$A \backslash B = A^* \cup D = A^* \cup \{d_1, d_2, d_3, \ldots\}$$

Define $f\colon A \to A \backslash B$ as in Fig. 6-6, that is,

$$\begin{array}{ll} f(a) = a & \text{if } a \in A^* \\ f(d_n) = d_{2n} - 1 & n \in \mathbf{P} \\ f(b_n) = d_{2n} & n \in \mathbf{P} \end{array}$$

Then f is one-to-one and onto; hence $A \backslash B \approx A$.

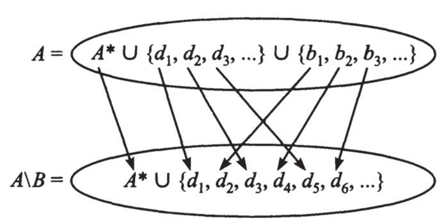


Fig. 6-6

6.14. Prove: The plane $\mathbf{R}^2$ is not the union of a countable number of lines.

Let $\mathscr{L}$ be any countable collection of lines. Since there are $\mathbf{c}$ vertical lines and $\mathscr{L}$ is countable, there is a vertical line T such that $T \notin \mathscr{L}$. Now each line in $\mathscr{L}$ can intersect T in at most one point. Thus there are only a countable number of points in T which lie on lines in $\mathscr{L}$. Hence there is a point $p \in T \subseteq \mathbf{R}^2$ which does not line on any line in $\mathscr{L}$.

6.15. Prove Theorem 6.8: The unit interval $\mathbf{I} = [0, 1]$ is not denumerable.

Method 1: Assume $\mathbf{I}$ is denumerable. Then

$$\mathbf{I} = \{x_1, x_2, x_3, \ldots\}$$

that is, the elements of $\mathbf{I}$ can be written in a sequence.

Now each element in $\mathbf{I}$ can be written in the form of an infinite decimal as follows:

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots a_{1n}\cdots \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots a_{2n}\cdots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= 0.a_{n1}a_{n2}a_{n3}\cdots a_{nn}\cdots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where $a_{ij} \in \{0, 1, \ldots, 9\}$ and where each decimal contains an infinite number of nonzero elements. Thus we write 1 as 0.999... and, for those numbers which can be written in the form of a decimal in two ways, for example,

$$1/2 = 0.5000\ldots = 0.4999\ldots$$

(in one of them there is an infinite number of nines and in the other all except a finite set of digits are zeros), we write the infinite decimal in which an infinite number of nines appear.

Now construct the real number

$$y = 0.b_1b_2b_3\cdots b_n\cdots$$

which will belong to $\mathbf{I}$, in the following way:

Choose b_1 so $b_1 \neq a_{11}$ and $b_1 \neq 0$. Choose b_2 so $b_2 \neq a_{22}$ and $b_2 \neq 0$. And so on.

Note $y \neq x_1$ since $b_1 \neq a_{11}$ (and $b_1 \neq 0$); $y \neq x_2$ since $b_2 \neq a_{22}$ (and $b_2 \neq 0$), and so on. That is, $y \neq x_n$ for all $n \in \mathbf{P}$. Thus $y \notin \mathbf{I}$, which contradicts the fact that $y \in \mathbf{I}$. Thus the assumption that $\mathbf{I}$ is denumerable has led to a contradiction. Consequently, $\mathbf{I}$ is nondenumerable.

Method 2: [This second proof of Theorem 6.8 uses Problem 6.17(*b*).]

Assume $\mathbf{I}$ is denumerable. Then, as above,

$$\mathbf{I} = \{x_1, x_2, x_3, \ldots\}$$

that is, the elements of $\mathbf{I}$ can be written in a sequence.

Now construct a sequence of closed intervals $I_1, I_2, \ldots$ as follows. Consider the following three closed subintervals of $[0, 1]$:

$$[0, 1/3], \qquad [1/3, 2/3], \qquad [2/3, 1] \tag{1}$$

where each has length $1/3$. Now x_1 cannot belong to all three intervals. (If x_1 is one of the endpoints, then it could belong to two of the intervals, but not all three.) Let $I_1 = [a_1, b_1]$, be one of the intervals in (*1*) such that $x_1 \notin I_1$. Now consider the following three closed subintervals of $I_1 = [a_1, b_1]$:

$$[a_1, a_1 + 1/9], \qquad [a_1 + 1/9,\ a_1 + 2/9], \qquad [a_1 + 2/9,\ b_1] \tag{2}$$

where each has length $1/9$. Similarly, let I_2 be one of the intervals in (*2*) with the property that x_2 does not belong to I_2. Continue in this manner. Thus we obtain a sequence of closed intervals,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \tag{3}$$

such that $x_n \notin I_n$ for all $n \in \mathbf{P}$.

By the above property of real numbers, there exists a real number $y \in \mathbf{I} = [0, 1]$ such that y belongs to every interval in (*3*). But since

$$y \in \mathbf{I} = \{x_1, x_2, x_3, \ldots\}$$

we must have $y = x_m$ for some $m \in \mathbf{P}$. By our construction $y = x_m \notin I_m$, which contradicts the fact that y belongs to every interval in (*3*). Thus our assumption that $\mathbf{I}$ is denumerable has led to a contradiction. Accordingly, $\mathbf{I}$ is nondenumerable.

6.16. Prove that $\mathbf{R}^2 \approx \mathbf{R}$ and, more generally, that $\mathbf{R}^n \approx \mathbf{R}$.

Since $\mathbf{R} \approx S = (0, 1)$, it suffices to show that the open unit square

$$S^2 = \{(x, y) : 0 < x < 1,\ 0 < y < 1) = (0, 1) \times (0, 1)$$

has the same cardinality as $S = (0, 1)$. Any point $(x, y) \in S$ can be written in the decimal form

$$(x, y) = (0.d_1 d_2 d_3, \cdots, 0.e_1 e_2 e_3 \cdots)$$

where each decimal expansion contains an infinite number of nonzero digits (e.g., for 1/2 write 0.4999... instead of 0.5000... 0. The function

$$f(x, y) = 0.d_1 e_1 d_2 e_2 d_3 e_3 \cdots$$

is one-to-one by the uniqueness of decimal expansions. Furthermore, the function $g: S \to S^2$ defined by $g(x) = (x, 1/2)$ is one-to-one. Accordingly, by the Schroeder–Bernstein Theorem 6.10, $S^2 \approx S$. Thus $\mathbf{R}^2 \approx \mathbf{R}$.

Therefore, $\mathbf{R}^3 \approx \mathbf{R}^2 \times \mathbf{R} \approx \mathbf{R} \times \mathbf{R} \approx \mathbf{R}$. Similarly, by induction, $\mathbf{R}^n \approx \mathbf{R}$.

6.17. A sequence $I_1, I_2, \ldots$ of intervals is said to be "nested" if $I_1 \supseteq I_2 \supseteq \ldots$.

(*a*) Give an example of a nested sequence of open intervals I_k whose intersection is empty.

(*b*) Prove the Nested Interval Property of the real numbers **R**: A nested sequence $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2], \ldots$ of closed intervals is not empty.

(*a*) Let $I_k = (0, 1/k)$. Then $\bigcap(I_k : k \in P) = \varnothing$. [This follows from the fact that, for any $\mathbf{c} > 0$ there exists a k such that $1/k < \mathbf{c}$.]

(*b*) Let $A = \{a_1, a_2, \ldots\}$. Since the intervals are nested, A is bounded and every b_k is an upper bound of A. By the completion property of $\mathbf{R}$, $y = \sup(A)$ exists. Thus, for every k, $a_k \le y \le b_k$. Thus y belongs to every interval, and hence $\bigcap_k I_k \neq \varnothing$.

CARDINAL NUMBERS AND THE INEQUALITY OF CARDINAL NUMBERS

6.18. Prove Cantor's Theorem 6.9: For any set A, we have $|A| < |\mathscr{P}(A)|$.

The function $g: A \to \mathscr{P}(A)$ which sends each element $a \in A$ into the set consisting of a alone, i.e., which is defined by $g(a) = \{a\}$, is one-to-one. Thus $|A| \le |\mathscr{P}(A)|$.

If we now show that $|A| \neq |\mathscr{P}(A)|$, then the theorem will follow. Suppose the contrary, that is, suppose $|A| = |\mathscr{P}(A)|$ and that $f: A \to \mathscr{P}(A)$ is one-to-one and onto. Let $a \in A$ be called a "bad" element if a is not a member of the set which is its image, i.e., if $a \notin f(a)$. Now let B be the set of "bad" elements. That is,

$$B = \{x : x \in A, x \notin f(x)\}$$

Now B is a subset of A, that is, $B \in \mathscr{P}(A)$. Since $f: A \to \mathscr{P}(A)$ is onto, there exists an element $b \in A$ such that $f(b) = B$. Is b a "bad" element or a "good" element? If $b \in B$ then, by definition of B, $b \notin f(b) = B$, which is impossible. Likewise, if $b \notin B$, then $b \in f(b) = B$, which is also impossible. Thus the original assumption, that $|A| = |\mathscr{P}(A)|$, has led to a contradiction. Hence the assumption is false, and so the theorem is true.

6.19. Prove Theorem 6.11 (which is an equivalent formulation of the Schroeder–Bernstein theorem 6.10): Let X, Y, X_1 be sets such that $X \supseteq Y \supseteq X_1$ and $X \approx X_1$. Then $X \approx Y$.

Since $X \approx X_1$, there exists a one-to-one correspondence (bijection) $f: X \to X_1$. Since $X \supseteq Y$, the restriction of f to Y, which we also denote by f, is also one-to-one. Let $f(Y) = Y_1$. Then Y and Y_1 are equipotent,

$$X \supseteq Y \supseteq X_1 \supseteq Y_1$$

and $f: Y \to Y_1$ is bijective. But now $Y \supseteq X_1 \supseteq Y_1$ and $Y \approx Y_1$. For similar reasons, X_1 and $f(X_1) = X_2$ are equipotent,

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2$$

and $f: X_1 \to X_2$ is bijective. Accordingly, there exist equipotent sets $X, X_1, X_2, \ldots$ and equipotent sets $Y, Y_1, Y_2, \ldots$ such that

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2 \supseteq Y_2 \supseteq X_3 \supseteq Y_3 \supseteq \ldots$$

and $f: X_k \to X_{k+1}$ and $f: Y_k \to Y_{k+1}$ are bijective.
Let

$$B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \cap \ldots$$

Then

$$X = (X \backslash Y) \cup (Y \backslash X_1) \cup (X_1 \backslash Y_1) \cup \cdots \cup B$$
$$Y = (Y \backslash X_1) \cup (X_1 \backslash Y_1) \cup (Y_1 \backslash X_2) \cup \cdots \cup B$$

Furthermore, $X \backslash Y, X_1 \backslash Y_1, X_2 \backslash Y_2, \ldots$ are equipotent. In fact, the function

$$f: (X_k \backslash Y_k) \to (X_{k+1} \backslash Y_{k+1})$$

is one-to-one and onto.

Consider the function $g: X \to Y$ defined by the diagram in Fig. 6-7. That is,

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_k \backslash Y_k \text{ or } x \in X \backslash Y \\ x & \text{if } x \in Y_k \backslash X_k \text{ or } x \in B \end{cases}$$

Then g is one-to-one and onto. Therefore $X \approx Y$.

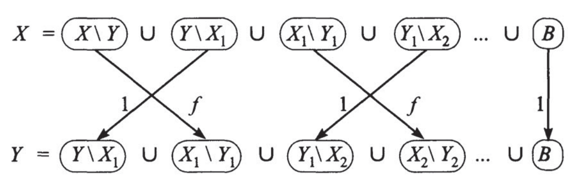


Fig. 6-7

6.20. Prove Theorem 6.13: $\mathbf{c} = 2^{\aleph_0}$.

Let $\mathbf{R}$ be the set of real numbers and let $\mathscr{P}(\mathbf{Q})$ be the power set of the set $\mathbf{Q}$ of rational numbers, i.e., the family of subsets of $\mathbf{Q}$. Furthermore, let the function $f: \mathbf{R} \to \mathscr{P}(\mathbf{Q})$ be defined by

$$f(a) = \{x : x \in \mathbf{Q}, x < a\}$$

That is, f maps each real number a into the set of rational numbers less than a. We shall show that f is one-to-one. Let $a, b \in \mathbf{R}$, $a \neq b$ and, say, $a < b$. By a property of the real numbers, there exists a rational number r such that

$$a < r < b$$

Then $r \in f(b)$ and $r \notin f(a)$; hence $f(b) \neq f(a)$. Therefore, f is one-to-one. Thus $|\mathbf{R}| \leq |\mathscr{P}(\mathbf{Q})|$. Since $|\mathbf{R}| = \mathbf{c}$ and $|\mathbf{Q}| = \aleph_0$, we have

$$\mathbf{c} \leq 2^{\aleph_0}$$

Now let $C(\mathbf{P})$ be the family of characteristic functions $f: \mathbf{P} \to \{0, 1\}$ which, as proven in Problem 6.8, is equivalent to $\mathscr{P}(\mathbf{P})$. Here $\mathbf{P} = \{1, 2, \ldots\}$. Let $\mathbf{I} = [0, 1]$, the closed unit interval, and let the function $F : C(\mathbf{P}) \to \mathbf{I}$ be defined by

$$F(f) = 0.f(1)f(2)f(3)\cdots$$

an infinite decimal consisting of zeros or ones. Suppose $f, g \in C(\mathbf{P})$ and $f \neq g$. Then the decimals would be different, and so $F(f) \neq F(g)$. Accordingly, F is one-to-one. Therefore,

$$|\mathscr{P}(\mathbf{Q})| = |C(\mathbf{P})| \leq |\mathbf{I}|$$

Since $|\mathbf{Q}| = \aleph_0$ and $|\mathbf{I}| = \mathbf{c}$, we have

$$2^{\aleph_0} \leq \mathbf{c}$$

Both inequalities give us

$$\mathbf{c} = 2^{\aleph_0}$$

6.21. Let $S = (0, 1)$, the open unit interval, and let T be the set of real numbers in S which have an infinite number of threes in their decimal expansion. Show that $|T| = |S|$.

Let $x \in S$ and suppose $x = 0.d_1d_2d_3 \cdots d_n \cdots$. Let the function $f \colon S \to T$ be defined by

$$f(x) = 0.d_1 3 d_2 3 d_3 3 \cdots 3 d_n 3 \cdots$$

Then f is one-to-one and hence $|S| \leq |T|$. Since T is a subset of S, we have $|T| \leq |S|$. By the Schroeder–Bernstein theorem, $|T| = |S|$.

6.22. Let S denote the open unit interval $(0, 1)$, and let S^ω denote the set of all denumerable sequences $(x_1, x_2, x_3, \ldots)$ where $x_i \in S$. (*a*) Prove $|S^\omega| \approx |S|$. (*b*) Prove the set $\mathbf{R}^\omega$ of all denumerable sequences of real numbers has cardinality $\mathbf{c}$.

(*a*) Let $(x_1, x_2, x_3, \ldots) \in S^\omega$. Consider the decimal expansions:

$$x_1 = 0.d_{11}d_{12}d_{13}d_{14}\cdots$$
$$x_2 = 0.d_{21}d_{22}d_{23}d_{24}\cdots$$
$$x_3 = 0.d_{31}d_{32}d_{33}d_{34}\cdots$$

And so on

Associate the sequence $(x_1, x_2, x_3, \ldots)$ with the decimal number

$$0.d_{11} : d_{21}d_{12} : d_{13}d_{22}d_{31} : \cdots$$

where the subscripts in the successive blocks of digits $d_{11}, d_{22}d_{12}, d_{13}d_{22}d_{31}, \ldots$ are obtained by "following the arrows" in Fig. 6-1. (This procedure was used to show that $\mathbf{P} \times \mathbf{P}$ is countable.) This association defines a one-to-one function from S^ω into S. The function $g : S \to S^\omega$ defined by $f(x) = (x, x, x, \ldots)$ is also one-to-one. By the Schroeder–Bernstein theorem $|S^\omega| \approx |S|$.

(*b*) Since $\mathbf{R} \approx S$, it follows that $|\mathbf{R}^\omega| = |S^\omega| = |S| = \mathbf{c}$.

CARDINAL ARITHMETIC

6.23. Let A_1, A_2, A_3, A_4 be any sets. Define sets B_1, B_2, B_3, B_4 such that

$$|A_1| + |A_2| + |A_3| + |A_4| = |B_1 \cup B_2 \cup B_3 \cup B_4|$$

Let $B_1 = A_1 \times \{1\}$, $B_2 = A_2 \times \{2\}$, $B_3 = A_3 \times \{3\}$, $B_4 = A_4 \times \{4\}$. Then $B_k \approx A_k$ for $k = 1, 2, 3, 4$. Also, the B_k are disjoint, that is, $B_i \cap B_j = \varnothing$ if $i \neq j$. Consequently, the above will be true.

6.24. Let $\{A_i : i \in I\}$ be any family of sets. Define a family of sets $\{B_i : i \in I\}$ such that $B_i \approx A_i$, for $i \in I$, and $B_i \cap B_j = \emptyset$ for $i \neq j$.

Let $B_i = A_i \times \{i\}$. Then the family $\{B_i : i \in I\}$ has the required properties.

6.25. Prove Theorem 6.14: The addition and multiplication of cardinal numbers satisfy the properties in Table 6-1. That is, for cardinal numbers α, β, γ:

(1) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

(2) $\alpha+\beta = \beta+\alpha$

(3) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

(4) $\alpha\beta = \beta\alpha$

(5) $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$

(6) If $\alpha \leq \beta$, then $\alpha+\gamma \leq \beta+\gamma$

(7) If $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$

Let A, B, C be pairwise disjoint sets such that $\alpha = |A|$, $\beta = |B|$, $\gamma = |C|$.

(1) We have:

$$(\alpha+\beta)+\gamma = |A \cup B| + |C| = |(A \cup B) \cup C|$$
$$\alpha+(\beta+\gamma) = |A| + |B \cup C| = |A \cup (B \cup C)|$$

However, the union of sets is associative, i.e., $(A \cup B) \cup C = A \cup (B \cup C)$. Hence

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$$

(2) Since $A \cup B = B \cup A$, we have

$$\alpha+\beta = |A \cup B| = |B \cup A| = \beta+\alpha$$

(3) We have:

$$(\alpha\beta)\gamma = |A \times B||C| = |(A \times B) \times C|$$
$$\alpha(\beta\gamma) = |A||B \times C| = |A \times (B \times C)|$$

However, by Problem 6.6(*b*), $(A \times B) \times C \approx A \times (B \times C)$. Hence

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

(4) By Problem 6.6(*a*), $A \times B \approx B \times A$; hence

$$\alpha\beta = |A \times B| = |B \times A| = \beta\alpha$$

(5) Note first that $B \cap C = \emptyset$ implies $(A \times B) \cap (A \times C) = \emptyset$. Then:

$$\alpha(\beta+\gamma) = |A||B \cup C| = |A \times (B \cup C)|$$
$$\alpha\beta+\alpha\gamma = |A \times B| + |A \times C| = |(A \times B) \cup (A \times C)|$$

However, $A \times (B \cup C) = (A \times B) \cup (A \times C)$. Therefore,

$$\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$$

(6) Suppose $\alpha \leq \beta$. Then there exists a one-to-one mapping $f: A \to B$. Let $g: A \cup C \to B \cup C$ be defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in C \end{cases}$$

Then g is one-to-one. Accordingly, $|A \cup C| \leq |B \cup C|$ and so

$$\alpha+\gamma \leq \beta+\gamma$$

(7) Suppose $\alpha \leq \beta$. Then there exists a one-to-one mapping $f: A \to B$. Let $g: A \times C \to B \times C$ be defined by

$$g(a,c) = (f(a), c)$$

Then g is one-to-one. Accordingly, $|A \times C| \leq |B \times C|$ and so

$$\alpha\gamma \leq \beta\gamma$$

6.26. Prove: $\aleph_0 \mathbf{c} = \mathbf{c}$.

Consider the integers $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and the half-open interval $A = [0, 1)$. Furthermore, let $f: \mathbf{Z} \times A \to \mathbf{R}$ be defined by

$$f(n, a) = n + a$$

In other words, $f(\{n\} \times [0, 1))$ is mapped onto $[n, n+1)$. Then f is a one-to-one correspondence between $\mathbf{Z} \times A$ and $\mathbf{R}$. Since $|\mathbf{Z}| = \aleph_0$ and $|A| = |\mathbf{R}| = \mathbf{c}$, we have

$$\aleph_0 \mathbf{c} = |\mathbf{Z} \times A| = |\mathbf{R}| = \mathbf{c}$$

6.27. Prove: Let α be any infinite cardinal number. Then $\aleph_0 + \alpha = \alpha$.

We have shown that $\aleph_0 + \aleph_0 = \aleph_0$. Suppose α is uncountable, and $\alpha = |A|$. By Problem 6.13, $A \backslash B \approx A$ where B is a denumerable subset of A. Recall $A = (A \backslash B) \cup B$ and the union is disjoint. Hence

$$\alpha = |A| = |(A \backslash B) \cup B| = |A \backslash B| + |B| = \alpha + \aleph_0 = \aleph_0 + \alpha$$

MISCELLANEOUS PROBLEMS

6.28. Prove: The set $\mathscr{P}$ of all polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_m x^m \tag{1}$$

with integral coefficients, that is, where $a_0, a_1, \ldots, a_m$ are integers, is denumerable.

For each pair of nonnegative integers (n, m), let $P(n, m)$ be the set of polynomials in *(1)* of degree m in which

$$|a_0| + |a_1| + \cdots + |a_m| = n$$

Note that $P(n, m)$ is finite. Therefore

$$\mathscr{P} = \bigcup (P(n, m) : (n, m) \in \mathbf{N} \times \mathbf{N})$$

is countable since it is a countable family of countable sets. But $\mathscr{P}$ is not finite; hence $\mathscr{P}$ is denumerable.

6.29. A real number r is called an *algebraic* number if r is a solution to a polynomial equation

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_m x^m = 0$$

with integral coefficients. Prove the set A of algebraic numbers is denumerable.

By the preceding Problem 6.28, that the set E of polynomial equations is denumerable:

$$E = \{p_1(x) = 0,\ p_2(x) = 0,\ p_3(x) = 0, \ldots\}$$

Define

$$A_k = \{x : x \text{ is a solution of } p_k(x) = 0\}$$

Since a polynomial of degree n can have at most n roots, each A_k is finite. Therefore

$$A = \bigcup \{A_k : k \in \mathbf{P}\}$$

is a countable family of countable sets. Accordingly, A is countable and, since A is not finite, A is denumerable.

6.30. Explicitly exhibit $\aleph_0$ pairwise-disjoint denumerable subsets of $\mathbf{P} = \{1, 2, 3, \ldots\}$.

Let p and q be distinct prime numbers. The sets

$$S_p = \{p, p^2, p^3, \ldots\} \quad \text{and} \quad S_q = \{q, q^2, q^3, \ldots\}$$

are pairwise disjoint. One can show that the set $\{p_1, p_2, p_3, \ldots\}$ of prime numbers is an infinite subset of $\mathbf{P}$ and hence has cardinality $\aleph_0$. Thus the family $\{S_{p_1}, S_{p_2}, S_{p_3}, \ldots\}$ has the desired properties.

Supplementary Problems

EQUIPOTENT SETS, COUNTABLE SETS, CONTINUUM

6.31. The set **Z** of integers can be put into a one-to-one correspondence with $\mathbf{P} = \{1, 2, 3, \ldots\}$ as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \ldots \end{array}$$

Find a formula for the function $f\colon \mathbf{P} \to \mathbf{Z}$ which gives the above correspondence between **P** and **Z**.

6.32. $\mathbf{P} \times \mathbf{P}$ was written as a sequence by considering the diagram in Fig. 6-1. This is not the only way to write $\mathbf{P} \times \mathbf{P}$ as a sequence. Write $\mathbf{P} \times \mathbf{P}$ as a sequence in two other ways by drawing appropriate diagrams.

6.33. Prove that the set S of rational points in the plane $\mathbf{R}^2$ is denumerable. [A point $p = (x, y)$ in $\mathbf{R}^2$ is rational if x and y are rational.]

6.34. Let S be the set of rational points in the plane $\mathbf{R}^2$. Show that S can be partitioned into two sets V and H such that the intersection of V with any vertical line is finite and the intersection of H with any horizontal line is finite.

6.35. Let $\mathscr{A} = \{A_i : i \in I\}$ be a set of pairwise disjoint intervals in the line **R**. Show that $\mathscr{A}$ is countable.

6.36. Let $\mathscr{B} = \{B_i : i \in I\}$ be a set of pairwise disjoint circles in the plane $\mathbf{R}^2$. Show that $\mathscr{B}$ is countable.

6.37. A function $f\colon \mathbf{P} \to \mathbf{P}$ is said to have finite support if $f(n) = 0$ for all but a finite number of n. Show that the set of all such functions is denumerable.

6.38. A real number x is called *transcendental* if x is not algebraic, i.e., if x is not a solution to a polynomial equation

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

with integral coefficients. (See Problem 6.29.) For example, π and e are transcendental numbers. Prove that the set T of transcendental numbers has the power of the continuum.

6.39. Recall that a *permutation* of $\mathbf{P} = \{1, 2, 3, \ldots\}$ is a bijective function $\sigma : \mathbf{P} \to \mathbf{P}$. Show that the set PERM(**P**) of all permutations of **P** has the power of the continuum.

CARDINAL NUMBERS, CARDINAL ARITHMETIC

6.40. Suppose α and β are cardinal numbers such that $\alpha \le \beta$. Show that there exists a set S with a subset A such that $\alpha = |A|$ and $\beta = |S|$.

6.41. Show that Theorems 6.10 and 6.11 are equivalent. (Hence each proves the Schroeder–Bernstein theorem.)

6.42. Prove $\mathbf{c}^{\aleph_0} = \mathbf{c}$.

6.43. Show that there are only **c** continuous functions from **R** into **R**. (Assume that if f and g are such continuous functions and $f(q) = g(q)$ for all rational numbers q in **R**, then $f = g$, that is, $f(x) = g(x)$ for all x in **R**.)

6.44. Prove Theorem 6.16(2): Let α, β, γ be cardinal numbers. Then $\alpha^\beta \alpha^\gamma = \alpha^{\beta+\gamma}$.

6.45. Let α, β, γ be cardinal numbers such that $\alpha \le \beta$. Prove: (*a*) $\alpha^\gamma \le \beta^\gamma$, (*b*) $\gamma^\alpha \le \gamma^\beta$.

6.46. Show that the cardinal inequality relations are well defined; that is, if $A \approx A'$ and $B \approx B'$, show that:

(*a*) $|A| \leq |B|$ if and only if $|A'| \leq |B'|$. (*b*) $|A| < |B|$ if and only if $|A'| < |B'|$.

6.47. Show that cardinal addition and multiplication are well defined, that is:

(*a*) *Cardinal Addition*: If $A \approx A'$ and $B \approx B'$, where A and B are disjoint and A' and B' are disjoint, show that $|A \cup B| = |A' \cup B'|$.

(*b*) *Cardinal Multiplication*: If $A \approx A'$ and $B \approx B'$, show that $|A \times B| = |A' \times B'|$.

6.48. Let $\mathscr{C}$ be the collection of all circles in the plane $\mathbf{R}^2$. Show that $\mathscr{C}$ has cardinality **c**.

MISCELLANEOUS PROBLEMS

6.49. (Heine–Borel Property of the real numbers **R**.) Let $\mathscr{C} = \{I_k : k \in K\}$ be a collection of open intervals which covers a closed interval $A = [a, b]$. Show that $\mathscr{C}$ contains a finite subcover of A, that is, a finite subcollection of $\mathscr{C}$ is a cover of A. [A collection $\{I_k : k \in K\}$ of intervals is called a "cover" of a set A if $A \subseteq \bigcup_k I_k$.]

Answers to Supplementary Problems

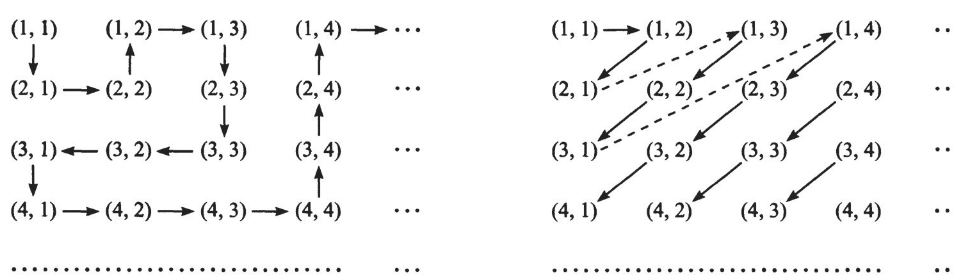
6.31. The following function f: $\mathbf{P} \to \mathbf{P}$ has the required property:

$$f(n) = \begin{cases} -n/2 + 1/2 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

6.32. Each diagram in Fig. 6-8 shows that $\mathbf{P} \times \mathbf{P}$ can be written as an infinite sequence of distinct elements as follows:

(*a*) $\mathbf{P} \times \mathbf{P} = \{(1,1),\ (2,1),\ (2,2),\ (1,2),\ (1,3),\ (2,3), \ldots\}$

(*b*) $\mathbf{P} \times \mathbf{P} = \{(1,1),\ (1,2),\ (2,1),\ (1,3),\ (2,2),\ (3,1),\ (1,4), \ldots\}$



(*a*) (*b*)

Fig. 6-8

6.33. $|\mathbf{Q} \times \mathbf{Q}| = |\mathbf{P} \times \mathbf{P}| = |\mathbf{P}| = \aleph_0$

6.35. *Hint*: Each interval contains a distinct rational number.

6.36. *Hint*: Each circle contains a distinct rational point in $\mathbf{R}^2$.

6.38. *Hint*: **R** is the union of the algebraic and transcendental numbers.

6.42. *Hint*: Use Problem 6.22

6.43. *Hint*: Use Problem 6.22 or 6.42.

6.44. *Hint*: Let $\alpha = |A|, \beta = |B|, \gamma = |C|$ where B and C are disjoint. Let $D = B \cup C$. Then $\beta + \gamma = |B \cup C| = |D|$. Associate with each function $f\colon D \to A$ the pair $f_1\colon B \to A$ and $f_2\colon C \to A$ where $f_1 = f|_B$ and $f_2 = f|_C$. Show that the map $F(f) = (f_1, f_2)$ is bijective.

6.45. *Hint*: Let $\alpha = |A|, \beta = |B|, \gamma = |C|$ where we can assume $A \subseteq B$ since $\alpha \leq \beta$.

(*a*) For each function $f\colon C \to A$ associate the function $f'\colon C \to B$ defined by $f'(x) = f(x)$. Show that the map $F(f) = g$ is one-to-one.

(*b*) For each function $f\colon A \to C$ associate a function $f'\colon B \to C$ which extends f, i.e., for each $a \in A$, $f'(a) = f(a)$. Show that the map $F(f) = f'$ is one-to-one.

6.48. Since each circle in $\mathscr{C}$ is determined by its center (x, y) and radius r, $\mathscr{C} \approx \mathbf{R} \times \mathbf{R} \times \mathbf{R}^+ \approx \mathbf{R}$.

6.49. Suppose no finite subcollection of $\mathscr{C}$ is a cover of A. Let p_1 be the midpoint of the interval $A = A_1 = [a_1, b_1]$. At least one of $[a_1, p_1]$ and $[p_1, b_1]$ cannot be covered by a finite subcollection of $\mathscr{C}$ or else the whole interval A_1 will be, and let $A_2 = [a_2, b_2]$ be that subinterval. Similarly, let p_2 be the midpoint of the interval $A_2 = [a_2, b_2]$, and let $A_3 = [a_3, b_3]$ be one of the two intervals $[a_2, p_2]$ and $[p_2, b_2]$ which cannot be covered by a finite subcollection of $\mathscr{C}$, and so on. Thus we have a sequence $A_1, A_2, \ldots$ of nested closed intervals, and each cannot be covered by a finite subcollection of $\mathscr{C}$. Furthermore, $\lim_{n\to\infty} d_n = 0$ where $d_n = b_n - a_n$ is the length of A_n. By Problem 6.17(*b*), there exists a real number y in every A_k. Since $\mathscr{C}$ is a cover of A, y belongs to some element of $\mathscr{C}$, say $y \in I_j$ where $I_j = (c, d)$. Let e be the distance from y to the closest endpoint of I_j. Then there exists d_j such that $d_j < e$. This means $A_j \subseteq I_j$. This contradicts the fact that A_j cannot be covered by a finite subcollection of $\mathscr{C}$. Thus the original assumption that no finite subcollection of $\mathscr{C}$ covers A leads to a contradiction, and so a finite subcollection of $\mathscr{C}$ covers A.