Chapter 4

Functions

4.1 INTRODUCTION

One of the most important concepts in mathematics is that of a function. The terms "map", "mapping", "transformation", and many others mean the same thing; the choice of which word to use in a given situation is usually determined by tradition and the mathematical background of the person using the term.

4.2 FUNCTIONS

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a *function* from A into B. The set A is called the *domain* of the function, and the set B is called the *target set*.

Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f\colon A \to B$$

which is read: "f is a function from A into B", or "f takes A into B", or "f maps A into B".

Suppose $f\colon A \to B$ and $a \in A$. Then $f(a)$ [read: "f of a"] will denote the unique element of B which f assigns to a. This element $f(a)$ in B is called the *image* of a under f or the *value* of f at a. We also say that f *sends* or *maps* a into $f(a)$. The set of all such image values is called the *range* or *image* of f, and it is denoted by $\text{Ran}(f)$, $\text{Im}(f)$ or $f(A)$. That is,

$$\text{Im}(f) = \{b \in B : \text{there exists } a \in A \text{ for which } f(a) = b\}$$

We emphasize that $\text{Im}(f)$ is a subset of the target set B.

Frequently, a function can be expressed by means of a mathematical formula. For example, consider the function which sends each real number into its square. We may describe this function by writing

$$f(x) = x^2 \qquad \text{or} \qquad x \mapsto x^2 \qquad \text{or} \qquad y = x^2$$

In the first notation, x is called a *variable* and the letter f denotes the function. In the second notation, the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value of x.

Furthermore, suppose a function is given by a formula in terms of a variable x. Then we assume, unless otherwise stated, that the domain of the function is $\mathbf{R}$ or the largest subset of $\mathbf{R}$ for which the formula has meaning, and that the target set is $\mathbf{R}$.

Remark: Suppose $f\colon A \to B$. If A' is a subset of A, then $f(A')$ denotes the set of images of elements in A'; and if B' is a subset of B, then $f^{-1}(B')$ denotes the set of elements of A each whose image belongs to B'. That is,

$$f(A') = \{f(a) : a \in A'\} \qquad \text{and} \qquad f^{-1}(B') = \{a \in A : f(a) \in B'\}$$

We call $f(A')$ the *image* of A', and we call $f^{-1}(B')$ the *inverse image* or *preimage* of B'.

EXAMPLE 4.1

(*a*) Consider the function $f(x) = x^3$, i.e., f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$. Similarly, $f(-3) = -27$, and $f(0) = 0$.

(*b*) Let g assign to each country in the world its capital city. Here the domain of g is the set of all the countries in the world, and the target set is the list of cities in the world. The image of France under g is Paris; that is $g(\text{France}) = \text{Paris}$. Similarly, $g(\text{Denmark}) = \text{Copenhagen}$ and $g(\text{England}) = \text{London}$.

(*c*) Figure 4-1 defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way; that is,

$$f(a) = s, \qquad f(b) = u, \qquad f(c) = r, \qquad f(d) = s$$

The image of f is the set $\{r, s, u\}$. Note that t does not belong to the image of f because t is not the image of any element of A under f.

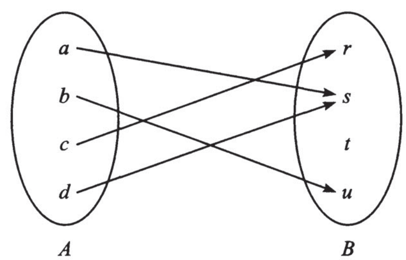


Fig. 4-1

Identity Function

Consider any set A. Then there is a function from A into A which sends each element into itself. It is called the *identity function* on A and it is usually denoted by 1_A or simply 1. In other words, the identity function $1_A : A \to A$ is defined by

$$1_A(a) = a$$

for every element $a \in A$.

Functions as Relations

There is another point of view from which functions may be considered. First of all, every function $f: A \to B$ gvies rise to a relation from A to B called the *graph* of f and defined by

$$\text{Graph of } f = \{(a, b) : a \in A, b = f(a)\}$$

Two functions $f: A \to B$ and $g: A \to B$ are defined to be equal, written $f = g$, if $f(a) = g(a)$ for every $a \in A$; that is, if they have the same graph. Accordingly, we do not distinguish between a function and its graph. Now, such a graph relation has the property that each a in A belongs to a unique ordered pair (a, b) in the relation. On the other hand, any relation f from A to B that has this property gives rise to a function $f: A \to B$, where $f(a) = b$ for each (a, b) in f. Consequently, one may equivalently define a function as follows:

Definition: A function $f: A \to B$ is a relation from A to B (i.e., a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Although we do not distinguish between a function and its graph, we will still use the terminology "graph of f" when referring to f as a set of ordered pairs. Moreover, since the graph of f is a relation, we can draw its picture as was done for relations in general, and this pictorial representation is itself sometimes called the graph of f. Also, the defining condition of a function, that each $a \in A$ belongs to a unique pair (a, b) in f, is equivalent to the geometrical condition of each vertical line intersecting the graph in exactly one point.

EXAMPLE 4.2

(*a*) Let $f: A \to B$ be the function in Example 4.1(*c*). Then the graph of f is the following set of ordered pairs:

$$f = \{(a,s),\ (b,u),\ (c,r),\ (d,s)\}$$

(*b*) Consider the following relations on the set $A = \{(1,2,3)\}$

$$f = \{(1,3),\ (2,3),\ (3,1)\}, \qquad g = \{(1,2),\ (3,1)\}, \qquad h = \{(1,3),\ (2,1),\ (1,2),\ (3,1)\}$$

f is a function from A into A since each member of A appears as the first coordinate in exactly one ordered pair in f; here $f(1) = 3$, $f(2) = 3$ and $f(3) = 1$. g is not a function from A into A since $2 \in A$ is not the first coordinate of any pair in g and so g does not assign any image to 2. Also h is not a function from A into A since $1 \in A$ appears as the first coordinate of two distinct ordered pairs in h, $(1,3)$ and $(1,2)$. If h is to be a function it cannot assign both 3 and 2 to the element $1 \in A$.

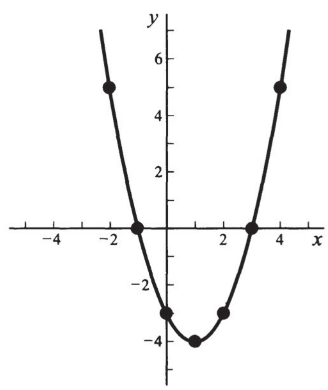
(*c*) By a *real polynomial function*, we mean a function $f : \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where the a_i are real numbers. Since $\mathbf{R}$ is an infinite set, it would be impossible to plot each point of the graph. However, the graph of such a function can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The points are usually obtained from a table where various values are assigned to x and the corresponding values of $f(x)$ computed.

Figure 4-2 illustrates this technique using the function $f(x) = x^2 - 2x - 3$.

x	$f(x)$
−2	5
−1	0
0	−3
1	−4
2	−3
3	0
4	5



Graph of $f(x) = x^2 - 2x - 3$

Fig. 4-2

4.3 COMPOSITION OF FUNCTIONS

Consider functions $f: A \to B$ and $g: B \to C$, that is, where the target set B of f is the domain of g. This relationship can be pictured by the following diagram:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Let $a \in A$; then its image $f(a)$ under f is in B which is the domain of g. Accordingly, we can find the image of $f(a)$ under the function g, that is, we can find $g(f(a))$. Thus we have a rule which assigns to each element a in A an element $g(f(a))$ in C or, in other words, f and g give rise to a well defined function from A to C. This new function is called the *composition* of f and g, and it is denoted by

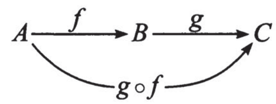
$$g \circ f$$

More briefly, if $f\colon A \to B$ and $g\colon B \to C$, then we define a new function $g \circ f\colon A \to C$ by

$$(g \circ f)(a) \equiv g(f(a))$$

Here $\equiv$ is used to mean equal by definition.

Note that we can now add the function $g \circ f$ to the above diagram of f and g as follows:



We emphasize that the composition of f and g is written $g \circ f$, and not $f \circ g$; that is, the composition of functions is read from right to left, and not from left to right.

EXAMPLE 4.3

(*a*) Let $f\colon A \to B$ and $g\colon B \to C$ be the functions defined by Fig. 4-3. We compute $g \circ f : A \to C$ by its definition:

$$(g \circ f)(a) \equiv g(f(a)) = g(y) = t, \quad (g \circ f)(b) \equiv g(f(b)) = g(z) = r, \quad (g \circ f)(c) \equiv g(f(c)) = g(y) = t$$

Observe that the composition $g \circ f$ is equivalent to "following the arrows" from A to C in the diagrams of the functions f and g.

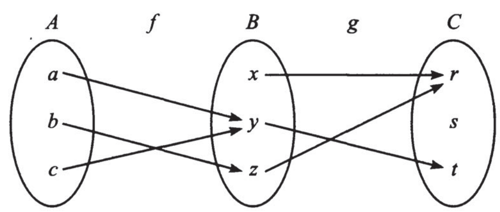


Fig. 4-3

(*b*) Let $f\colon \mathbf{R} \to \mathbf{R}$ and $g\colon \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$ and $g(x) = x + 3$. Then

$$(g \circ f)(2) \equiv g(f(2)) = g(4) = 7; \quad (f \circ g)(2) \equiv f(g(2)) = f(5) = 25$$

Thus the composition functions $g \circ f$ and $f \circ g$ are not the same function. We compute a general formula for these functions:

$$(g \circ f)(x) \equiv g(f(x)) = g(x^2) = x^2 + 3$$
$$(f \circ g)(x) \equiv f(g(x)) = f(x + 3) = (x + 3)^2 = x^2 + 6x + 9$$

(*c*) Consider any function $f : A \to B$. Then one can easily show that

$$f \circ 1_A = f \qquad \text{and} \qquad 1_B \circ f = f$$

where 1_A and 1_B are the identity functions on A and B, respectively. In other words, the composition of any function with the appropriate identity function is the function itself.

Associativity of Composition of Functions

Consider functions $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$. Then, as pictured in Fig. 4-4(a), we can form the composition $g \circ f\colon A \to C$, and then the composition $h \circ (g \circ f)\colon A \to D$. Similarly, as pictured in Fig. 4-4(b), we can form the composition $h \circ g\colon B \to D$, and then the composition

$(h \circ g) \circ f$: $A \to D$. Both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are functions with domain A and target set D. The next theorem on functions (proved in Problem 4.15) states that these two functions are equal. That is:

Theorem 4.1: Let f: $A \to B$, g: $B \to C$, and h: $C \to D$. Then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Theorem 4.1 tells us that we can write $h \circ g \circ f$: $A \to D$ without any parentheses.

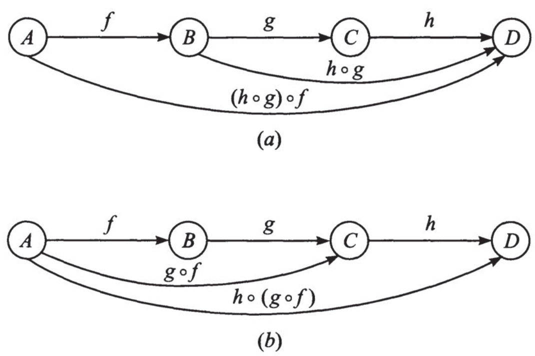


Fig. 4-4

Remark: The above definition of the composition of functions and Theorem 4.1 are not really new. Specifically, viewing the functions f and g as relations, then the composition function $g \circ f$ is the same as the composition of f and g as relations (Section 3.5) and Theorem 4.1 is the same as Theorem 3.1. One main difference is that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for relations.

4.4 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

A function f: $A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing follows:

f is one-to-one if $f(a) = f(a')$ implies $a = a'$

A function f: $A \to B$ is said to be an *onto* function if every element of B is the image of some element in A or, in other words, if the image of f is the entire target set B. In such a case we say that f is a function of A onto B or that f maps A onto B. That is:

f maps A onto B if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$

Here

$\forall$ means "for every", and $\exists$ means "there exist"

(These quantifiers are discussed in Chapter 10.)

A function $f: A \to B$ is said to be *invertible* if its inverse relation f^{-1} is a function from B to A. Equivalently, $f: A \to B$ is *invertible* if there exists a function $f^{-1}: B \to A$, called the *inverse* of f, such that

$$f^{-1} \circ f = 1_A \quad \text{and} \quad f \circ f^{-1} = 1_B$$

In general, an inverse function f^{-1} need not exist or, equivalently, the inverse relation f^{-1} may not be a function. The following theorem (proved in Problem 4.23) gives simple criteria which tell us when it is.

Theorem 4.2: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

If $f: A \to B$ is both one-to-one and onto, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will correspond to a unique element of B and vice versa.

Some texts use the term *injective* for a one-to-one function, *surjective* for an onto function, and *bijective* for a one-to-one correspondence.

EXAMPLE 4.4 Consider functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$, and $f_4: D \to E$ defined by Fig. 4-5. Now f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$.

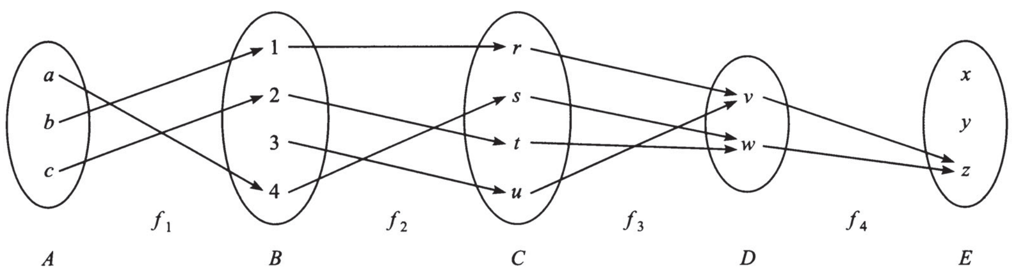


Fig. 4-5

As far as being onto is concerned, f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, i.e., $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto since $3 \in B$ but 3 is not the image under f_1 of any element of A, and f_4 is not onto since, for example, $x \in E$ but x is not the image under f_4 of any element of D.

Thus f_1 is one-to-one but not onto, f_3 is onto but not one-to-one, and f_4 is neither one-to-one nor onto. However, f_2 is both one-to-one and onto, i.e., f_2 is a one-to-one correspondence between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

Geometrical Characterization of One-to-One and Onto Functions

Consider now a real-valued function $f: \mathbf{R} \to \mathbf{R}$. Since f may be identified with its graph and the graph may be plotted in the cartesian plane $\mathbf{R}^2$, we might wonder whether the concepts of being one-to-one and onto have some geometrical meaning. The answer is yes. Specifically:

(*a*) The function $f: \mathbf{R} \to \mathbf{R}$ is one-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in the graph of f; hence each vertical line in $\mathbf{R}^2$ can intersect the graph of f in at most one point.

(*b*) The function $f: \mathbf{R} \to \mathbf{R}$ is onto means that for every $b \in \mathbf{R}$ there is at least one point $a \in \mathbf{R}$ such that (a, b) belongs to the graph of f; hence each vertical line in $\mathbf{R}^2$ must intersect the graph of f at least once.

(*c*) Accordingly, the function $f: \mathbf{R} \to \mathbf{R}$ is one-to-one and onto, i.e., f is invertible, if and only if each horizontal line in $\mathbf{R}^2$ will intersect the graph of f in exactly one point.

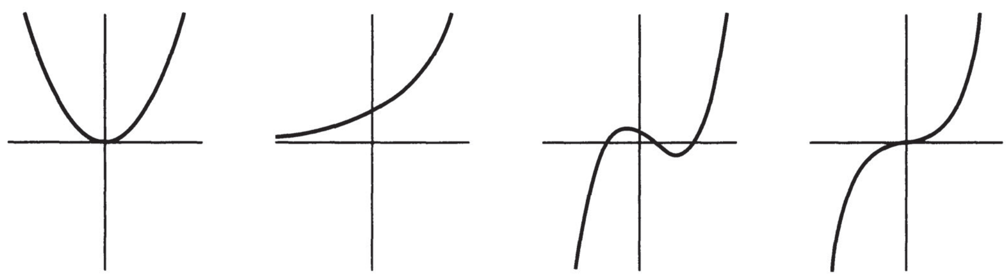
We illustrate the above properties in the next example.

EXAMPLE 4.5 Consider the following four functions from **R** into **R** whose graphs appear in Fig. 4-6:

$$f_1(x) = x^2, \qquad f_2(x) = 2^x, \qquad f_3(x) = x^3 - 2x^2 - 5x + 6, \qquad f_4(x) = x^3$$

Observe that there are horizontal lines which intersect the graph of f_1 twice and there are horizontal lines which do not intersect the graph of f_1 at all; hence f_1 is neither one-to-one nor onto. Similarly, f_2 is one-to-one but not onto, f_3 is onto but not one-to-one, and f_4 is both one-to-one and onto. The inverse of f_4 is the cube root function, that is,

$$f_4^{-1}(x) = \sqrt[3]{x}$$



$f_1(x) = x^2$ $\qquad$ $f_2(x) = 2^x$ $\qquad$ $f_3(x) = x^3 - 2x^2 - 5x + 6$ $\qquad$ $f_4(x) = x^3$

Fig. 4-6

Remark: Sometimes we restrict the domain and/or target set of a function f in order to obtain an inverse function f^{-1}. For example, suppose we restrict the domain and target set of the function $f_1(x) = x^2$ to be the set D of nonnegative real numbers. Then f_1 is one-to-one and onto and its inverse is the square root function, that is,

$$f_1^{-1}(x) = \sqrt{x}$$

Similarly, suppose we restrict the target set of the exponential function $f_2(x) = 2^x$ to be the set $\mathbf{R}^+$ of positive real numbers. Then f_1 is one-to-one and onto and its inverse is the logarithmic function (to the base 2), that is,

$$f_2^{-1}(x) = \log_2 x$$

(Exponential and logarithmic functions are investigated in Section 4.5.)

4.5 MATHEMATICAL FUNCTIONS, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This section presents various mathematical functions which appear often in mathematics and computer science, together with their notation. We also discuss the exponential and logarithmic functions, and their relationship.

Integer and Absolute Value Functions

Let x be any real number. The *integer value* of x, written $\mathrm{INT}(x)$, converts x into an integer by deleting (truncating) the fractional part of the number. Thus

$$\mathrm{INT}(3.14) = 3, \qquad \mathrm{INT}(\sqrt{5}) = 2, \qquad \mathrm{INT}(-8.5) = -8, \qquad \mathrm{INT}(7) = 7$$

The *absolute value* of the real number x, written $\mathrm{ABS}(x)$ or $|x|$, is defined as the greater of x or $-x$. Hence $\mathrm{ABS}(0) = 0$, and, for $x \neq 0$, $\mathrm{ABS}(x) = x$ or $\mathrm{ABS}(x) = -x$, depending on whether x is positive or negative. Thus

$$|-15| = 15, \qquad |7| = 7, \qquad |-3.33| = 3.33, \qquad |4.44| = 4.44, \qquad |-0.975| = 0.075$$

We note that $|x| = |-x|$ and, for $x \neq 0$, $|x|$ is positive.

Remainder Function; Modular Arithmetic

Let k be any integer and let M be a positive integer. Then

$$k \pmod{M}$$

(read *k modulo M*) will denote the integer remainder when k is divided by M. More exactly, $k \pmod{M}$ is the unique integer r such that

$$k = Mq + r \qquad \text{where} \qquad 0 \le r < M$$

When k is positive, simply divide k by M to obtain the remainder r. Thus

$$25 \pmod{7} = 4, \qquad 25 \pmod{5} = 0, \qquad 35 \pmod{11} = 2, \qquad 3 \pmod{8} = 3$$

Problem 4.25 shows a method to obtain $k \pmod{M}$ when k is negative.

The term "mod" is also used for the mathematical congruence relation, which is denoted and defined as follows:

$$a \equiv b \pmod{M} \qquad \text{if and only if} \qquad M \text{ divides } b - a$$

M is called the *modulus*, and $a \equiv b \pmod{M}$ is read "a is congruent to b modulo M". The following aspects of the congruence relation are frequently useful:

$$0 \equiv M \pmod{M} \qquad \text{and} \qquad a \pm M \equiv a \pmod{M}$$

Arithmetic modulo M refers to the arithmetic operations of addition, multiplication, and subtraction where the arithmetic value is replaced by its equivalent value in the set

$$\{0, 1, 2, \ldots, M - 1\}$$

or in the set

$$\{1, 2, 3, \ldots, M\}$$

For example, in arithmetic modulo 12, sometimes called "clock" arithmetic,

$$6 + 9 \equiv 3, \qquad 7 \times 5 \equiv 11, \qquad 1 - 5 \equiv 8, \qquad 2 + 10 \equiv 0 \equiv 12$$

(The use of 0 or M depends on the application.)

Exponential Functions

Recall the following definitions for integer exponents (where m is a positive integer):

$$a^m = a \cdot a \ldots a \,(m \text{ times}), \qquad a^0 = 1, \qquad a^{-m} = \frac{1}{a^m}$$

Exponents are extended to include all rational numbers by defining, for any rational number m/n,

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

For example,

$$2^4 = 16, \qquad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \qquad 125^{2/3} = 5^2 = 25$$

In fact, exponents are extended to include all real numbers by defining, for any real number x,

$$a^x = \lim_{r \to x} a^r \qquad \text{where } r \text{ is a rational number}$$

Accordingly, the exponential function $f(x) = a^x$ is defined for all real numbers.

Logarithmic Functions

Logarithms are related to exponents as follows. Let b be a positive number. The logarithm of any positive number x to the base b, written

$$\log_b x$$

represents the exponent to which b must be raised to obtain x. That is,

$$y = \log_b x \quad \text{and} \quad b^y = x$$

are equivalent statements. Accordingly,

$$\log_2 8 = 3 \quad \text{since} \quad 2^3 = 8; \qquad \log_{10} 100 = 2 \quad \text{since} \quad 10^2 = 100$$
$$\log_2 64 = 6 \quad \text{since} \quad 2^6 = 64; \qquad \log_{10} 0.001 = -3 \quad \text{since} \quad 10^{-3} = 0.001$$

Furthermore, for any base b,

$$\log_b 1 = 0 \quad \text{since} \quad b^0 = 1$$
$$\log_b b = 1 \quad \text{since} \quad b^1 = b$$

The logarithm of a negative number and the logarithm of 0 are not defined.

Frequently, logarithms are expressed using approximate values. For example, using tables or calculators, one obtains

$$\log_{10} 300 = 2.4771 \quad \text{and} \quad \log_e 40 = 3.6889$$

as approximate answers. (Here $e = 2.718\,281\cdots$.)

Three classes of logarithms are of special importance: logarithms to base 10, called *common logarithms*; logarithms to base e, called *natural logarithms*; and logarithms to base 2, called *binary logarithms*. Some texts write:

$$\ln x \text{ for } \log_e x \quad \text{and} \quad \lg x \text{ or } \log x \text{ for } \log_2 x$$

The term $\log x$, by itself, usually means $\log_{10} x$; but it is also used for $\log_e x$ in advanced mathematical texts and for $\log_2 x$ in computer science texts.

Relationship between the Exponential and Logarithmic Functions

The basic relationship between the exponential and the logarithmic functions

$$f(x) = b^x \quad \text{and} \quad g(x) = \log_b x$$

is that they are inverses of each other; hence the graphs of these functions are related geometrically. This relationship is illustrated in Fig. 4-7 where the graphs of the exponential function $f(x) = 2^x$, the logarithmic function $g(x) = \log_2 x$, and the linear function $h(x) = x$ appear on the same coordinate axis. Since $f(x) = 2^x$ and $g(x) = \log_2 x$ are inverse functions, they are symmetric with respect to the linear function $h(x) = x$ or, in other words, the line $y = x$.

Figure 4-7 also indicates another important property of the exponential and logarithmic functions. Specifically, for any positive c, we have

$$g(c) < h(c) < f(c)$$

In fact, as c increases in value, the vertical distances $h(c) - g(c)$ and $f(c) - g(c)$ increase in value. Moreover, the logarithmic function $g(x)$ grows very slowly compared with the linear function $h(x)$, and the exponential function $f(x)$ grows very quickly compared with $h(x)$.

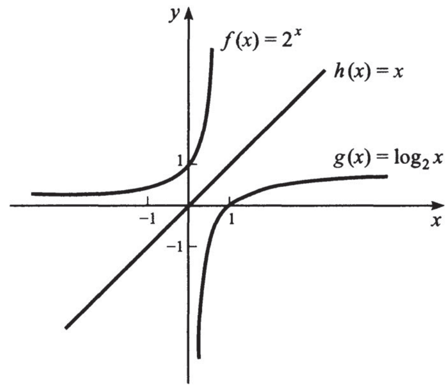


Fig. 4-7

4.6 RECURSIVELY DEFINED FUNCTIONS

A function is said to be *recursively defined* if the function definition refers to itself. In order for the definition not to be circular, the function definition must have the following two properties:

(1) There must be certain arguments, called *base values*, for which the function does not refer to itself.
(2) Each time the function does refer to itself, the argument of the function must be closer to a base value.

A recursive function with these two properties is said to be *well-defined.*

The following examples should help clarify these ideas.

Factorial Function

The product of the positive integers from 1 to n, inclusive, is called "n factorial" and is usually denoted by $n!$:

$$n! = 1 \cdot 2 \cdot 3 \ldots (n-2)(n-1)n$$

It is also convenient to define $0! = 1$, so that the function is defined for all nonnegative integers. Thus we have

$$0! = 1, \quad 1! = 1, \quad 2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6, \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24,$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120, \quad 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

and so on. Observe that

$$5! = 5 \cdot 4! = 5 \cdot 24 = 120 \quad \text{and} \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720$$

This is true for every positive integer n; that is,

$$n! = n \cdot (-1)!$$

Accordingly, the factorial function may also be defined as follows:

Definition 4.1: (Factorial Function)
(*a*) If $n = 0$, then $n! = 1$.
(*b*) If $n > 0$, then $n! = n \cdot (n-1)!$

Observe that the above definition of $n!$ is recursive, since it refers to itself when it uses $(n-1)!$ However:

(1) The value of $n!$ is explicitly given when $n = 0$ (thus 0 is a base value).
(2) The value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0.

Accordingly, the definition is not circular, or, in other words, the function is well-defined.

Fibonacci Sequence

The celebrated Fibonacci sequence (usually denoted by $F_0, F_1, F_2, \ldots$) is as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

That is, $F_0 = 0$ and $F_1 = 1$ and each succeeding term is the sum of the two preceding terms. For example, the next two terms of the sequence are

$$34 + 55 = 89 \qquad \text{and} \qquad 55 + 89 = 144$$

A formal definition of this function follows:

Definition 4.2: (Fibonacci Sequence)
(*a*) If $n = 0$ or $n = 1$, then $F_n = n$.
(*b*) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

This is another example of a recursive definition, since the definition refers to itself when it uses F_{n-2} and F_{n-1}. However:

(1) The base values are 0 and 1.
(2) The value of F_n is defined in terms of smaller values of n which are closer to the base values.

Accordingly, this function is well-defined.

Solved Problems

FUNCTIONS

4.1. State whether or not each diagram in Fig. 4-8 defines a function from $A = \{a, b, c\}$ into $B = \{x, y, z\}$.

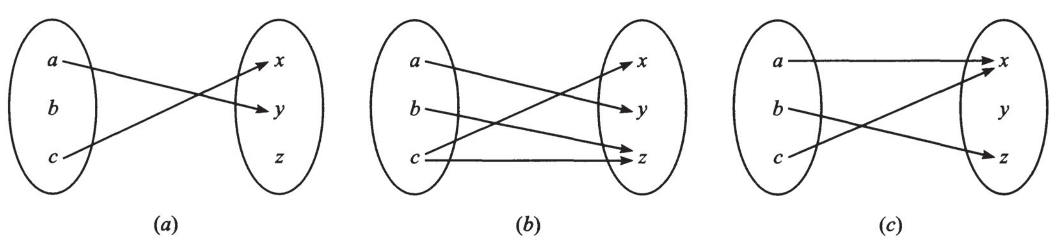


Fig. 4-8

(*a*) No. There is nothing assigned to the element $b \in A$.
(*b*) No. Two elements, x and z, are assigned to $c \in A$.
(*c*) Yes. Every element in the domain $A = \{a, b, c\}$ is assigned a unique element in the target set B.

4.2. Let $X = \{1, 2, 3, 4\}$. Determine whether or not each relation below is a function from X into X.

(*a*) $f = \{(2,3),\ (1,4),\ (2,1),\ (3,2),\ (4,4)\}$
(*b*) $g = \{(3,1),\ (4,2),\ (1,1)\}$
(*c*) $h = \{(2,1),\ (3,4),\ (1,4),\ (2,1),\ (4,4)\}$

Recall that a subset f of $X \times X$ is a function $f: X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.

(*a*) No. Two different ordered pairs $(2,3)$ and $(2,1)$ in f have the same number 2 as their first coordinate.
(*b*) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.
(*c*) Yes. Although $2 \in X$ appears as the first coordinate in two ordered pairs in h, these two ordered pairs are equal.

4.3. Let A be the set of students in a school. Determine which of the following assignments defines a function on A:

(*a*) To each student assign his age.
(*b*) To each student assign his teacher.
(*c*) To each student assign his sex.
(*d*) To each student assign his spouse.

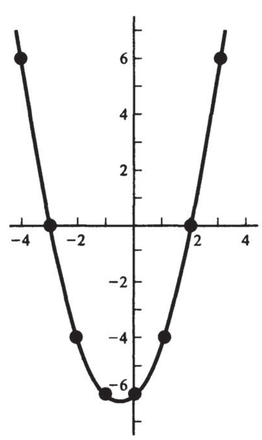
A collection of assignments is a function on A if and only if each element a in A is assigned exactly one element. Thus:

(*a*) Yes, because each student has one and only one age.
(*b*) Yes, if each student has only one teacher; no, if any student has more than one teacher.
(*c*) Yes.
(*d*) No, unless every student is married.

4.4. Sketch the graph of: (*a*) $f(x) = x^2 + x - 6$; (*b*) $g(x) = x^3 - 3x^2 - x + 3$.

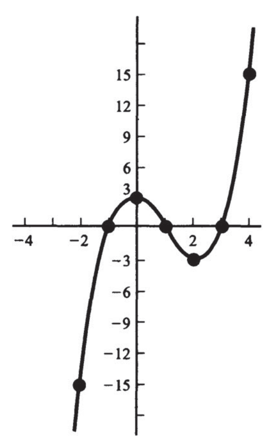
Set up a table of values for x and then find the corresponding values of the function. Since the functions are polynomials, plot the points in a coordinate diagram and then draw a smooth continuous curve through the points. See Fig. 4-9.

x	$f(x)$
−4	6
−3	0
−2	−4
−1	−6
0	−6
1	−4
2	0
3	6



Graph of f

x	$g(x)$
−2	−15
−1	0
0	3
1	0
2	−3
3	0
4	15



Graph of g

Fig. 4-9

4.5. Determine which of the graphs in Fig. 4-10 are functions from **R** into **R**.

Geometrically speaking, a set of points in the plane $\mathbf{R}^2$ is a function if and only if every vertical line contains exactly one point of the set. Thus: (*a*) Yes. (*b*) No. (*c*) No; however the graph does define a function from D into **R** where $D = [-2, 2] = \{x : -2 \le x \le 2\}$.

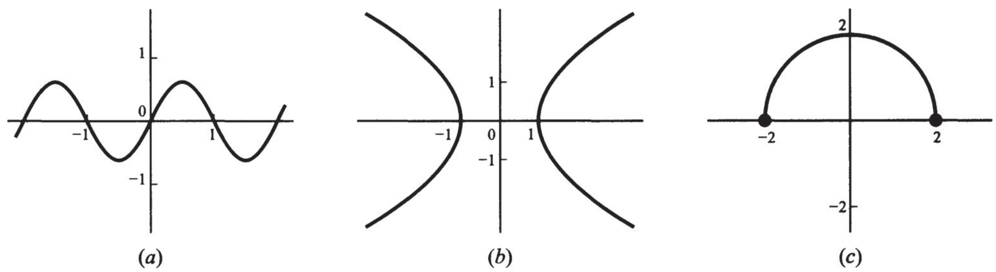


Fig. 4-10

4.6. Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined as follows:

$$f(x) = \begin{cases} 3x - 1 & \text{if } x > 3 \\ x^2 - 2 & \text{if } -2 \le x \le 3 \\ 2x + 3 & \text{if } x < -2 \end{cases}$$

Find: (*a*) $f(2)$, (*b*) $f(4)$, (*c*) $f(-1)$, (*d*) $f(-3)$

Note that there are three formulas used to define the single function f. (The reader should not confuse formulas and functions.)

(*a*) Since 2 belongs to the closed interval $[-2, 3]$, we use the formula $f(x) = x^2 - 2$. Hence

$$f(2) = 2^2 - 2 = 4 - 2 = 2$$

(*b*) Since 4 belongs to $(3, \infty)$, we use the formula $f(x) = 3x - 1$. Thus $f(4) = 3(4) - 1 = 12 - 1 = 11$.

(*c*) Since -1 is in the interval $[-2, 3]$, we use the formula $f(x) = x^2 - 2$. Computing,

$$f(-1) = (-1)^2 - 2 = 1 - 2 = -1$$

(*d*) Since -3 is less than -2, i.e., -3 belongs to $(-\infty, -2)$, we use the formula $f(x) = 2x + 3$. Thus

$$f(-3) = 2(-3) + 3 = -6 + 3 = -3$$

4.7. Find the domain D of each of the following real-valued functions:

(*a*) $f(x) = 1/(x - 2)$; (*b*) $g(x) = x^2 - 3x - 4$; (*c*) $h(x) = \sqrt{25 - x^2}$.

(*a*) f is not defined for $x - 2 = 0$ or $x = 2$; hence $D = \mathbf{R}\backslash\{2\}$.

(*b*) g is defined for every real number; hence $D = \mathbf{R}$.

(*c*) h is not defined when $25 - x^2$ is negative; hence $D = [-5, 5] = \{x : -5 \le x \le 5\}$.

4.8. Let $A = \{1, 2, 3, 4, 5\}$ and let $f : A \to A$ be defined by the diagram in Fig. 4-11.

(*a*) Find the graph of f, i.e., write f as a set of ordered pairs.

(*b*) Find $f(A)$, the image of f.

(*c*) Find $f(S)$ where $S = \{1, 3, 5\}$.

(*d*) Find $f^{-1}(T)$ where $T = \{2, 3\}$.

(*a*) The graph of f consists of all pairs $(a, f(a))$ where $a \in A$. Hence

$$f = \{(1,3),\ (2,5),\ (3,5),\ (4,2),\ (5,3)\}$$

(*b*) $f(A)$ consists of all image points. Since only 2, 3, 5 appear as image points, $f(A) = \{2,3,5\}$.

(*c*) $f(S) = f(\{1,3,5\}) = \{f(1), f(3), f(5)\} = \{3,5,3\} = \{3,5\}$.

(*d*) The element 4 has image 2, and the elements 1 and 5 have image 3; hence $f^{-1}(T) = f^{-1}(\{2,3\}) = \{1,4,5\}$.

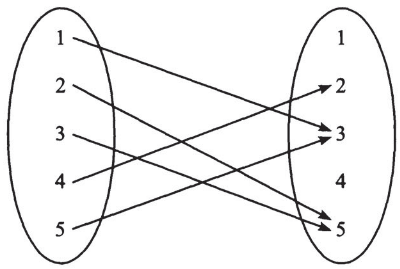


Fig. 4-11

4.9. Suppose $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Find the number m of functions:
(*a*) from A into B, (*b*) from B into A.

(*a*) There are three choices, 1, 2, or 3 for the image of a, and three choices for the image of b. Hence there are $m = 3 \cdot 3 = 9$ functions from A into B.

(*b*) There are two choices, a or b, for each of the three elements of B. Hence there are $m = 2 \cdot 2 \cdot 2 = 2^3 = 8$ functions from B into A.

4.10. Suppose A and B are finite sets with $|A|$ elements and $|B|$ elements, respectively. Show there are $|B|^{|A|}$ functions from A into B. (For this reason, one sometimes writes B^A for the collection of all functions from A into B.)

There are $|B|$ choices for each of the $|A|$ elements of A; hence there are $|B|^{|A|}$ possible functions from A into B.

COMPOSITION OF FUNCTIONS

4.11. Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 4-12. Find the composition function $g \circ f: A \to C$.

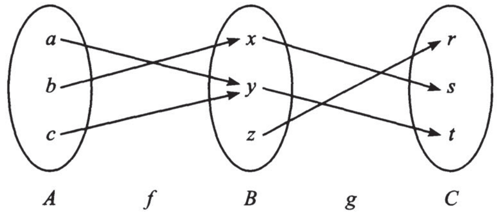


Fig. 4-12

Use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t, \quad (g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \rightarrow y \rightarrow t, \qquad b \rightarrow x \rightarrow s, \qquad c \rightarrow y \rightarrow t$$

4.12. Let the functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula defining the composition functions: (*a*) $g \circ f$, (*b*) $f \circ g$.

(*a*) Compute $g \circ f$ as follows:

$$(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1.$$

Observe that the same answer can be found by writing

$$y = f(x) = 2x + 1 \qquad \text{and} \qquad z = g(y) = y^2 - 2$$

and then eliminating y from both equations:

$$z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

(*b*) Compute $f \circ g$ as follows:

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 3$$

4.13. Let $f: A \rightarrow B$. When is $f \circ f$ defined?

The composition $f \circ f$ is defined when the domain of f is the same as the target set of f; that is, when $A = B$.

4.14. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = x^2 + 2x$.

(*a*) Find $(f \circ f)$ (2) and $(f \circ f)$ (3). (*b*) Find a formula for $f \circ f$.

(*a*) $(f \circ f)\ (2) = f(f(2)) = f(8) = (8)^2 + 16 = 80$

$(f \circ f)\ (3) = f(f(3)) = f(15) = (15)^2 + 30 = 255$

(*b*) $(f \circ f)\ (x) = f(f(x)) = f(x^2 + 2x) = (x^2 + 2x)^2 + 2(x^2 + 2x)$

$= x^4 + 4x^3 + 4x^2 + 2x^2 + 4x$

$= x^4 + 4x^3 + 6x^2 + 4x$

4.15. Prove Theorem 4.1: Let $f: A \rightarrow B$, $g: B \rightarrow C$, and $h: C \rightarrow D$. Then $(f \circ g) \circ h = f \circ (g \circ h)$.

Consider any element $a \in A$. Then:

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) \qquad \text{and} \qquad ((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$$

Thus $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for every $a \in A$, and so $h \circ (g \circ f) = (h \circ g) \circ f$.

ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

4.16. Suppose $f: A \rightarrow B$. Determine conditions under which:

(*a*) f is not one-to-one (injective); (*b*) f is not onto (surjective).

(*a*) f is not one-to-one if there exist $a, a' \in A$ for which $f(a) = f(a')$ but $a \neq a'$.

(*b*) f is not onto if there exists $b \in B$ such that $f(x) \neq b$ for every $x \in A$.

4.17. Determine if each function is one-to-one.

(*a*) To each person on the earth assign the number which corresponds to his age.
(*b*) To each country in the world assign the latitude and longitude of its capital.
(*c*) To each book written by only one author assign the author.
(*d*) To each country in the world which has a prime minister assign its prime minister.

(*a*) No. Many people in the world have the same age.
(*b*) Yes.
(*c*) No. There are different books with the same author.
(*d*) Yes. Different countries in the world have different prime ministers.

4.18. Let the functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 4-13.

(*a*) Determine if each function is one-to-one.
(*b*) Determine if each function is onto.
(*c*) Determine if each function is invertible.
(*d*) Find the composition $h \circ g \circ f$.

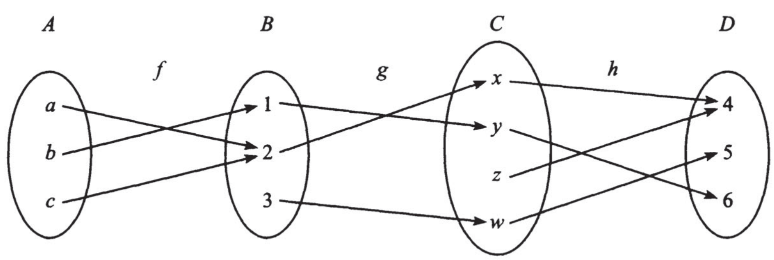


Fig. 4-13

(*a*) f is not one-to-one since $f(a) = f(c)$ but $a \neq c$. h is not one-to-one since $h(x) = h(z)$ but $x \neq z$. g is one-to-one, the elements $1, 2, 3 \in B$ have distinct images.
(*b*) $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A.
$g: B \to C$ is not onto since $z \in C$ is not the image of any element in B.
$h: C \to D$ is onto since each element in D is the image of some element of C.
(*c*) None of the functions are both one-to-one and onto; hence none of the functions are invertible.
(*d*) Now $a \to 2 \to x \to 4$, $b \to 1 \to y \to 6$, $c \to 2 \to x \to 4$. Hence $h \circ g \circ f = \{(a,4),\ (b,6),\ (c,4)\}$.

4.19. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse function f^{-1}. Find a formula for f^{-1}.

Let y be the image of x under the function f; that is, set

$$y = f(x) = 2x - 3 \qquad (1)$$

Consequently, x will be the image of y under the inverse function f^{-1}.

Method 1: Solve for x in terms of y in equation (*1*) obtaining

$$x = (y+3)/2$$

Then $f^{-1}(y) = (y+3)/2$. Replace y by x to obtain

$$f^{-1}(x) = (x+3)/2$$

which is the formula for f^{-1} using the usual independent variable x.

Method 2: First interchange x and y in (*1*) obtaining

$$x = 2y - 3$$

Then solve for y in terms of x to obtain

$$y = (x+3)/2 \qquad \text{and so} \qquad f^{-1}(x) = (x+3)/2$$

4.20. Find a formula for the inverse of $g(x) = \dfrac{2x-3}{5x-7}$.

Set $y = g(x)$ and then interchange x and y as follows:

$$y = \frac{2x-3}{5x-7} \qquad \text{and then} \qquad x = \frac{2y-3}{5y-7}$$

Now solve for y in terms of x:

$$5xy - 7x = 2y - 3 \qquad \text{or} \qquad 5xy - 2y = 7x - 3 \qquad \text{or} \qquad (5x-2)y = 7x - 3$$

Thus

$$y = \frac{7x-3}{5x-2} \qquad \text{and so} \qquad g^{-1}(x) = \frac{7x-3}{5x-2}$$

(Here the domain of g^{-1} excludes $x = 2/5$.)

4.21. Consider functions $f\colon A \to B$ and $g\colon B \to C$. Prove the following:

(*a*) If f and g are one-to-one, then the composition function $g \circ f$ is one-to-one.

(*b*) If f and g are onto functions, then $g \circ f$ is an onto function.

(*a*) Suppose $(g \circ f)(x) = (g \circ f)(y)$; then $g(f(x)) = g(f(y))$. Hence $f(x) = f(y)$ because g is one-to-one. Furthermore, $x = y$ since f is one-to-one. Accordingly, $g \circ f$ is one-to-one.

(*b*) Let c be any arbitrary element of C. Since g is onto, there exists a $b \in B$ such that $g(b) = c$. Since f is onto, there exists an $a \in A$ such that $f(a) = b$. But then

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

Hence each $c \in C$ is the image of some element $a \in A$. Accordingly, $g \circ f$ is an onto function.

4.22. Consider functions $f\colon A \to B$ and $g\colon B \to C$. Prove the following:

(*a*) If $g \circ f$ is one-to-one, then f is one-to-one.

(*b*) If $g \circ f$ is onto, then g is onto.

(*a*) Suppose f is not one-to-one. Then there exist distinct elements $x, y \in A$ for which $f(x) = f(y)$. Thus $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$; hence $g \circ f$ is not one-to-one. Therefore, if $g \circ f$ is one-to-one, then f must be one-to-one.

(*b*) If $a \in A$, then $(g \circ f)(a) = g(f(a)) \in g(B)$; hence $(g \circ f)(A) \subseteq g(B)$. Suppose g is not onto. Then $g(B)$ is properly contained in C and so $(g \circ f)(A)$ is properly contained in C; thus $g \circ f$ is not onto. Accordingly, if $g \circ f$ is onto, then g must be onto.

4.23. Prove Theorem 4.2: A function $f: A \to B$ is invertible if and only if f is bijective (one-to-one and onto).

Suppose f has an inverse, i.e., there exists a function $f^{-1}: B \to A$ for which $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. Since 1_A is one-to-one, f is one-to-one by Problem 4.22; and since 1_B is onto, f is onto by Problem 4.22. That is, f is both one-to-one and onto.

Now suppose f is both one-to-one and onto. Then each $b \in B$ is the image of a unique element in A, say $\hat{b}$. Thus if $f(a) = b$, then $a = \hat{b}$; hence $f(\hat{b}) = b$. Now let g denote the mapping from B to A defined by $g(b) = \hat{b}$. We have:

(i) $(g \circ f)(a) = g(f(a)) = g(b) = \hat{b} = a$, for every $a \in A$; hence $g \circ f = 1_A$.

(ii) $(f \circ g)(b) = f(g(b)) = f(\hat{b}) = b$, for every $b \in B$; hence $f \circ g = 1_B$.

Accordingly, f has an inverse. Its inverse is the mapping g.

SPECIAL MATHEMATICAL FUNCTIONS, RECURSIVELY DEFINED FUNCTIONS

4.24. Find: (*a*) $\lfloor 7.5 \rfloor, \lfloor -7.5 \rfloor, \lfloor -18 \rfloor$, where $\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer that does not exceed x; (*b*) $\lceil 7.5 \rceil, \lceil -7.5 \rceil, \lceil -18 \rceil$, where $\lceil x \rceil$, called the *ceiling* of x, denotes the least integer that does not exceed x.

(*a*) $\lfloor 7.5 \rfloor = 7, \lfloor -7.5 \rfloor = -8, \lfloor -18 \rfloor = -18$.

(*b*) $\lceil 7.5 \rceil = 8, \lceil -7.5 \rceil = -7, \lceil -18 \rceil = -18$.

4.25. Find: (*a*) 26 (mod 7), 25 (mod 5), 35 (mod 11);
(*b*) −26 (mod 7), −371 (mod 8), −2345 (mod 6).

(*a*) When k is positive, divide k by the modulus M to obtain the remainder r. Then $k \pmod M = r$. Thus:

$$26 \pmod 7 = 5, \qquad 25 \pmod 5 = 0, \qquad 35 \pmod{11} = 2$$

(*b*) When k is negative, divide $|k|$ by the modulus M to obtain the remainder r'. Then, when $r' \neq 0$, $k \pmod M = M - r'$. Thus:

$$-26 \pmod 7 = 7 - 5 = 2, \qquad -371 \pmod 8 = 8 - 3 = 5, \qquad -2345 \pmod 6 = 6 - 5 = 1$$

4.26. Using arithmetic modulo $M = 15$, evaluate: (*a*) $9 + 13$, (*b*) $7 + 11$, (*c*) $4 - 9$, (*d*) $2 - 10$.

Use $a \pm M \equiv a \pmod M$:

(*a*) $9 + 13 = 22 \equiv 22 - 15 = 7$ (*c*) $4 - 9 = -5 \equiv -5 + 15 = 10$

(*b*) $7 + 11 = 18 \equiv 18 - 15 = 3$ (*d*) $2 - 10 = -8 \equiv -8 + 15 = 7$

4.27. Evaluate: (*a*) $\log_2 8$; (*b*) $\log_2 64$; (*c*) $\log_{10} 100$; (*d*) $\log_{10} 0.001$.

(*a*) $\log_2 8 = 3$ since $2^3 = 8$ (*c*) $\log_{10} 100 = 2$ since $10^2 = 100$

(*b*) $\log_2 64 = 6$ since $2^6 = 64$ (*d*) $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$

4.28. Show that: (*a*) $\log_b AB = \log_b A + \log_b B$; (*b*) $\log_b A^n = n \log_b A$.

Let $\log_b A = x$ and $\log_b B = y$. Then $A = b^x$ and $B = b^y$.

(*a*) We have $AB = b^x b^y = b^{x+y}$. Hence

$$\log_b AB = x + y = \log_b A + \log_b B$$

(*b*) We have $A^n = (b^x)^n = b^{nx}$. Hence

$$\log_b A^n = nx = n \log_b A$$

4.29. Evaluate: (*a*) 2^5, (*b*) 3^{-4}, (*c*) $8^{2/3}$, (*d*) $25^{-3/2}$.

(*a*) $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

(*b*) $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

(*c*) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$

(*d*) $25^{-3/2} = \dfrac{1}{25^{3/2}} = \dfrac{1}{(\sqrt{25})^3} = \dfrac{1}{5^3} = \dfrac{1}{125}$

4.30. Let n denote a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

Find $L(25)$ and describe what this function does. (The *floor* function $\lfloor x \rfloor$ is defined in the above Problem 4.24.)

Find $L(25)$ recursively as follows:

$$\begin{aligned} L(25) &= L(12) + 1 \\ &= [L(6) + 1] + 1 = L(6) + 2 \\ &= [L(3) + 1] + 2 = L(3) + 3 \\ &= [L(1) + 1] + 3 = L(1) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

Each time n is divided by 2, the value of L is increased by 1. Hence L is the greatest integer such that

$$2^L \leq n$$

Accordingly, $L(n) = \lfloor \log_2 n \rfloor$

Supplementary Problems

FUNCTIONS

4.31. Define each of the following functions from **R** into **R** by a formula:

(*a*) To each number let f assign its square plus 3.

(*b*) To each number let g assign its cube plus twice the number.

(*c*) To each number greater than or equal to 3 let h assign the number squared; and to each number less than 3 let h assign the number -2.

4.32. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 3x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$$

Find $f(5)$, $f(0)$, and $f(-2)$.

4.33. Let $W = \{a, b, c, d\}$. Determine whether each set of ordered pairs is a function from W into W:

(*a*) $\{(b,a),\ (c,d),\ (d,a),\ (c,d),\ (a,d)\}$, (*c*) $\{(a,b),\ (b,b),\ (c,b),\ (d,b)\}$,
(*b*) $\{(d,d),\ (c,a),\ (a,b),(d,b)\}$, (*d*) $\{(a,a),\ (b,a),\ (a,b),\ (c,d)\}$.

4.34. Let the function g assign to each name in the following set S the number of different letters needed to spell the name:

$$S = \{\text{Britt, Martin, Alan, Audrey, Julianna}\}$$

Find the graph of g, i.e., write g as a set of ordered pairs.

4.35. Let $A = \{1,2,3,4,5\}$ and let $f\colon A \to A$ be defined by Fig. 4-14. (*a*) Write f as a set of ordered pairs. (*b*) Find the image of f. (*c*) Find $f(S)$ where $S = \{1,2,4\}$. (*d*) Find $f^{-1}(T)$ where $T = \{1,2,3\}$.

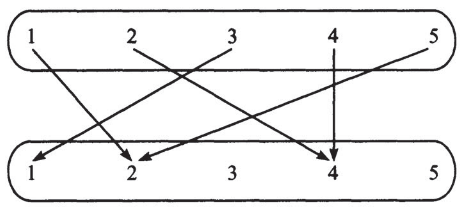


Fig. 4-14

4.36. Let $A = \{a, b, c\}$ and $B = \{1,2,3,4\}$. Find the number of functions from: (*a*) A into B; (*b*) B into A.

4.37. Consider any function $f\colon A \to B$. Show $f^{-1}[f[A]] = A$.

4.38. A function with domain A is called a *constant* function if every $a \in A$ is assigned the same element. Find the number of constant functions from A into B.

COMPOSITION FUNCTION

4.39. Figure 4-15 defines functions f, g, h from $A = \{1,2,3,4\}$ into itself.

(*a*) Find the images of f, g, h.
(*b*) Find the composition functions $f \circ g$, $h \circ f$, $g^2 = g \circ g$.
(*c*) Find the composition functions $h \circ g \circ f$ and $f \circ g \circ h$.

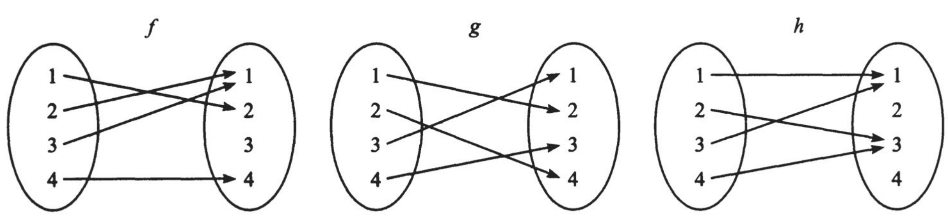


Fig. 4-15

4.40. Consider the functions $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$. Find a formula defining the composition function: (*a*) $f \circ g$; (*b*) $g \circ f$.

4.41. Let $V = \{1, 2, 3, 4\}$ and let

$$f = \{(1,3),\ (2,1),\ (3,4),\ (4,3)\} \quad \text{and} \quad g = \{(1,2),\ (2,3),\ (3,1),\ (4,1)\}$$

Find: (*a*) $f \circ g$; (*b*) $g \circ f$; (*c*) $f \circ f$.

4.42. Suppose $f\colon A \to B$ and $g\colon B \to C$. Show that $g \circ f$ is a constant function (Problem 4.38) if either f or g is a constant function.

ONE-TO-ONE, ONTO AND INVERTIBLE FUNCTIONS

4.43. Which of the functions in Fig. 4-15 are: (*a*) one-to-one, (*b*) onto, (*c*) invertible?

4.44. Consider the formula $f(x) = x^2$.

(*a*) Find the largest interval D such that $f\colon D \to \mathbf{R}$ is a one-to-one function.
(*b*) Find the smallest target set T such that $f\colon \mathbf{R} \to T$ is an onto function.

4.45. Find the domain D and a formula defining the inverse f^{-1} of each function:
(*a*) $f(x) = x^3 + 5$; (*b*) $f(x) = \dfrac{x-2}{x-3}$.

4.46. Suppose $f\colon A \to B$ is a constant function (Problem 4.38). When will f be: (*a*) one-to-one, (*b*) onto?

4.47. Suppose $f\colon A \to B$ and $g\colon B \to C$ are invertible functions. Show that $g \circ f\colon A \to C$ is invertible, and $(g \circ f)^{-1} = g^{-1} \circ f^{-1}$.

4.48. Let $W = [0, \infty) = \{x : x \geq 0\}$. Let $f\colon W \to W$, $g\colon W \to W$, $h\colon W \to W$ be defined as follows:

$$f(x) = x^4, \qquad g(x) = x^3 + 1, \qquad h(x) = x + 2$$

Which of the functions are (*a*) one-to-one, (*b*) onto, (*c*) invertible?

SPECIAL MATHEMATICAL FUNCTIONS, RECURSIVELY DEFINED FUNCTIONS

4.49. Find: (*a*) $\lfloor 13.2 \rfloor$, $\lfloor -0.17 \rfloor$, $\lfloor 34 \rfloor$; (*b*) $\lceil 13.2 \rceil$, $\lceil -0.17 \rceil$, $\lceil 34 \rceil$. (See Problem 4.24.)

4.50. Find: (*a*) 10 (mod 3), 200 (mod 20), 29 (mod 6); (*b*) −10 (mod 3), −29 (mod 6), −345 (mod 11).

4.51. Find: (*a*) $3! + 4!$; (*b*) $3!(3! + 2!)$; (*c*) $6!/5!$; (*d*) $30!/28!$.

4.52. Evaluate: (*a*) $\log_2 16$; (*b*) $\log_3 27$; (*c*) $\log_{10} 0.01$.

4.53. Find: (*a*) 6^3; (*b*) 7^{-2}; (*c*) $4^{5/2}$; (*d*) $27^{-4/3}$.

4.54. Let a and b be positive integers. Suppose a function Q is defined recursively as follows:

$$Q(a,b) = \begin{cases} 0 & \text{if } a < b \\ Q(a-b, b) + 1 & \text{if } a \geq b \end{cases}$$

(*a*) Find $Q(2,3)$ and $Q(14,3)$. (*b*) What does the function do? Find $Q(5861, 7)$.

MISCELLANEOUS PROBLEMS

4.55. Find the domain D of each of the following functions:

(*a*) $f(x) = 1/(x+3)$,
(*b*) $f(x) = 1/(x-3)$ where $x > 0$,
(*c*) $f(x) = \sqrt{16 - x^2}$,
(*d*) $f(x) = \log(x+3)$.

4.56. Sketch the graph of each function:

(*a*) $f(x) = \frac{1}{2}x - 1$

(*b*) $g(x) = x^3 - 3x + 2$

(*c*) $h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x \neq 0 \end{cases}$

Answers to Supplementary Problems

4.31. (*a*) $f(x) = x^2 + 3$ (*b*) $g(x) = x^3 + 2x$;
(*c*) $h(x) = \begin{cases} x^2 & \text{if } x \geq 3 \\ -2 & \text{if } x < 3 \end{cases}$

4.32. $f(5) = 10$; $f(0) = 2$; $f(-2) = 0$

4.33. (*a*) Yes; (*b*) no; (*c*) yes; (*d*) no

4.34. $g = \{$(Britt, 4), (Martin, 6), (Alan, 3), (Audrey, 6), (Julianna, 6)$\}$

4.35. (*a*) $f = \{(1,2), (2,4), (3,1), (4,4), (5,2)\}$; (*b*) $\text{Im}(f) = \{1,2,4\}$; (*c*) $f(S) = \{2,4\}$;
(*d*) $f^{-1}(T) = \{1,3,5\}$

4.36. (*a*) $4^3 = 64$; (*b*) $3^4 = 81$

4.37. $f^{-1}(f(A)) = A$

4.38. Number of elements in B.

4.39. (*a*) $\text{Im}(f) = \{1,2,4\}$, $\text{Im}(g) = \{1,2,3,4\}$, $\text{Im}(h) = \{1,3\}$
(*b*) $f \circ g = \{(1,1), (2,4), (3,2), (4,1)\}$
$h \circ f = \{(1,3), (2,1), (3,1), (4,3)\}$
$g^2 = g \circ g = \{(1,4), (2,3), (3,2), (4,1)\}$
(*c*) $h \circ g \circ f = \{(1,3), (2,3), (3,3), (4,1)\}$
$f \circ g \circ h = \{(1,1), (2,2), (3,1), (4,2)\}$

4.40. (*a*) $(f \circ g)(x) = 4x^2 - 6x + 1$; (*b*) $(g \circ f)(x) = 2x^2 + 6x - 1$

4.41. (*a*) $f \circ g = \{(1,1), (2,4), (3,3), (4,3)\}$
(*b*) $g \circ f = \{(1,1), (2,2), (3,1), (4,1)\}$
(*c*) $f^2 = f \circ f = \{(1,4), (2,3), (3,3), (4,4)\}$

4.43. (*a*) Only g; (*b*) only g; (*c*) only g

4.44. (*a*) $D = [0, \infty)$ or $D = (-\infty, 0]$; (*b*) $T = [0, \infty)$

4.45. (*a*) $f^{-1}(x) = \sqrt[3]{x - 5}$, $D = \mathbf{R}$; (*b*) $f^{-1}(x) = (2 - 3x)/(1 - x)$, $D = \mathbf{R} \backslash \{1\}$

4.46. (*a*) A has one element; (*b*) B has one element

4.48. (*a*) f, g, h; (*b*) f; (*c*) f

4.49. (*a*) 13, −1, 34; (*b*) 14, 0, 34

4.50. (*a*) 1, 0, 5; (*b*) 2, 1, 7

4.51. (*a*) 30; (*b*) 48; (*c*) 6; (*d*) 870

4.52. (*a*) 4; (*b*) 3; (*c*) −2

4.53. (*a*) 216; (*b*) 1/49; (*c*) 32; (*d*) 1/81

4.54. (*a*) $Q(2,3) = 0$, $Q(14,3) = 2$; (*b*) $Q(a,b)$ is the remainder when a is divided by b, so $Q(5861,7) = 2$.

4.55. (*a*) $R\backslash\{-3\}$; (*b*) $D = [0,\infty)\backslash\{-3\}$; (*c*) $D = [-4,4]$; (*d*) $D = (-3,\infty)$

4.56. See Fig. 4-16.

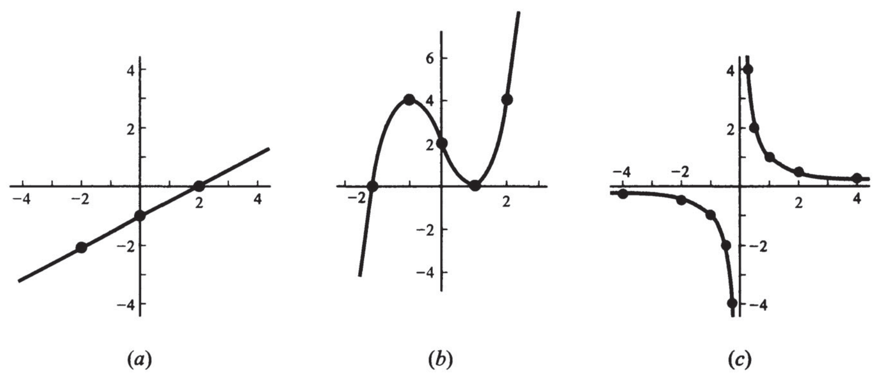


Fig. 4-16