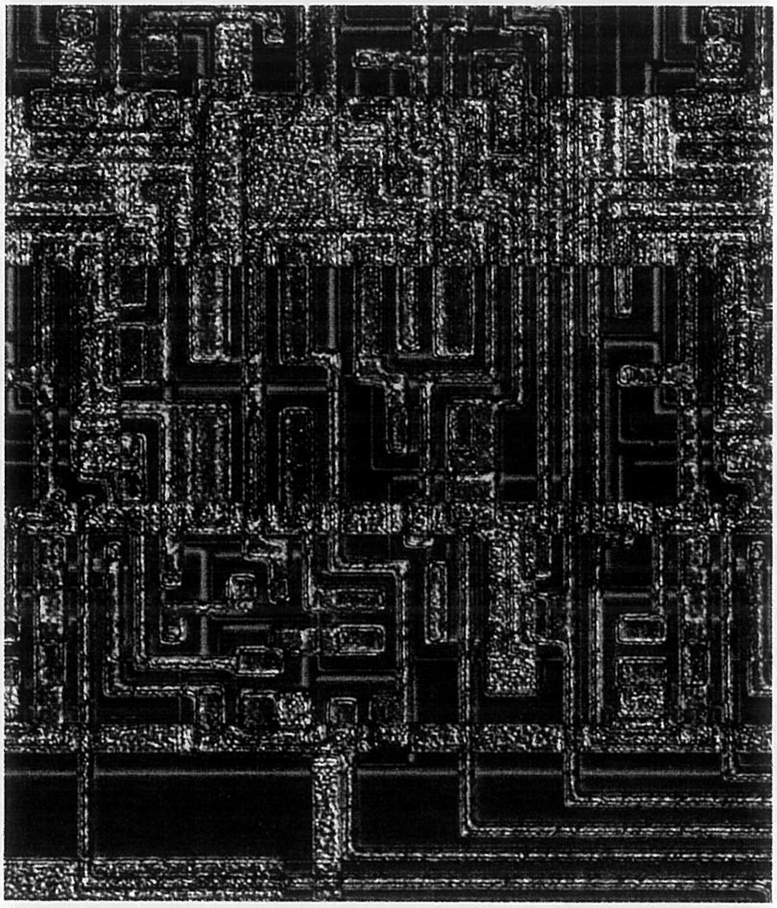
5

Using Congruent Triangles and Parallel Lines



Circuits in a computer chip form a parallelogram. If one angle of the parallelogram is five times as large as another, find the measures of the angles of the parallelogram.

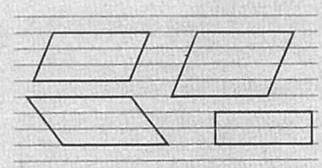
QUADRILATERALS

OBJECTIVE: *Apply the definition of a parallelogram and theorems about properties of parallelograms.*

5-1 Properties of Parallelograms

EXPLORE

Use ruled paper and a ruler to draw four different quadrilaterals in which both pairs of opposite sides are parallel. Use a ruler and protractor to measure the sides and angles of each quadrilateral. Make generalizations about properties you find to be true for all four quadrilaterals.

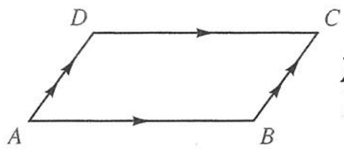


In Chapter 3 you learned that a quadrilateral is a polygon with four sides. In this chapter you will apply theorems you have learned about parallel lines and congruent triangles in your study of special quadrilaterals.

DEFINITION

A **parallelogram** (▱) is a quadrilateral with two pairs of parallel sides.

In ▱$ABCD$, $\overline{AB}$ and $\overline{CD}$ are opposite sides and so are $\overline{AD}$ and $\overline{BC}$. Sides $\overline{AB}$ and $\overline{BC}$ are adjacent sides. $\angle A$ and $\angle C$ are opposite angles and so are $\angle B$ and $\angle D$. $\angle A$ and $\angle B$ are consecutive angles since they have a common side.



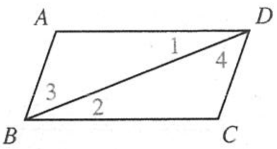
$\overline{AD} \parallel \overline{BC}$
$\overline{AB} \parallel \overline{CD}$

The theorems and corollaries that follow state properties about parallelograms that you may have discovered while completing the Explore.

THEOREM 5.1

Opposite sides of a parallelogram are congruent.

Given: $ABCD$ is a parallelogram.
Prove: $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{BC}$



Proof Draw $\overline{BD}$. $\angle 1$ and $\angle 2$ are congruent alternate interior angles as are $\angle 3$ and $\angle 4$, since $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{CD}$. Since $\overline{BD} \cong \overline{BD}$, $\triangle ABD \cong \triangle CDB$ by the ASA Postulate. So, $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$ because corresponding parts of congruent triangles are congruent.

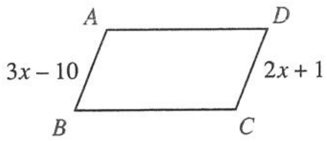
▶ COROLLARY 5.1a

Two parallel lines are equidistant at all points.

You will be asked to prove Corollary 5.1a in Exercise 28.

Example 1

Find the lengths of $\overline{AB}$ and $\overline{DC}$ in ▱$ABCD$.



Solution

$3x - 10 = 2x + 1$ *Opposite sides of a ▱ have equal measure.*

$x = 11$

$2(11) + 1 = 23$ *Replace x with 11 in 2x + 1.*

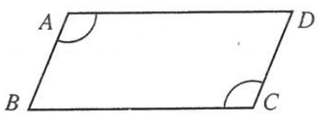
$AB = DC = 23$

Try This

Find the lengths of $\overline{AD}$ and $\overline{BC}$ if $AD = 2y + 22$ and $BC = 4y - 10$.

◆ THEOREM 5.2

Opposite angles of a parallelogram are congruent.



The following corollary can be proved using Theorem 5.2.

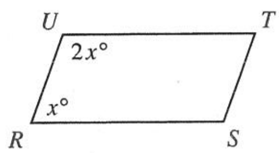
▶ COROLLARY 5.2a

Consecutive angles of a parallelogram are supplementary.

Convincing arguments to support Theorem 5.2 and Corollary 5.2a are asked for in Discussion Exercises 16–17.

Example 2

Find the measures of the four angles of ▱$RSTU$ if the measure of $\angle U$ is twice the measure of $\angle R$.



Solution

$x + 2x = 180$ *Consecutive angles of a ▱ are supplementary.*

$3x = 180$

$x = 60$

$m\angle R = m\angle T = 60 \quad m\angle U = m\angle S = 120$

Try This

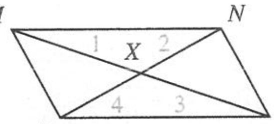
Find the measures of the four angles of ▱$RSTU$ if the measure of $\angle S$ is ten less than four times the measure of $\angle T$.

◆ THEOREM 5.3

The diagonals of a parallelogram bisect each other.

Given: *MNOP* is a parallelogram.

Prove: $\overline{MO}$ and $\overline{PN}$ bisect each other.



Plan Prove $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$, since $\overline{MN} \parallel \overline{PO}$. Prove $\triangle MNX \cong \triangle OPX$ by ASA. Then $\overline{MX} \cong \overline{XO}$ and $\overline{NX} \cong \overline{PX}$ by corresponding parts of congruent triangles.

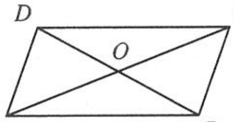
Summary of Properties of Parallelograms

1. Opposite sides are parallel.
2. Opposite sides are congruent.
3. Opposite angles are congruent.
4. Consecutive angles are supplementary.
5. The diagonals bisect each other.

Class Exercises

Short Answer

***ABCD* is a parallelogram.**



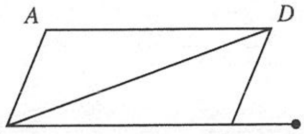
1. Name two pairs of congruent sides.
2. Name two pairs of congruent angles.
3. Name pairs of congruent segments that are not sides.
4. Name two pairs of supplementary angles.
5. If $m\angle CDB = 30$, find $m\angle ABD$.
6. If $m\angle ADC = 100$, find $m\angle ABC$ and $m\angle BAD$.

Sample Exercises

State the definition or theorem that justifies each conclusion.

7. $\overline{AB} \cong \overline{CD}$
8. $AO = OC$
9. $\overline{AD} \parallel \overline{BC}$
10. $\angle DAB$ and $\angle ADC$ are supplementary.

***ABCD* is a parallelogram. Complete each statement.**



11. If $AB = 3x$, $CD = x + 10$, $AB =$ ___
12. If $AD = 3x + 15$, $BC = 21$, $AD =$ ___
13. If $AD = \frac{x}{2}$, $BC = 2x - 12$, $BC =$ ___
14. If $m\angle BAD = 100$, $m\angle DCE =$ ___
15. If $m\angle ADC = 135$, $m\angle ABD = 80$, $m\angle DBC =$ ___

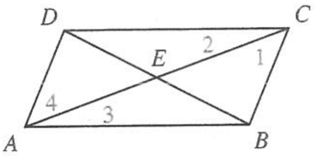
Discussion

16. Explain how Theorem 5.1 can be used to prove Theorem 5.2.
17. Explain how Theorem 5.2 can be used to prove Corollary 5.2a.

Exercises

A

***ABCD* is a parallelogram. State the theorem that justifies each conclusion.**

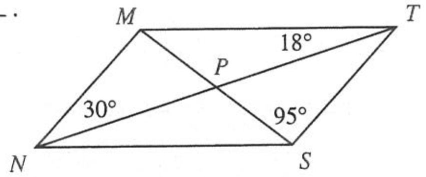


1. $\angle DAB \cong \angle DCB$
2. $\overline{BE} \cong \overline{ED}$
3. $\overline{AD} \cong \overline{BC}$
4. $\overline{DC} \cong \overline{AB}$

Complete each statement.

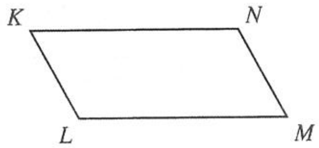
5. If $AD = 20$, $BC =$ ___.
6. If $m\angle ADC = 115$, $m\angle ABC$ ___.
7. If $DB = 22$, $DE =$ ___.
8. If $AE = 18$, $AC =$ ___.
9. If $m\angle DAB = 75$, $m\angle ADC =$ ___.
10. If $m\angle 2 = 30$, $m\angle 3 =$ ___.
11. If $BD = 10$ and $AE = 8$, $AC =$ ___.
12. If $m\angle ABC = 2(m\angle BCD)$, $m\angle ADC =$ ___.
13. If $m\angle ADC = 130$, $m\angle 1 = 35$, $m\angle 2 =$ ___.

Find the measure of each angle in ▱ *MNST*.



14. $m\angle TMN$
15. $m\angle TSN$
16. $m\angle MSN$
17. $m\angle SPN$

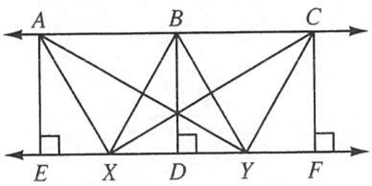
Complete each statement.



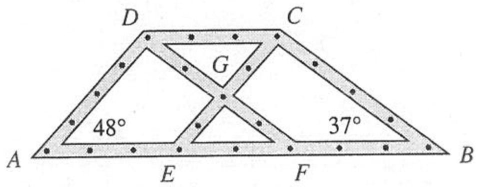
18. If $KN = 3x - 5$, $LM = x + 9$, $KN =$ ___.
19. If $KL = \frac{x}{2}$, $MN = 2x - 9$, $KL =$ ___.
20. If $KL = 8$, $MN = \frac{x^2}{2}$, $x =$ ___.
21. If $m\angle K = 4x + 11$, $m\angle L = 6x - 1$, $m\angle K =$ ___.
22. If $m\angle K = 31$, $m\angle M = 2x^2 - 1$, $x =$ ___.
23. If $m\angle L = x - 40$, $m\angle N = \frac{3x}{4}$, $m\angle L =$ ___.

B

24. If $\overleftrightarrow{AC} \parallel \overleftrightarrow{XY}$ and the altitude, $\overline{BD}$, of $\triangle BXY$ is 8, what are the lengths of the altitudes of $\triangle AXY$ and $\triangle CXY$ from vertices A and C respectively?



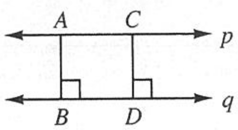
25. Part of the structural support system for a bridge is shown. $\overline{AB} \parallel \overline{CD}$, $\overline{DF} \parallel \overline{CB}$, and $\overline{AD} \parallel \overline{EC}$. Find $m\angle CGF$.



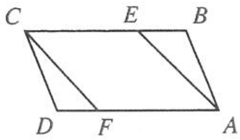
26. Given: ▱ $ABCD$ with $AB = x + 5$ and $CD = 2x - 7$ Find the length of $\overline{AB}$.

27. Given: ▱ $ABCD$ with $AB = 2x$, $CD = 3y + 4$, $BC = x + 7$, and $AD = 2y$ Find the lengths of the sides of the parallelogram.

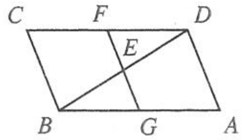
28. Use the figure to the right to prove Corollary 5.1a.
Given: $p \parallel q$, $\overline{AB} \perp q$, $\overline{CD} \perp q$
Prove: $\overline{AB} \cong \overline{CD}$



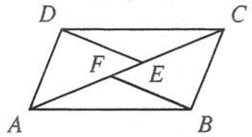
29. Given: ▱ $ABCD$, ▱ $AECF$
Prove: $\triangle CDF \cong \triangle ABE$



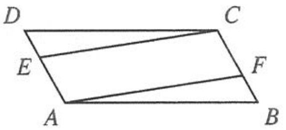
30. Given: ▱ $ABCD$, $\overline{FG}$ bisects $\overline{DB}$.
Prove: $\overline{DB}$ bisects $\overline{FG}$.



31. Given: ▱ $ABCD$, A, F, E, and C are collinear. $AF = CE$
Prove: $\overline{DE} \parallel \overline{BF}$

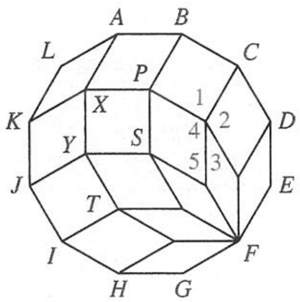


32. Given: ▱ $ABCD$, $AE = CF$
Prove: $CE = AF$

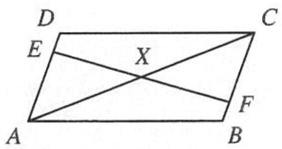


Assume the figure to the right is a regular 12-gon that has been divided into parallelograms and that $m\angle ALK = 150$. Use the definition of a regular polygon and the theorems from this section for each exercise.

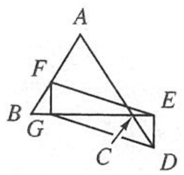
33. Find $m\angle BPX$.
34. Find $m\angle 1$.
35. Find $m\angle 2$.
36. Find $m\angle 3$.
37. Find $m\angle 4$.
38. Find $m\angle 5$.



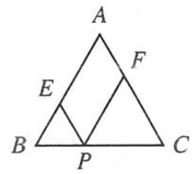
39. Given: ▱ $ABCD$, $\overline{DE} \cong \overline{FB}$
Prove: X bisects $\overline{EF}$.



40. Given: $EFGD$ is a ▱. $\overline{ED} \perp \overline{BE}$, $\overline{BF} \cong \overline{CD}$
Prove: $\triangle ABC$ is isosceles.

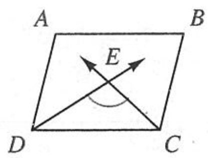


41. Prove: If from any point on the base of an isosceles triangle lines are drawn parallel to the congruent sides of the triangle, a parallelogram is formed whose perimeter is equal to the sum of the lengths of the congruent sides.



Critical Thinking

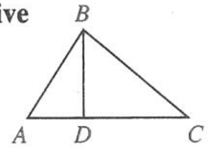
42. Use a ruler to draw several parallelograms. Use a protractor to bisect a pair of consecutive angles in each parallelogram. What generalization can you make about the angle bisectors? Write a paragraph proof to justify your generalization.



43. Write a statement similar to Corollary 5.1a but involving three dimensions. Give a convincing argument that your statement is true.

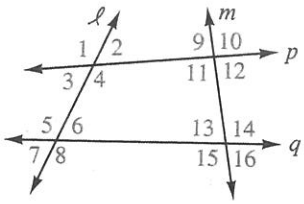
Mixed Review

Use the given information to state which triangles are congruent. Give the postulate or theorem that justifies your answer.



1. $\overline{BD} \perp \overline{AC}$, $\overline{AD} \cong \overline{DC}$
2. $\overline{AB} \cong \overline{BC}$, $\overline{BD}$ bisects $\overline{AC}$.
3. $\overline{AB} \cong \overline{BC}$, $\overline{BD} \perp \overline{AC}$
4. $\angle A \cong \angle C$, $\angle ABD \cong \angle CBD$

Use the given information to state which lines are parallel. Give the postulate or theorem that justifies your answer.



5. $\angle 4 \cong \angle 5$
6. $\angle 3 \cong \angle 10$
7. $\angle 2 \cong \angle 6$
8. $\angle 7 \cong \angle 15$
9. $\angle 1 \cong \angle 8$
10. $\angle 10 \cong \angle 14$
11. $m\angle 4 + m\angle 6 = 180$
12. $m\angle 8 + m\angle 15 = 180$

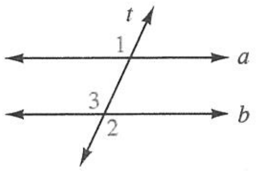
Study Skills

Working Backward to Check a Proof

While checking a proof you have just completed, it is helpful to start at the conclusion and work backward, as demonstrated with Questions 1–4 with the proof below.

Given: $\angle 1 \cong \angle 2$

Prove: $a \parallel b$



Proof

	Statements	Reasons
Question 4	1. $\angle 1 \cong \angle 2$	1. Given
Question 3	2. $\angle 2 \cong \angle 3$	2. Vertical $\angle$s are $\cong$.
Question 2	3. $\angle 1 \cong \angle 3$	3. Congruence of $\angle$s is transitive.
Question 1	4. $a \parallel b$	4. If two lines are cut by a transversal and corr. $\angle$s are $\cong$, the lines are $\parallel$.

Question 1: How can I prove the lines parallel? (Postulate 14, Theorem 3.5, Theorem 3.6, or Theorem 3.7)

Question 2: Which method did I use? (Postulate 14)

Question 3: How do I know that $\angle 1 \cong \angle 3$? (Statements 1–2 and Theorem 1.1)

Question 4: Why is each of those statements true? (Statement 1 is given; statement 2 is justified by the Vertical Angle Theorem.)

In addition to being a good method for checking a proof, working backward is also a helpful problem-solving strategy if you are having difficulty completing a proof.

5-2 Proving Quadrilaterals Are Parallelograms

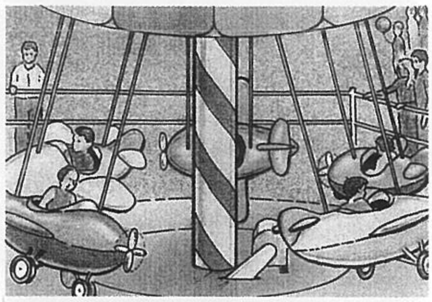
The theorems in the last section can be given in conditional form as in the following general statement.

If $ABCD$ is a parallelogram, *then* $ABCD$ has . . . (a given property).

This lesson deals with the converses of those theorems, stating ways of proving that some quadrilaterals are parallelograms, as in the following conditional form.

If . . . (a given property) is true, *then* $ABCD$ is a parallelogram.

One way to prove that a quadrilateral is a parallelogram is to show that both pairs of opposite sides are parallel and then use the definition of a parallelogram. The following theorems state other ways.

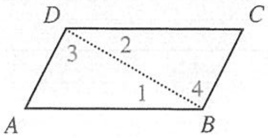


Why will the car of this amusement park ride always be parallel to the top frame?

THEOREM 5.4

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Given: $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{BC}$
Prove: $ABCD$ is a parallelogram.

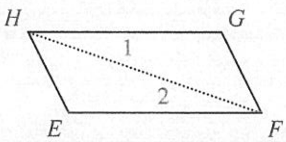


Proof Draw $\overline{BD}$ so $\triangle ABD \cong \triangle CDB$ by the SSS Postulate. $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$ by corresponding parts of congruent triangles are congruent. Then $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$ since the alternate interior angles are congruent. Hence, $ABCD$ is a parallelogram by definition.

THEOREM 5.5

If one pair of opposite sides of a quadrilateral are both parallel and congruent, then it is a parallelogram.

Given: $EF = GH$, $\overline{EF} \parallel \overline{GH}$
Prove: $EFGH$ is a parallelogram.

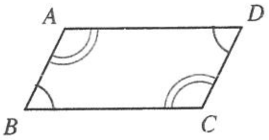


Proof Draw $\overline{HF}$. Since $\overline{EF} \parallel \overline{GH}$, $\angle 1 \cong \angle 2$. Since $\overline{HF} \cong \overline{HF}$, $\triangle HGF \cong \triangle FEH$ by the SAS Postulate. $\overline{HE} \cong \overline{FG}$ by corresponding parts of congruent triangles are congruent. Therefore, $EFGH$ is a parallelogram since both pairs of opposite sides are congruent.

The following two theorems give methods of proving that a quadrilateral is a parallelogram by using the angles of the quadrilateral.

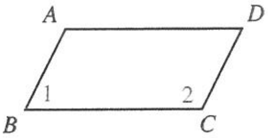
THEOREM 5.6

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



THEOREM 5.7

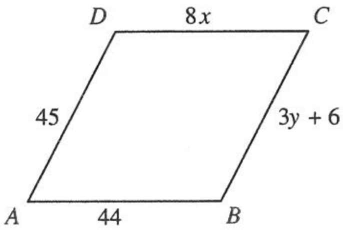
If the consecutive angles of a quadrilateral are supplementary, then the quadrilateral is a parallelogram.



You will be asked to prove Theorems 5.6 and 5.7 in Exercises 33–34.

Example

For what values of x and y will $ABCD$ be a parallelogram?



Solution

$8x = 44$ *If opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.*

$x = 5\frac{1}{2}$

$3y + 6 = 45$

$3y = 39$

$y = 13$

Try This

If $m\angle A = 80$, $m\angle B = 6y + 4$, $m\angle C = 5x$, and $m\angle D = 100$, for what values of x and y will $ABCD$ be a parallelogram?

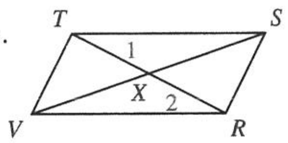
The following method of proving that a quadrilateral is a parallelogram involves the diagonals of a quadrilateral.

THEOREM 5.8

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Given: $\overline{VS}$ and $\overline{RT}$ bisect each other.

Prove: $RSTV$ is a parallelogram.



Plan Prove $\triangle TXS = \triangle RXV$ by the SAS postulate. Conclude $\overline{TS} \cong \overline{RV}$ and $\angle 1 \cong \angle 2$ since they are corresponding parts of congruent triangles. Since $\angle 1 \cong \angle 2$, prove $\overline{TS} \parallel \overline{VR}$. Use Theorem 5.5 to conclude $TSRV$ is a parallelogram.

The methods of proving that a quadrilateral is a parallelogram are summarized below.

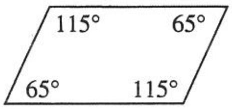
Methods of Proving a Quadrilateral a Parallelogram

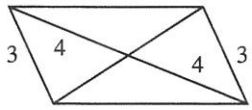
1. Prove that both pairs of opposite sides are parallel.
2. Prove that both pairs of opposite sides are congruent.
3. Prove that one pair of opposite sides is both congruent and parallel.
4. Prove that both pairs of opposite angles are congruent.
5. Prove that the consecutive angles are supplementary.
6. Prove that the diagonals bisect each other.

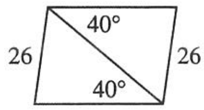
Class Exercises

Short Answer

State whether or not the figure is a parallelogram. If it is, state a theorem or definition that justifies your conclusion.

1. 

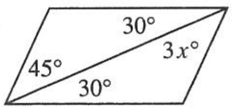
2. 

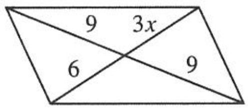
3. 

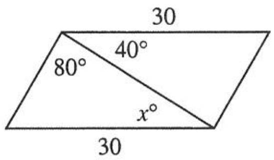
4. Of the six ways to prove that a quadrilateral is a parallelogram summarized in the lesson, which one is not a theorem? What is it?

Sample Exercises

Find the value of x for which each quadrilateral is a parallelogram.

5. 

6. 

7. 

Draw a rectangular coordinate system, plot each point, and find the lengths of AB, BC, CD, and AD. Use Theorem 5.4 to decide whether $ABCD$ is a parallelogram.

8. $A(0, 0)$, $B(0, 3)$, $C(5, 3)$, $D(5, 0)$
9. $A(0, 0)$, $B(3, 4)$, $C(9, 4)$, $D(6, 0)$
10. $A(0, 0)$, $B(-4, 0)$, $C(-7, 4)$, $D(-3, 4)$
11. $A(3, -2)$, $B(8, -2)$, $C(8, -4)$, $D(3, -4)$

Discussion

Draw a figure, if needed, to support your argument.

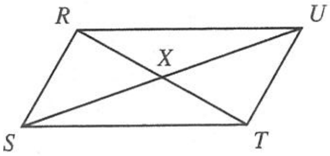
12. Is it possible for a quadrilateral to have two congruent opposite sides and two parallel sides but not be a parallelogram? Explain.
13. Is it possible for a polygon to have two sides that are both congruent and parallel but not be a parallelogram? Explain.

Exercises

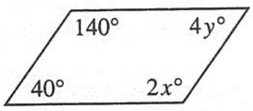
A

State whether or not you can conclude that the figure is a parallelogram, based on the given information. If it is, state the theorem or definition that justifies your conclusion.

1. $RU = ST$, $RS = TU$
2. $\overline{RS} \parallel \overline{TU}$, $\overline{RU} \parallel \overline{ST}$
3. $\overline{RU} \parallel \overline{ST}$, $\overline{RS} \cong \overline{TU}$
4. $\overline{RS} \parallel \overline{TU}$, $\overline{RS} \cong \overline{TU}$
5. $RX = \frac{1}{2}RT$, $XU = \frac{1}{2}SU$
6. $RX = XU$, $SX = XT$
7. $RU = UT = TS = SR$
8. $m\angle SRU = m\angle UTS$, $m\angle RST = m\angle RUT$



Determine the values of x and y so that the figure is a parallelogram.

9.


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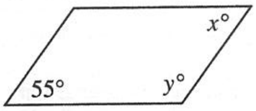

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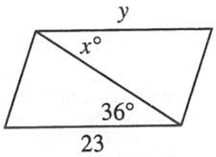

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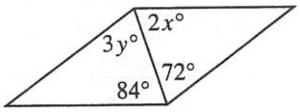

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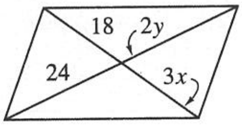
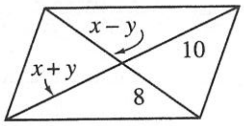
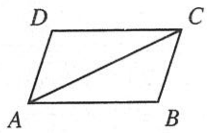


14.


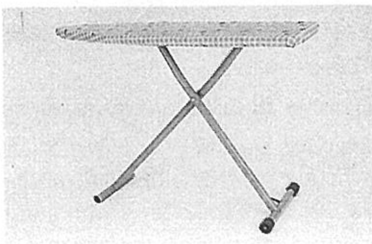
Draw a rectangular coordinate system, plot each point, and find the lengths of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Use Theorem 5.4 to decide whether $ABCD$ is a parallelogram.

15. $A(0, 0)$, $B(5, 0)$, $C(5, 4)$, $D(0, 4)$
16. $A(0, 0)$, $B(5, 0)$, $C(7, 2)$, $D(2, -2)$
17. $A(0, 0)$, $B(-4, 0)$, $C(-6, 3)$, $D(-1, 3)$
18. $A(0, 7)$, $B(3, 7)$, $C(5, 1)$, $D(2, 1)$
19. $A(1, 0)$, $B(0, 1)$, $C(-2, 0)$, $D(0, 2)$
20. $A(0, 1)$, $B(1, 0)$, $C(4, 3)$, $D(3, 4)$
21. **Given:** $\triangle ABC \cong \triangle CDA$
    Prove: $ABCD$ is a ▱.

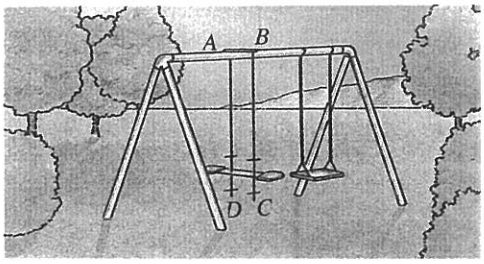


B

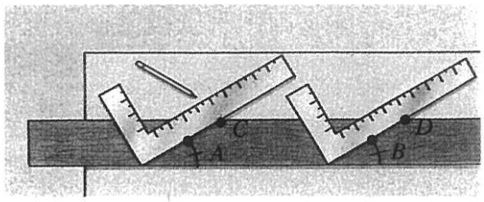
22. The ironing board shown on the right is supported by legs that are equal in length. The legs also bisect each other at the point where they cross. Explain why the ironing board will always be parallel to the floor, regardless of the height to which it happens to be adjusted.



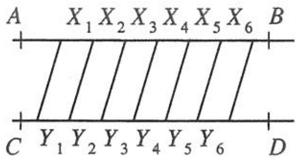
23. Consider the children's swing set shown to to right. The hangers $\overline{AD}$ and $\overline{BC}$ are the same length and the distances between the hangers, $\overline{AB}$ and $\overline{DC}$, are equal. Explain why the seat $\overline{DC}$ will always be parallel to the bar $\overline{AB}$ at the top of the swing.



24. A carpenter wants to construct parallel lines on a board. This can be done by using a carpenter square twice. Each time the tool is placed at the same angles with the board and equal units are marked off. Explain why this method assures that $\overline{AB}$ will be parallel to $\overline{CD}$.



25. A parking lot is to be marked for slant parking. A string is stretched from A to B with marks made every 9 ft at $X_1, X_2, \ldots, X_6$. A second string is stretched parallel to $\overline{AB}$ from C to D, with marks located every 9 ft at $Y_1, Y_2, \ldots, Y_6$. Explain why all the painted lines will be parallel.



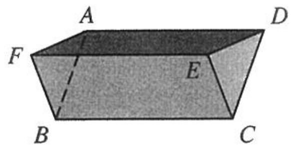
Draw a quadrilateral *ABCD* and determine the values of *x* and *y* for which *ABCD* is a parallelogram.

26. $AB = 2x + 4$, $CD = 4x - 20$, $AD = 2y$, $BC = y + 5$
27. $m\angle A = 2x - 60$, $m\angle D = x - 5$, $AB = 4y + 6$, $CD = 6y - 10$
28. $AB = 6x + 30$, $BC = 2x - 5$, $CD = 2y - 10$, $AD = y - 35$

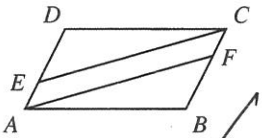
Determine the coordinates of *D* for which *ABCD* is a parallelogram.

29. $A(0, 0)$, $B(1, 4)$, $C(6, 5)$, $D(x, y)$
30. $A(0, 2)$, $B(2, 0)$, $C(-2, -4)$, $D(x, y)$

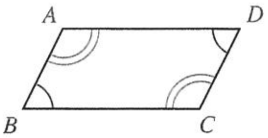
31. **Given:** *ABCD* is a ▱.
BCEF is a ▱.
Prove: *ADEF* is a ▱.



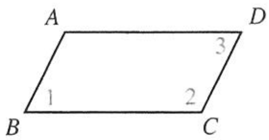
32. **Given:** *ABCD* is a ▱.
E is the midpoint of $\overline{AD}$.
F is the midpoint of $\overline{BC}$.
Prove: AFCE is a ▱.



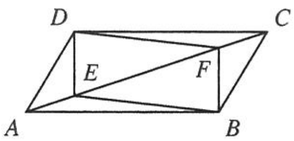
33. Prove Theorem 5.6.
Given: $\angle A \cong \angle C$
$\angle B \cong \angle D$
Prove: *ABCD* is a ▱.



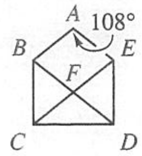
34. Prove Theorem 5.7.
Given: $\angle 1$ and $\angle 2$ are supplementary.
$\angle 2$ and $\angle 3$ are supplementary.
Prove: *ABCD* is a ▱.



35. **Given:** *ABCD* is a ▱.
$AE = CF$
Prove: *BFDE* is a ▱.



36. Polygon $ABCDE$ shown to the right is an equilateral pentagon. The perimeter of $ABCDE$ is 20. If $\overline{BD} \parallel \overline{AE}$ and $\overline{CE} \parallel \overline{BA}$, find the perimeter of $ABFE$.



37. If $ABCDEF$ is a regular hexagon, prove that $BCEF$ is parallelogram.

38. Draw $\triangle ABC$ with median $\overline{AM}$. Extend $\overline{AM}$ so that $\overline{AM} \cong \overline{MD}$. Prove that $ABDC$ is a parallelogram.

Critical Thinking

39. Suppose a parallelogram is defined in the following way:
A parallelogram is a quadrilateral with opposite sides congruent.
Use this definition to prove the following statement:
If a quadrilateral has both pairs of opposite sides parallel, then it is a parallelogram.

40. If the definition of a parallelogram that is given in Exercise 39 is accepted, would the statement you proved using that definition in the same exercise be classified as a definition, postulate, or theorem?

Algebra Review

Factor.

1. $x^2 - 4x - 45$ **2.** $y^2 + y - 42$ **3.** $a^2 + 2a - 15$

4. $a^2 - 7ab + 10b^2$ **5.** $x^2 + 5xy - 24y^2$ **6.** $b^2 + 9b - 90$

Simplify.

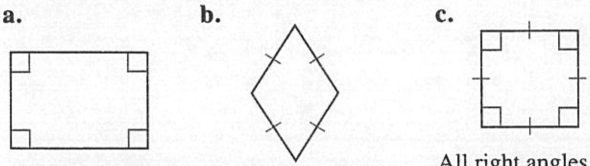
7. $\sqrt{x^2}$ **8.** $\sqrt{4y^2}$ **9.** $\sqrt{(-5)^2}$ **10.** $\sqrt{(-3a)^2}$

Computer Activity

Properties of Special Parallelograms

Use computer software to construct special parallelograms like the ones shown below.

a. **b.** **c.**



All right angles — All sides congruent — All right angles, All sides congruent

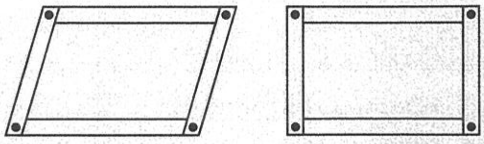
1. Measure the diagonals in each figure. What do you notice? State a generalization about your observation in each figure.
2. Measure the angles formed by the diagonals in each figure. What do you notice? State a generalization about your observation in each figure.
3. Measure the four segments formed by the diagonals in each figure. What do you notice? State a generalization about your observations.

OBJECTIVE: *Apply definitions and theorems about rectangles, rhombuses, and squares.*

5-3 Rectangles, Rhombuses, and Squares

EXPLORE

Take the linkage that forms a parallelogram (Position 1) and place it in the position of a rectangle (Position 2). What additional properties seem to be true about the rectangle?

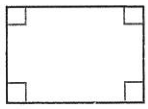


Position 1 Position 2

In the last two lessons, you learned about the quadrilateral called a parallelogram. Three special parallelograms, *rectangles*, *rhombuses*, and *squares*, are defined below. They are classified by the special relationships of their angles and sides.

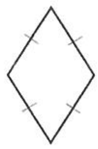
DEFINITION

A **rectangle** is a quadrilateral with four right angles.



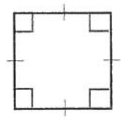
DEFINITION

A **rhombus** is a quadrilateral with four congruent sides.



DEFINITION

A **square** is a quadrilateral with four right angles and four congruent sides.



Based on the definitions of rectangle, rhombus, and square, the following statements are true.

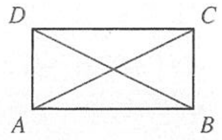
Every rectangle is a parallelogram.
Every rhombus is a parallelogram.
Every square is a parallelogram, a rectangle, and a rhombus.

This means that all of the properties of parallelograms—such as opposite sides that are congruent, opposite angles that are congruent, consecutive angles that are supplementary, and diagonals that bisect each other—are also properties of rectangles, rhombuses, and squares.

◆ THEOREM 5.9

The diagonals of a rectangle are congruent.

Given: *ABCD* is a rectangle.
Prove: $\overline{AC} \cong \overline{BD}$



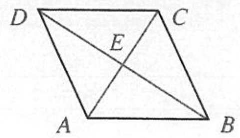
Proof

Statements	Reasons
1. *ABCD* is a rectangle.	**1.** Given
2. $\overline{AD} \cong \overline{BC}$	**2.** Opposite sides of a ▱ are ≅.
3. $\overline{AB} \cong \overline{AB}$	**3.** Congruence of segments is reflexive.
4. $\angle DAB$ and $\angle CBA$ are rt. ∠s.	**4.** Definition of a rectangle
5. $\angle DAB = \angle CBA$	**5.** All right angles are congruent.
6. $\triangle DAB \cong \triangle CBA$	**6.** SAS Congruence Postulate
7. $\overline{AC} \cong \overline{BD}$	**7.** Corres. parts of ≅ △s are ≅.

◆ THEOREM 5.10

The diagonals of a rhombus are perpendicular.

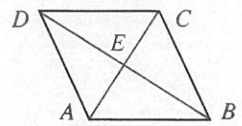
Given: *ABCD* is a rhombus.
Prove: $\overline{AC} \perp \overline{BD}$



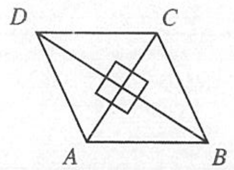
Plan Use the definition of a rhombus and the fact that the diagonals of a parallelogram bisect each other to show that $\triangle AED \cong \triangle CED$. Conclude that $\overline{AC} \perp \overline{BD}$ since $\angle EAC$ and $\angle CED$ are supplementary and congruent.

◆ THEOREM 5.11

Each diagonal of a rhombus bisects a pair of opposite angles.



You will be asked to prove Theorem 5.11 in Exercise 43.



Example 1

If *ABCD* is a rhombus and $m\angle BDC = 35$, find $m\angle ACD$.

Solution

$m\angle ACD = 90 - 35 = 55$ *The acute angles of a right triangle are complementary.*

Try This

If *ABCD* is a rhombus and $m\angle CAD = 68$, find $m\angle CBD$.

A summary of the properties of rectangles, rhombuses, and squares is given below.

Properties of Rectangles

A rectangle has all properties of a parallelogram as well as having
all angles, right angles and
congruent diagonals.

Properties of Rhombuses

A rhombus has all properties of a parallelogram as well as having
all sides congruent,
perpendicular diagonals, and
diagonals that bisect opposite angles.

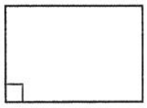
Properties of Squares

A square has all properties of a parallelogram, a rectangle, and a rhombus.

While it is the case that every rectangle, rhombus, and square is a parallelogram, it is not the case that every parallelogram is a rectangle, rhombus, or square. The following two theorems state conditions for which a parallelogram is either a rectangle or a rhombus.

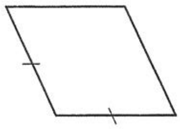
◆ **THEOREM 5.12**

If a parallelogram has a right angle, then it is a rectangle.



◆ **THEOREM 5.13**

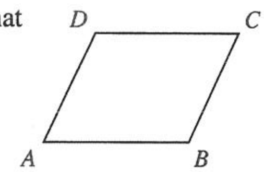
If a parallelogram has two adjacent sides congruent, then it is a rhombus.



You will be asked to prove Theorems 5.12 and 5.13 in Exercises 44–45. From Theorems 5.12 and 5.13, it follows that a parallelogram with a right angle and two adjacent sides congruent is a square.

Example 2

ABCD is a parallelogram. If $AD = 2x + 10$ and $CD = 4x - 20$, for what value of x will *ABCD* be a rhombus?



Solution

$4x - 20 = 2x + 10$ *If a parallelogram has two adjacent sides congruent, then it is a rhombus.*

$2x = 30$

$x = 15$

Try This

If $AB = 5 - 2x$ and $BC = 3x + 80$, for what value of x will *ABCD* be a rhombus?

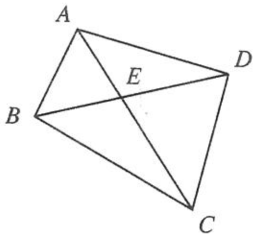
Class Exercises

Short Answer

Determine whether each statement is true or false.

1. The diagonals of a rectangle are perpendicular.
2. The diagonals of a rectangle bisect the opposite angles.
3. If *ABCD* is a square, then *ABCD* is a rhombus.
4. If *ABCD* is a parallelogram, then *ABCD* is a rhombus.
5. If *ABCD* is a rectangle, then *ABCD* is a rhombus.

Sample Exercises



6. If *ABCD* is a rectangle, name all pairs of congruent segments.
7. If *ABCD* is a rhombus, name all pairs of congruent segments.
8. If *ABCD* is a rhombus, name all the angles that are right angles.
9. If *ABCD* is a square, name all the angles that are congruent to $\angle CAB$.
10. If *ABCD* is a square, what is $m\angle AED$?
11. If *ABCD* is a rectangle and $BE = 6$, what is length AC?
12. If *ABCD* is a rhombus and $m\angle BAC = 35$, what is $m\angle BCD$?

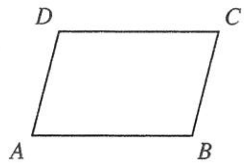
Discussion

Name two properties that are true for each of the following.

13. all squares but not all rectangles
14. all rhombuses but not all parallelograms
15. all rectangles but not all parallelograms

Exercises

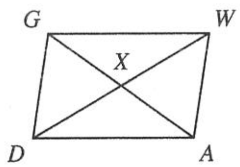
A



1. State three properties that are true about the diagonals of a square.
2. *ABCD* is a parallelogram. If $m\angle A = 90$, must *ABCD* be a rectangle, a rhombus, or a square?
3. *ABCD* is a parallelogram. If $\overline{AB} \cong \overline{BC}$, must *ABCD* be a rectangle, a rhombus, or a square?
4. *ABCD* is a parallelogram. If $m\angle A = 90$ and $\overline{AB} \cong \overline{BC}$, must *ABCD* be a rectangle, a rhombus, or a square?

Assume that *GWAD* is a rectangle. Find the indicated measure.

5. $GX = 6$ Find XW.
6. $XA = 9$ Find DW.
7. $m\angle GXD = 40$ Find $m\angle GDX$.
8. $m\angle WAG = 65$ Find $m\angle GXW$.



Assume that *GWAD* is a rhombus. Find the indicated measure.

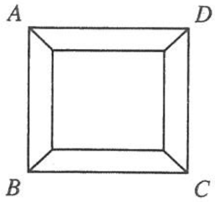
9. $m\angle GDX = 47$ Find $m\angle XGD$.
10. $m\angle GWA = 86$ Find $m\angle GWD$.

Assume that *GWAD* is a square. Find the indicated measure.

11. Find $m\angle XDA$.
12. $GX = 3x + 6$, $XW = 4x - 10$ Find GA.

Determine whether each statement is true or false.

13. Every rectangle is a parallelogram.
14. Every rhombus is a rectangle.
15. Every parallelogram is a rectangle.
16. Every property of a parallelogram is also a property of a square.
17. Every property of a square is also a property of a rectangle.
18. The diagonals of a square bisect each other.
19. A picture frame is constructed of four pieces of wood that are glued together so that $AB = BC = CD = DA$. The ends of each piece of wood are cut at a 45° angle. Explain why the completed frame will form a square.

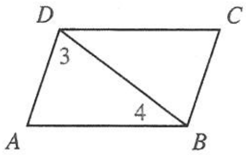


20. **Given:** $ABCD$ is a $\square$. $m\angle 3 = m\angle 4$

    Prove: $ABCD$ is a rhombus.

21. **Given:** $ABCD$ is a $\square$. $m\angle 3 + m\angle 4 = 90$

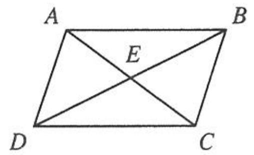
    Prove: $ABCD$ is a rectangle.



B

Draw a parallelogram for the given conditions or write *not possible*.

22. all angles congruent
23. diagonals that bisect each other
24. no right angles, diagonals congruent
25. congruent and perpendicular diagonals
26. all sides congruent with diagonals that are not perpendicular
27. $ABCD$ is a parallelogram. $AB = 2x + 4$, $DC = 3x - 11$, $AD = x + 19$ Show that $ABCD$ is a rhombus.
28. $ABCD$ is a rhombus. $m\angle DEC = 4x + 10$, $m\angle DAB = 3x + 4$ Find $m\angle ABC$.



***ABCD* is a rectangle with the given coordinates. Find the lengths of the diagonals and verify that they are congruent.**

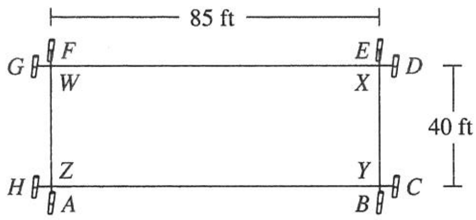
29. $A(2, 5)$, $B(2, 1)$, $C(7, 1)$, $D(7, 5)$
30. $A(-2, 3)$, $B(-4, 3)$, $C(-4, -2)$, $D(-2, -2)$

Determine the coordinates of *D* for which *ABCD* is a rectangle.

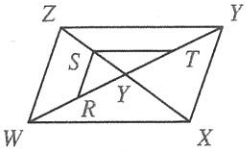
31. $A(-2, 5)$, $B(1, 5)$, $C(1, -2)$, $D(x, y)$
32. $A(-3, -2)$, $B(4, -2)$, $C(4, -3)$, $D(x, y)$

Determine the coordinates of *D* for which *ABCD* is a square.

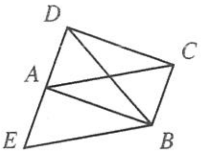
33. $A(3, -1)$, $B(-1, 3)$, $C(-5, -1)$, $D(x, y)$
34. $A(3, 0)$, $B(1, 2)$, $C(3, 4)$, $D(x, y)$
35. A contractor is measuring for the foundation of a building that is to be 85 ft by 40 ft. Stakes and string are placed as shown. The outside corners of the building will be at the points where the strings cross. To make sure that $WXYZ$ is a rectangle, the contractor constructs $WX = ZY$ and $WZ = XY$. Prove that if $WY = ZX$, $WXYZ$ will be a rectangle.
36. If $WY = 93$ ft and $XZ = 94$ ft, is $WXYZ$ a rectangle? If not, which way should stakes E and F be moved to make $WXYZ$ a rectangle?



37. **Given:** *WXYZ* is a rhombus.
R is the midpoint of $\overline{WV}$.
T is the midpoint of $\overline{VY}$.
S is any point on $\overline{ZV}$.
Prove: $\triangle SRT$ is isosceles.

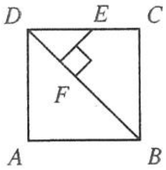


38. **Given:** *ABCD* is a rectangle.
ACBE is a parallelogram.
Prove: $\triangle DBE$ is isosceles.



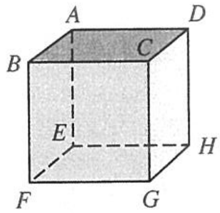
C

39. **Given:** *ABCD* is a square.
$FB = CB$
$\overline{EF} \perp \overline{BD}$
Prove: $DF = FE = EC$



What quadrilateral is determined by a cross section of the cube containing the given vertices? Write a paragraph proof to justify your choice.

40. vertices *A*, *B*, *G*, and *H*
41. vertices *A*, *C*, *G*, and *E*



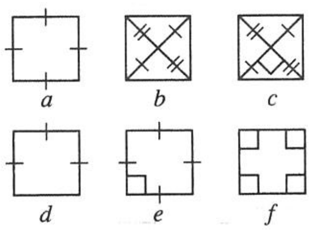
42. Draw and label a rectangle *ABCD* and equilateral triangles *ABX* and *BCY* entirely outside the rectangle. Prove that $\triangle XDY$ is isosceles.

43. Prove Theorem 5.11
44. Prove Theorem 5.12
45. Prove Theorem 5.13

Critical Thinking

Answer the questions in Exercises 46–50 for each figure (a–f).

46. Must the figure be a quadrilateral?
47. Must the figure be a parallelogram?
48. Must the figure be a rectangle?
49. Must the figure be a rhombus?
50. Must the figure be a square?



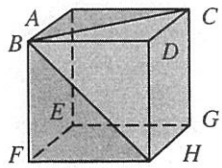
Algebra Review

Solve.

1. $x^2 - 9 = 6$
2. $25a^2 = 16$
3. $y^2 = \frac{4}{9}$

Math Contest Problem

The figure shown is a cube. $\overline{BC}$ and $\overline{BH}$ are diagonals of faces of the cube. Find the measure of $\angle CBH$.

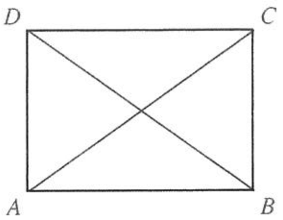


COMPARING APPROACHES IN GEOMETRY

Synthetic and Coordinate Geometry

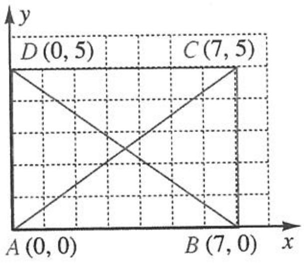
In the last lesson, we proved that the diagonals of a rectangle are congruent. The approach that was used, a synthetic approach, does not rely on algebra. Recall from its introduction on page 194 that a coordinate approach relies on a coordinate system. Compare the two approaches for a 5 by 7 rectangle.

Synthetic Approach



Plan Prove $\triangle ABC \cong \triangle DCB$ and use corresponding parts of congruent triangles are congruent to conclude that $AC = BD$.

Coordinate Approach



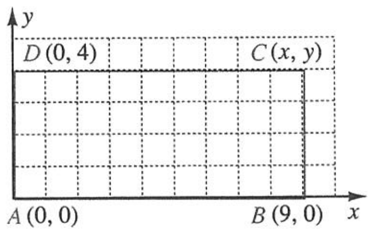
Plan Name the coordinates of the vertices and use the distance formula to conclude that $AC = BD$.

In the coordinate approach use the distance formula to find AC and BD. Observe that these lengths are equal.

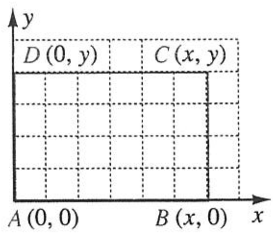
$$AC = \sqrt{(7-0)^2 + (5-0)^2} = \sqrt{49 + 25}$$
$$BD = \sqrt{(0-7)^2 + (5-0)^2} = \sqrt{49 + 25}$$

Exercises

When using a coordinate proof it is important to name the figure correctly after it has been positioned on a coordinate system.



1. Suppose that a coordinate system is selected as shown for a 4 by 9 rectangle. What are the x and y coordinates of vertex C?
2. Suppose that a coordinate system is placed on an 8 by 11 rectangle in the same manner as is shown for the 4 by 9 rectangle in the figure to the right. What are the coordinates of all four vertices A, B, C, and D?



3. A general rectangle is represented by using variable coordinates rather than numerical coordinates. For example, suppose that a coordinate system is placed on a rectangle that is a units long and b units wide as is shown in the figure. What are the coordinates of all four vertices A, B, C, and D in terms of a and b?
4. Use the distance formula to find the distances AC and BD in the figure for Exercise 3. What theorem does your work prove?

5-4 Trapezoids

EXPLORE

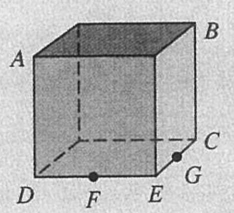
Take several styrofoam cubes and cut three cross sections.

a. a cross section containing vertices A, B, C, and D

b. a cross section containing vertices A and B and midpoints F and G

c. a cross section containing vertices A, B, and E

Sketch the figure formed by each of the three cross sections.

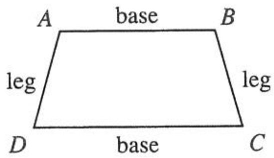


Another classification of quadrilateral is the trapezoid.

DEFINITION

A **trapezoid** is a quadrilateral with exactly one pair of parallel sides.

The parallel sides of a trapezoid are called **bases.** The nonparallel sides are called **legs.** The pairs of angles formed by a base and the legs are called **base angles.** In trapezoid $ABCD$, $\angle ADC$ and $\angle BCD$ are base angles and $\angle DAB$ and $\angle CBA$ are base angles. An **isosceles trapezoid** is a trapezoid with congruent nonparallel sides.

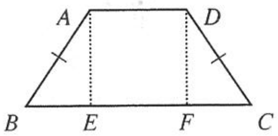


THEOREM 5.14

Each pair of base angles of an isosceles trapezoid are congruent.

Given: Isosceles trapezoid $ABCD$, $\overline{AB} \cong \overline{DC}$

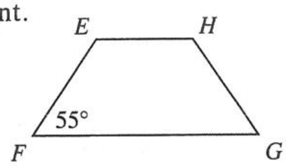
Prove: $\angle B \cong \angle C$, $\angle BAD \cong \angle CDA$



Plan Draw $\overline{AE}$ and $\overline{DF}$ perpendicular to $\overline{BC}$. Prove $\triangle ABE \cong \triangle DCF$ by the Hypotenuse-Leg Theorem. It follows that $\angle B \cong \angle C$. Prove that $\angle BAD \cong \angle CDA$ since supplements of congruent angles are congruent.

Example 1

Given isosceles trapezoid $EFGH$ with $m\angle EFG = 55$, find $m\angle FGH$.



Solution

$m\angle FGH = 55$ *Base angles of an isosceles trapezoid have the same measure.*

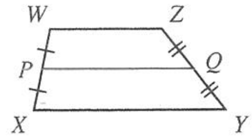
Try This

If $m\angle E$ is twice $m\angle F$, find the measures of the four angles of trapezoid $EFGH$.

◆ THEOREM 5.15

The diagonals of an isosceles trapezoid are congruent.

The **median of a trapezoid** is the segment joining the midpoints of the legs. $\overline{PQ}$ is the median of trapezoid $WXYZ$. The median and the bases of a trapezoid are related in a special way.



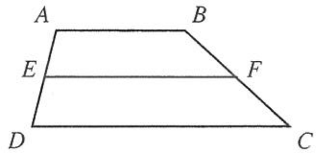
◆ THEOREM 5.16

The median of a trapezoid is parallel to the bases and has a length equal to half the sum of the lengths of the bases. That is, $PQ = \frac{1}{2}(XY + WZ)$.

You will be asked to prove Theorems 5.15 and 5.16 in Exercises 20 and 23.

Example 2

$ABCD$ is a trapezoid. $AB = 13$, $CD = 20$
Find the length EF of the median.



Solution

$EF = \frac{1}{2}(AB + CD)$ *Theorem 5.17*
$EF = \frac{1}{2}(13 + 20)$
$EF = 16.5$

Try This

Find the length of $\overline{CD}$ if $AB = 15$ and $EF = 21$.

Class Exercises

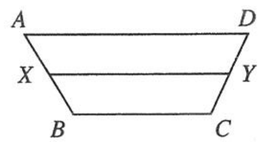
Short Answer

***ABCD* is an isosceles trapezoid with median $\overline{XY}$.**

1. Name the bases of trapezoid $ABCD$.
2. Name the legs of trapezoid $ABCD$.
3. Name two pairs of congruent angles.
4. Name two pairs of congruent segments.

Sample Exercises

Use isosceles trapezoid *ABCD* with median $\overline{XY}$ to complete each statement.



5. $AX = 4$, $CD =$ ___
6. $m\angle ABC = 110$, $m\angle BAD =$ ___
7. $m\angle BAD = 65$, $m\angle CDA =$ ___
8. $AD = 22$, $BC = 10$, $XY =$ ___
9. $BC = 20$, $XY = 32$, $AD =$ ___
10. $m\angle DCB = 105$, $m\angle DAB =$ ___

Discussion

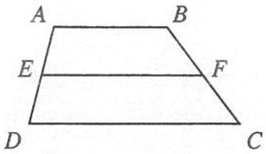
Draw a trapezoid satisfying the following conditions. If not possible, explain.

11. two right angles
12. three right angles
13. three congruent sides
14. four congruent sides
15. congruent legs
16. congruent bases

Exercises

A

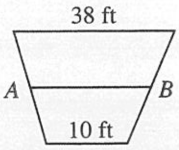
***ABCD* is a trapezoid with median $\overline{EF}$.**

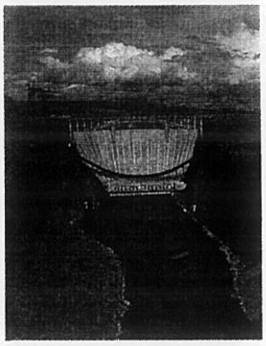


1. $AB = 24$, $CD = 43$ Find EF.
2. $AB = 2\frac{1}{2}$, $CD = 4\frac{1}{2}$ Find EF.
3. $AB = 9.7$, $CD = 24.6$ Find EF.
4. $CD = 32$, $EF = 23$ Find AB.
5. $AB = x + 1$, $EF = 5x$, $CD = 35$ Find AB.
6. $AB = x + 11$, $EF = x - 20$, $CD = 9$ Find EF.

Indicate whether each statement is always, sometimes, or never true.

7. If the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.
8. If a quadrilateral has more than two right angles, then it is not a trapezoid.
9. The opposite angles of an isosceles trapezoid are supplementary.
10. The diagonals of an isosceles trapezoid bisect each other.
11. If the consecutive angles of a quadrilateral are supplementary, it is a trapezoid.
12. A dam is constructed with a trapezoidal cross section that measures 38 ft across the top and 10 ft across the base. What is the average width of the dam?



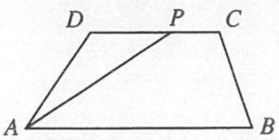


B

13. The length of one base of a trapezoid is four times the length of the other base. If the length of the median is 40, find the length of each base.
14. If the length of the shorter base of a trapezoid is doubled but the length of the median is not changed, how does the length of the longer base change?
15. If one base of a trapezoid is increased 12 cm and the other base remains the same length, by how much is the length of the median increased?
16. Suppose the lengths of the bases of a trapezoid are represented by x and y. If the length of the median is doubled, write an expression that represents the length of the median of the larger trapezoid.

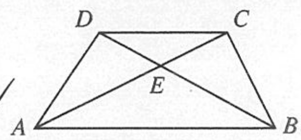
17. **Given:** $ABCD$ is a trapezoid.
$\overline{AB} \parallel \overline{CD}$
$\overline{AP}$ bisects $\angle A$.

Prove: $\triangle APD$ is isosceles.



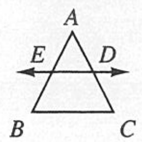
18. **Given:** $ABCD$ is a trapezoid.
$\overline{AB} \parallel \overline{CD}$, $\overline{AD} \cong \overline{BC}$
$\overline{AC}$ and $\overline{BD}$ intersect at E.

Prove: $\triangle CDE$ is isosceles.



19. **Given:** $\triangle ABC$ is isosceles with base $\overline{BC}$.
$\angle AED \cong \angle B$

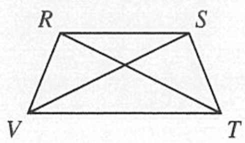
Prove: $BEDC$ is an isosceles trapezoid.



20. Prove Theorem 5.15

Given: $RSTV$ is a trapezoid with $\overline{RV} \cong \overline{ST}$.

Prove: $\overline{RT} \cong \overline{VS}$



Graph the given coordinates and determine if *ABCD* is an isosceles trapezoid.

21. $A(3, 3)$, $B(5, 3)$, $C(8, 1)$, $D(1, 1)$ **22.** $A(-4, 1)$, $B(-4, 6)$, $C(2, 4)$, $D(2, 3)$

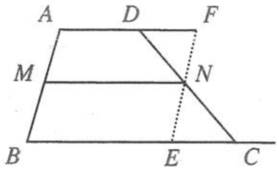
C

23. Prove Theorem 5.16.

Given: In trapezoid *ABCD*, *M* is a midpoint of leg $\overline{AB}$. *N* is a midpoint of leg $\overline{DC}$.

Prove: a. $\overline{MN} \parallel \overline{AD}$, $\overline{MN} \parallel \overline{BC}$ **b.** $MN = \frac{1}{2}(AD + BC)$

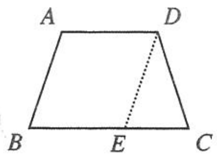
(HINT: Draw a line through *N* parallel to $\overline{AB}$. Prove that *AMNF* is a parallelogram.)



24. Prove Theorem 5.14 by constructing a line through *D* parallel to $\overline{AB}$. (NOTE: This is an alternate method of proving Theorem 5.15.)

Given: *ABCD* is an isosceles trapezoid with $\overline{AB} \cong \overline{CD}$.

Prove: $\angle B \cong \angle C$



25. Prove that if the base angles of a trapezoid are congruent, then the trapezoid is isosceles.

26. Prove that if *ABCDEF* is a regular hexagon, then *ABEF* is an isosceles trapezoid.

Critical Thinking

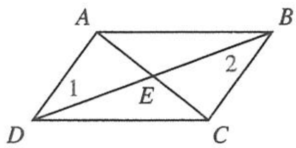
27. *P* is a point above the plane *M* determined by the equilateral triangle *ABC*. If a plane parallel to *M* cuts the segments $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$, how many trapezoids are formed? Where must *P* be located in order for the trapezoids to be isosceles?

Mixed Review

1. Find the sum of the measures of the angles of a regular hexagon.
2. Find the measure of each angle of a regular hexagon.
3. Find the measure of an exterior angle of a regular hexagon.

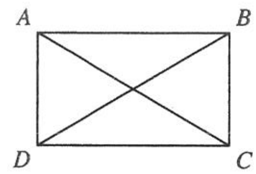
Quiz

***ABCD* is a parallelogram.**



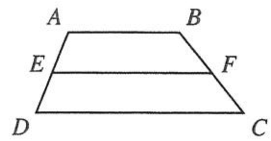
1. If $AD = 6$, find *BC*.
2. If $m\angle 1 = 32$, find $m\angle 2$.
3. If $m\angle ABC = 4x - 6$ and $m\angle BCD = 2x + 54$, find $m\angle CDA$.

Determine whether each statement is true or false.



4. If $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, then *ABCD* is a parallelogram.
5. If *ABCD* is a square, then $\overline{AC} \cong \overline{BD}$.
6. If *ABCD* is a rhombus and $m\angle ABC = 90$, then *ABCD* is also a square.

***ABCD* is an isosceles trapezoid with median $\overline{EF}$.**



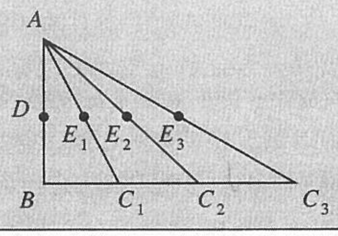
7. If $AB = 6$ and $DC = 18$, find *EF*.
8. If $AB = 2x$, $EF = 3x + 2$, and $DC = 24$, find *EF*.

OBJECTIVE: *Apply theorems about segments that join midpoints of sides of triangles and quadrilaterals.*

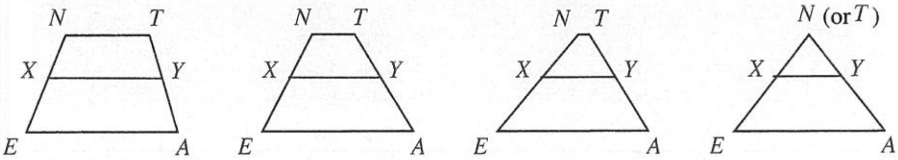
5-5 The Midsegment Theorem

EXPLORE

Draw right triangles ABC_1, ABC_2, ABC_3 with $BC_1 = 1$ in., $BC_2 = 2$ in., and $BC_3 = 3$ in. Let D, E_1, E_2, and E_3 be the midpoints of their respective segments. Compare the lengths of $\overline{DE_1}$, $\overline{DE_2}$, and $\overline{DE_3}$ with the lengths of $\overline{BC_1}$, $\overline{BC_2}$, and BC_3. State a conclusion about your findings.



Consider the trapezoids $NEAT$ and medians $\overline{XY}$ below. Note that in each figure the base $\overline{NT}$ gets smaller until finally the base is the point N (or T).



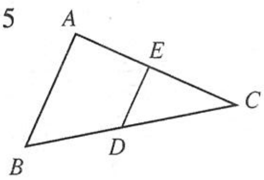
By Theorem 5.16, $\overline{XY} \parallel \overline{EA}$ and $XY = \frac{1}{2}(EA + NT)$. However, in the last case $\overline{NT}$ has length zero so $XY = \frac{1}{2}EA$. This suggests the next theorem. You will be asked to prove Theorem 5.17 in Exercise 31.

THEOREM 5.17 Midsegment Theorem

The segment that joins the midpoints of two sides of a triangle is parallel to the third side and has a length equal to half the length of the third side.

Example 1

D and E are midpoints of the sides of $\triangle ABC$. If $DE = x + 5$ and $AB = 4x - 16$, find the length of $\overline{AB}$.



Solution

$\frac{1}{2}(AB) = DE$ *Midsegment Theorem*

$\frac{1}{2}(4x - 16) = x + 5$

$2x - 8 = x + 5$

$x = 13$

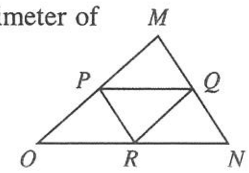
$AB = 4(13) - 16 = 36$

Try This

If D and E are midpoints and $AB + DE = 36$, find the lengths of $\overline{AB}$ and $\overline{DE}$.

Example 2

P, Q, and R are midpoints of the sides of $\triangle MNO$. Find the perimeter of $\triangle PQR$ if $MN = 12$, $NO = 16$, and $OM = 14$.



Solution

$PQ = \frac{1}{2}(\text{ON}) = \frac{1}{2}(16) = 8$ *Midsegment Theorem*

$QR = \frac{1}{2}(\text{OM}) = \frac{1}{2}(14) = 7$

$RP = \frac{1}{2}(\text{MN}) = \frac{1}{2}(12) = 6$

Perimeter $\triangle PQR = PQ + QR + RP$
$= 8 + 7 + 6$
$= 21$ The perimeter of $\triangle PQR$ is 21.

Try This

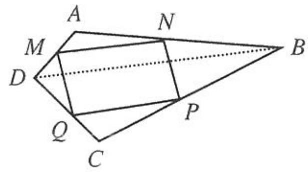
P, Q, and R are midpoints of the sides of $\triangle MNO$. If $MP = 3$, $QN = 5$ and $NR = 7$, find the perimeter of $\triangle PQR$.

The following theorem can be proved by using the Midsegment Theorem.

THEOREM 5.18

The quadrilateral formed by joining the midpoints of the consecutive sides of another quadrilateral is a parallelogram.

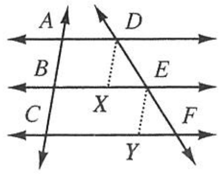
Given: M, N, P, and Q are midpoints of the sides of quadrilateral $ABCD$.
Prove: $MNPQ$ is a parallelogram.



Plan Draw $\overline{BD}$, forming triangles ABD and CBD. Prove that $MN = \frac{1}{2}BD$ and $\overline{MN} \parallel \overline{BD}$ by the Midsegment Theorem. Prove that $QP = \frac{1}{2}BD$ and $\overline{QP} \parallel \overline{BD}$ by the Midsegment Theorem. Conclude that $MN = QP$ and $\overline{MN} \parallel \overline{QP}$, and hence $MNPQ$ is a parallelogram.

THEOREM 5.19

If three or more parallel lines cut off congruent segments on one transversal, then they will cut off congruent segments on every transversal.



You will be asked to prove Theorem 5.19 in Exercise 35.

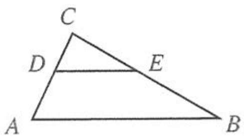
COROLLARY 5.19a

A line that contains the midpoint of one side of a triangle and is parallel to another side bisects the third side.

Class Exercises

Short Answer

***D* and *E* are midpoints of the sides of $\triangle ABC$. Complete each statement.**



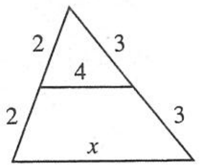
1. If $AB = 8$, then $DE =$ ___. **2.** If $AC = 9$, then $AD =$ ___.

3. If $BE = 5$, then $BC =$ ___. **4.** If $AB = 15$, then $DE =$ ___.

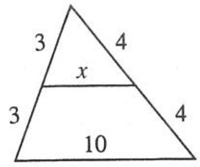
Sample Exercises

Find the value of x or state *cannot determine*.

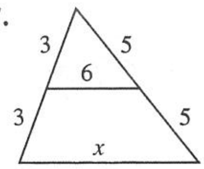
5.



6.

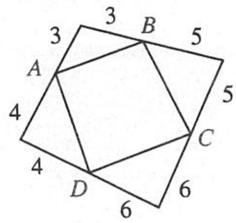


7.

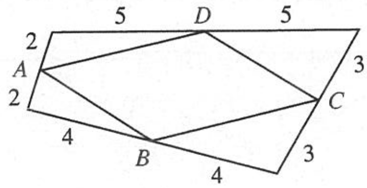


Determine whether Theorem 5.18 allows you to conclude that ABCD is a $\square$.

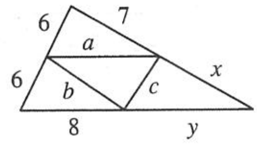
8.



9.



Discussion

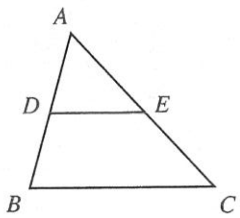


10. Determine values for x and y so that only one of the segments a, b, and c can be determined. Explain your choices.

Exercises

A

***D* and *E* are midpoints of the sides of $\triangle ABC$. Complete each statement.**



1. $DE = 10$, $BC =$ ___ **2.** $BC = 12$, $DE =$ ___ **3.** $DE = 8$, $BC =$ ___

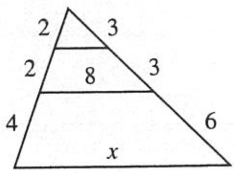
4. $DE = 2x$, $BC = x + 10$, $DE =$ ___

5. $DE = x - 4$, $BC = x + 12$, $BC =$ ___

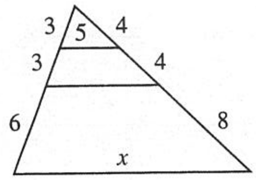
6. $DE = x + 2$, $BC = \frac{1}{2}x + 19$, $DE =$ ___

Find the value of x.

7.



8.



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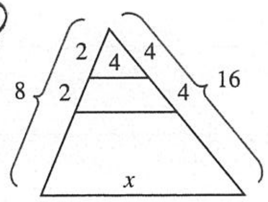
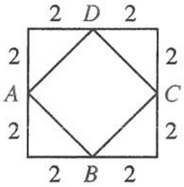


Determine whether Theorem 5.18 allows you to conclude that *ABCD* is a parallelogram.

10.



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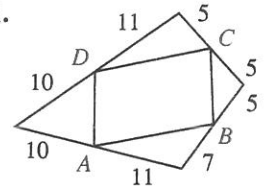

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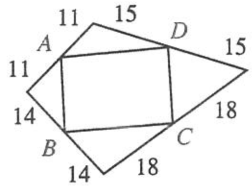
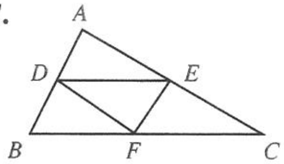


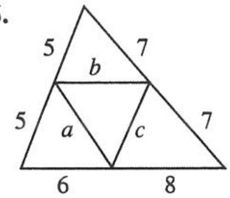
***D*, *E*, and *F* are midpoints of the sides of $\triangle ABC$. Find the perimeter of $\triangle DEF$.**



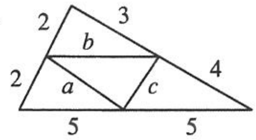
13. $AB = 20$, $BC = 36$, $AC = 24$
14. $AB = 15$, $BC = 25$, $AC = 17$
15. $AD = 3$, $BF = 6$, $EC = 4$

Exactly one of the values *a*, *b*, or *c* can be determined. Find it.

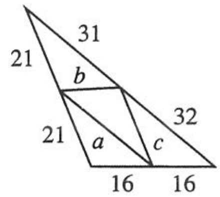
16.



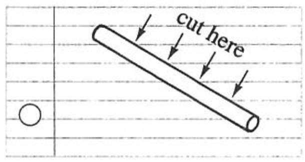
17.



18.



19. If the dowel is cut where the diagram indicates, will it be divided into five equal lengths? Why or why not?

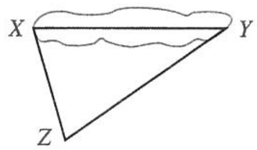


B

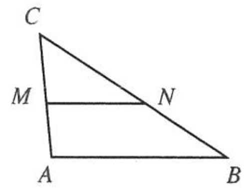
Determine the coordinates of the midpoint *X* of side $\overline{AB}$ and the midpoint *Y* of side $\overline{AC}$ of $\triangle ABC$. Find *XY* and *BC* and verify that $XY = \frac{1}{2}BC$.

20. $A(0, 0)$, $B(8, 0)$, $C(0, 4)$ **21.** $A(0, 5)$, $B(0, 1)$, $C(-8, 5)$
22. $A(-2, -1)$, $B(2, -1)$, $C(-2, -7)$

23. Explain how the distance across the pond (*XY*) can be found by using $\triangle ZXY$ where *Z* is any point other than *X* or *Y*. (HINT: Consider the midpoints of $\overline{XZ}$ and $\overline{ZY}$.)

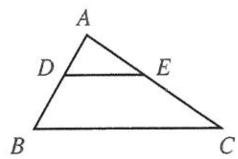


***M* and *N* are midpoints of sides $\overline{AC}$ and $\overline{BC}$.**



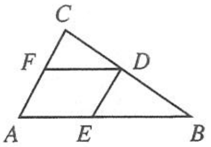
24. If $MN = x^2 + x - 8$ and $AB = x^2 + x + 14$, find *MN* and *AB*.
25. If $AM = x + 5$, $MC = 2y + 6$, $MN = 2x - 5$, and $AB = y + 8$, find *MN* and *AB*.

***D* is the midpoint of $\overline{AB}$ and $\overline{DE} \parallel \overline{BC}$.**



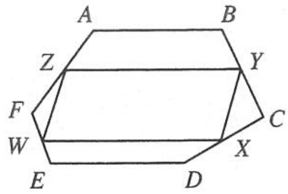
26. Find *AC* if $AE = 2x^2 - 4x$ and $EC = 2x + 20$.
27. Find *EC* if $AE = 3x + 4$ and $AC = x + 20$.

28. **Given:** F is the midpoint of $\overline{AC}$.
D is the midpoint of $\overline{BC}$.
E is the midpoint of $\overline{AB}$.
Prove: $AEDF$ is a parallelogram.

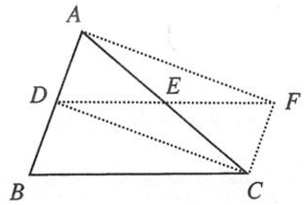


29. **Given:** $\overline{AB} \cong \overline{AC}$
E is the midpoint of $\overline{AB}$.
D is the midpoint of $\overline{CB}$.
Prove: $\triangle BDE$ is isosceles.

30. **Given:** $ABCDEF$ is a hexagon.
$\overline{AB} \parallel \overline{DE}$, $\overline{AB} \cong \overline{DE}$
W, X, Y, and Z are midpoints.
Prove: $WXYZ$ is a parallelogram.
(HINT: Draw $\overline{AE}$ and $\overline{BD}$.)

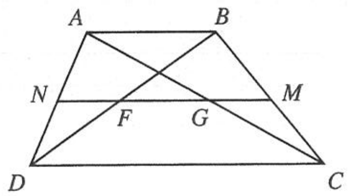


31. Prove Theorem 5.17.
Given: D and E are midpoints.
Prove: a. $\overline{DE} \parallel \overline{BC}$ **b.** $DE = \frac{1}{2}BC$
(HINT: Extend $\overline{DE}$ until $\overline{DE} \cong \overline{EF}$ as shown in the figure to the right. Prove $ADCF$ is a parallelogram. The prove $DBCF$ is a parallelogram.)

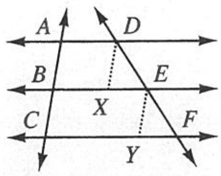


C

32. If the length of each diagonal of a given rectangle is ten, find the perimeter of the quadrilateral formed by joining the midpoints of the sides of the rectangle.
33. $ABCD$ is a trapezoid with diagonals $\overline{AC}$ and $\overline{BD}$ and median $\overline{MN}$. Find FG if $MN = 27$, $AB = 7x$ and $CD = 4x^2 - x$.

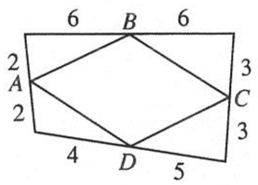


34. Given a regular pentagon $ABCDE$ in which M is the midpoint of side $\overline{AB}$ and N is the midpoint of side $\overline{BC}$, prove $\overline{MN} \parallel \overline{ED}$.
35. Prove Theorem 5.19.
Given: $\overleftrightarrow{AD} \parallel \overleftrightarrow{BE} \parallel \overleftrightarrow{CF}$, $\overline{AB} \cong \overline{BC}$
Prove: $\overline{DE} \cong \overline{EF}$
(HINT: Draw $\overline{DX}$ and $\overline{EY}$ parallel to $\overline{AC}$.)



Critical Thinking

36. Draw several isosceles trapezoids and connect the midpoints of the sides. In each case, what kind of quadrilateral is formed? Write a theorem that describes your discovery. Then write a paragraph proof.
37. A student studied the figure to the right and concluded that $ABCD$ is not a parallelogram, giving Theorem 5.19 as the reason for this conclusion. Do you agree? Why or why not?



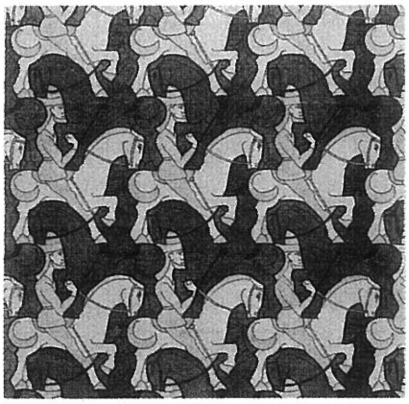
Algebra Review

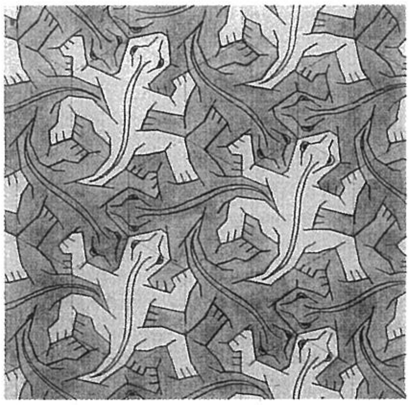
Solve.

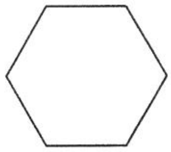
1. $x + 7 > 9$
2. $y + 12 < 6$
3. $2a + 3 - a > 5$
4. $b - 10 > -16$
5. $6x > 72$
6. $9a < 81$

Enrichment

Tessellations





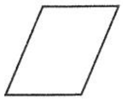


Regular hexagon

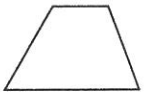
M. C. Escher, a Dutch graphic artist, created more than 150 drawings using tessellations. A **tessellation** is an arrangement of polygonal shapes that cover a plane such that no polygons overlap and there are no gaps. A question of interest to mathematicians is, "Which shapes or combinations of shapes tessellate the plane?" It is easy to show that a square tessellates the plane by using tracing paper to repeatedly trace the square until the entire sheet of paper is covered. As shown to the right, a plane can also be tessellated by a regular hexagon.

Exercises

Determine which of the polygons shown tessellate the plane.

1. 

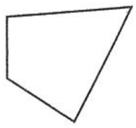
Parallelogram

2. 

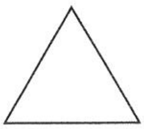
Trapezoid

3. 

Kite

4. 

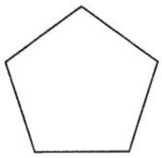
Any quadrilateral

5. 

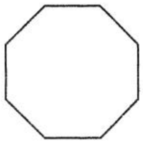
Equilateral triangle

6. 

Nonconvex quadrilateral

7. 

Regular pentagon

8. 

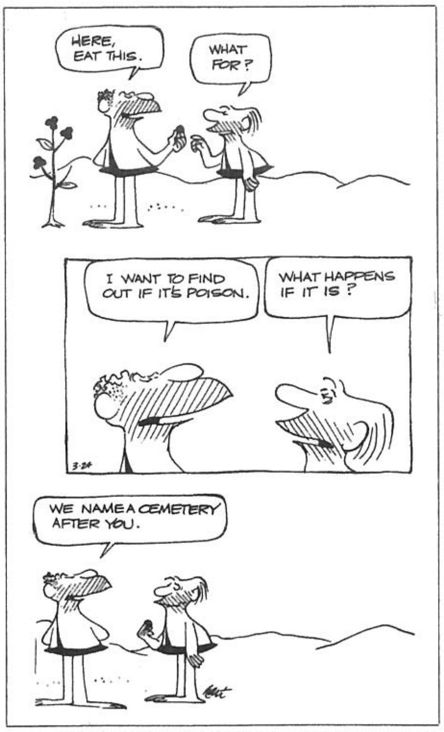
Regular octagon

9. Cut out several copies each of an equilateral triangle, a square, a regular hexagon, and a regular octagon so that each side of each polygon has the same length. Find several combinations of two or more of these polygons that will tessellate the plane.

OBJECTIVE: *Use the indirect method of proof.*

5-6 Indirect Proof

In some cases it is easier to use indirect reasoning rather than to use direct reasoning as you have been doing in the proofs you have written so far. Indirect reasoning involves assuming the opposite of what you want to conclude. You then reason logically until you reach a **contradiction** involving two conflicting statements.



B.C. by permission of Johnny Hart and Field Enterprises, Inc.

Indirect reasoning is illustrated by what might happen in this cartoon—if B.C. cooperates! Suppose B.C. eats the berry but does not die. The following illustrates the use of indirect reasoning to conclude that the berry is not poison.

Suppose the berry is poison.
If B.C. eats it, he will die.
B.C. ate it but didn't die.
Therefore, the berry is not poison!

In an **indirect proof** you use indirect reasoning to establish a statement or theorem. An important part of an indirect proof is assuming the negation of a statement. The **negation** of a statement is the opposite of the original statement. In an indirect proof you first assume the negation of what you want to prove. Then you reason logically until you have shown that the assumption leads to a contradiction involving two conflicting statements.

Example 1

State the negation of each statement.

a. $m\angle 1 = m\angle 2$
b. $\overline{AB} \not\perp \overline{CD}$ (NOTE: $\not\perp$ means "is not perpendicular to.")

Solution

a. $m\angle 1 \neq m\angle 2$ **b.** $\overline{AB} \perp \overline{CD}$

Try This

State the negation of "$\overline{AB}$ intersects $\overline{CD}$."

Another part of an indirect proof is reaching a contradiction. The following pairs of statements each form a contradiction since they cannot both be true.

$\overline{XY}$ is longer than $\overline{MN}$. $\overline{MN}$ is longer than $\overline{XY}$.

$\angle A$ and $\angle B$ are right angles. $\angle A$ and $\angle B$ are not congruent angles.

Example 2

Determine which pairs of statements form a contradiction.

a. $\overline{AB} \parallel \overline{CD}$
$\overline{AB}$ and $\overline{CD}$ are perpendicular.

b. In $\triangle ABC$, $m\angle A = m\angle B$.
$\triangle ABC$ is scalene.

c. *ABCD* is a rhombus.
ABCD is a square.

Solution

Statements in **a** and **b** form a contradiction.

Try This

Which two of the following three statements form a contradiction?

a. $\triangle ABC$ is isosceles. **b.** $\triangle ABC$ is equilateral. **c.** *ABC* is a right triangle.

The steps for indirect proof are summarized below.

Steps for Using Indirect Proof

Step 1 Assume the negation of the *Prove* statement.

Step 2 Reason logically to show that the assumption leads to the contradiction of a known fact (theorem, definition, postulate, given information, etc.).

Step 3 Conclude that the assumption is false and that the *Prove* statement is true.

An indirect proof can be written in paragraph form, as in Examples 3 and 4.

Example 3

Given: $\angle 1$ and $\angle 2$ are not congruent.

Prove: $\angle 1$ and $\angle 2$ are not vertical angles.

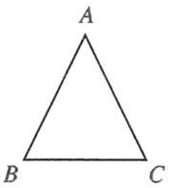
Proof Assume that $\angle 1$ and $\angle 2$ are vertical angles. Then $\angle 1 \cong \angle 2$ by the Vertical Angle Theorem. This statement contradicts the given information which says that $\angle 1 \not\cong \angle 2$. Therefore, this assumption must be false. It follows that $\angle 1$ and $\angle 2$ are not vertical angles.

Example 4 presents an indirect proof for the statement: The base angles of an isosceles triangle must be acute.

Example 4

Given: $AB = AC$

Prove: $m\angle B < 90$



Proof Assume $m\angle B \geq 90$. Since $AB = AC$ then $m\angle B = m\angle C$ by the Isosceles Triangle Theorem. Therefore, $m\angle C \geq 90$. By addition, $m\angle B + m\angle C \geq 180$. This contradicts the fact that the sum of the measures of the angles of a triangle is 180. The assumption that $m\angle B = 90$ must be false. Therefore, $m\angle B < 90$.

Class Exercises

Short Answer

State the negation of each statement.

1. The sun is shining.
2. It will not rain.
3. $m \perp n$
4. The adjacent sides are not parallel.
5. $\overline{AB} \cong \overline{CD}$
6. $\angle A$ is not acute.
7. $\triangle ABC$ is an isosceles triangle.
8. $\triangle ABC$ is congruent to $\triangle DEF$.

Sample Exercises

Identify which two of the three statements, a, b, and c, form a contradiction.

9. a. $\triangle ABC$ is equilateral.
b. $\triangle ABC$ is acute.
c. $\triangle ABC$ is scalene.

10. a. $\angle A \cong \angle B$
b. $\angle A$ and $\angle B$ are supplementary.
c. $\angle A$ and $\angle B$ are complementary.

11. a. $\angle A$ and $\angle B$ are adjacent angles.
b. $\angle A$ and $\angle B$ form a linear pair of angles.
c. $\angle A$ and $\angle B$ are acute angles.

12. a. $\angle A$ and $\angle B$ are supplementary.
b. $\angle A$ and $\angle B$ are both acute angles.
c. $\angle A$ and $\angle B$ are vertical angles.

13. a. Lines p and q are not parallel.
b. Lines p and q have no points in common.
c. Lines p and q lie in the same plane and have no points in common.

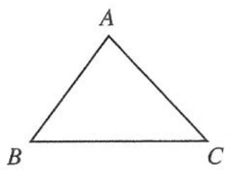
14. a. $\angle A$ and $\angle B$ are not right angles.
b. $\angle A$ and $\angle B$ are congruent angles.
c. $m\angle A \neq m\angle B$

Discussion

Discuss how an indirect proof could be used to prove the following.

15. Given: $m\angle B \neq m\angle C$
Prove: $AB \neq AC$
(HINT: Begin by assuming the negation $AB = AC$.)

16. Given: $AB > AC$
Prove: $\triangle ABC$ is not equiangular.



Exercises

A

Write the negation you would use to begin an indirect proof.

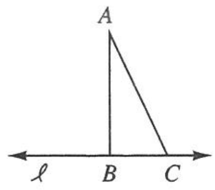
1. Prove: $\angle A$ is supplementary to $\angle B$.
2. Prove: $\angle A$ is not a right angle.
3. Prove: $\angle A$ and $\angle B$ are not vertical angles.
4. Prove: There is at most one line through P that is parallel to m.
5. Prove: ABC is not an equilateral triangle.

Indicate whether each pair of statements would enable you to arrive at a contradiction in an indirect proof.

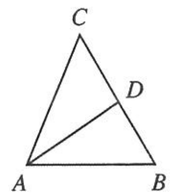
6. Lines p and q are parallel.
   Lines p and q do not intersect.
7. $\angle A \cong \angle B$
   $m\angle A > m\angle B$
8. $m \perp n$
   $m \not\perp n$
9. $\angle A$ and $\angle B$ form a linear pair.
   $m\angle A < 90$, $m\angle B < 90$
10. $\angle A$ and $\angle B$ are congruent.
    $\angle A$ and $\angle B$ are supplementary.
11. $\angle A$ and $\angle B$ are obtuse angles.
    $\angle A$ and $\angle B$ are supplementary.
12. Arrange statements **a–f** in the correct order to write the indirect proof in paragraph form.

    Given: $\overline{AB} \perp \ell$

    Prove: Any other line $\overline{AC}$ is not perpendicular to ℓ.

    a. $m\angle ABC = 90$
    b. Contradiction of Angle Sum Theorem for Triangles; therefore, $\overline{AC} \not\perp \ell$.
    c. $\overline{AB} \perp \ell$ **d.** $m\angle ABC + m\angle ACB = 180$
    e. Assume $\overline{AC} \perp \ell$. **f.** $m\angle ACB = 90$
13. Complete the following indirect proof.

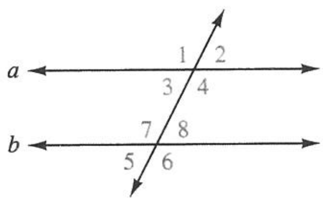
    

    Given: In $\triangle ABC$, $AC \neq AB$, D is the midpoint of $\overline{BC}$.

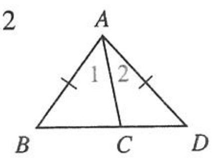
    Prove: $\overline{AD} \not\perp \overline{BC}$

    Proof Assume $\overline{AD} \perp \overline{BC}$. Then $\angle ADC \cong \angle ADB$. Since D is the midpoint of $\overline{BC}$, then $\overline{CD} \cong \overline{BD}$. By the Reflexive Property, $\overline{AD} \cong \overline{AD}$. Therefore, $\triangle ADC \cong \triangle ADB$.
    (HINT: Obtain a statement that contradicts a statement in the given information.)

Write an indirect proof in paragraph form.

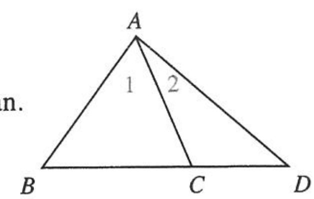


14. **Given:** $\angle 4 \ncong \angle 6$
    Prove: $a \nparallel b$
15. **Given:** $a \nparallel b$
    Prove: $\angle 3 \ncong \angle 5$
16. **Given:** $\angle 1 \ncong \angle 8$
    Prove: $\angle 4 \ncong \angle 5$
17. **Given:** $\overline{AB} \cong \overline{AD}$, $\angle 1 \ncong \angle 2$
    Prove: $\overline{BC} \ncong \overline{CD}$

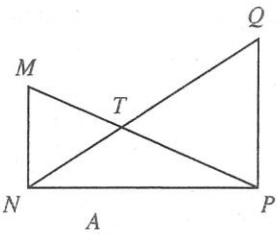


B

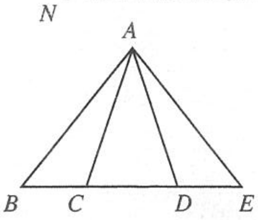
18. **Given:** $\overline{AB} \cong \overline{AD}$, $\overline{BC} \ncong \overline{CD}$
    Prove: $\angle 1 \ncong \angle 2$
19. **Given:** $\triangle ABD$ is scalene. $\overline{AC}$ is a median.
    Prove: $\overline{AC} \not\perp \overline{BD}$
20. **Given:** $\overline{AC}$ bisects $\angle BAD$. $\overline{AC} \not\perp \overline{BD}$
    Prove: $AB \neq AD$



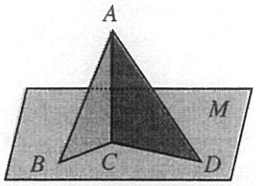
21. **Given:** $\overline{MN} \perp \overline{NP}$
$\overline{QP} \perp \overline{NP}$
$\angle M \not\cong \angle Q$
Prove: $\overline{MN} \not\cong \overline{QP}$



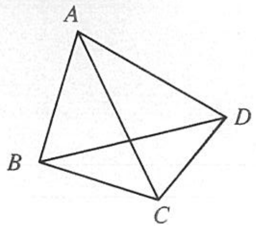
22. **Given:** $AB = AE$
$AC \neq AD$
Prove: $BC \neq DE$



23. **Given:** $\overline{AC} \perp M$
$\angle B \not\cong \angle D$
Prove: $\angle BAC \not\cong \angle DAC$



24. **Given:** $\triangle ABC$ is equilateral.
$\triangle DAC$ is equilateral.
$\triangle ABD$ is not equilateral.
Prove: $\triangle BCD$ is not equilateral.



C

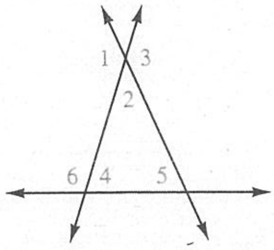
25. Write an indirect proof in paragraph form to prove the following statement. If a line is perpendicular to one side of an angle, it is not perpendicular to the other side of the angle.
26. Write an indirect proof in paragraph form to prove the following statement. Two lines perpendicular to the same plane do not intersect.
27. Write an indirect proof in paragraph form to prove the following statement. If point D is not in the same plane as $\triangle ABC$, then $\overline{BD}$ and $\overline{AC}$ are skew lines.

Critical Thinking

28. A store owner claimed that an expensive radio had been stolen from her store. She was convinced that either Anna, Brenda, Carlos, or Dee had stolen the radio. Each person made a statement but only one of the four statements was true.
Anna said, "I didn't take it."
Brenda said, "Anna is lying."
Carlos said, "Brenda is lying."
Dee said, "Brenda took it."
Who told the truth? Who took the radio? Use indirect reasoning and write a paragraph proof to justify your choice.

Mixed Review

Find the measure of the indicated angle.



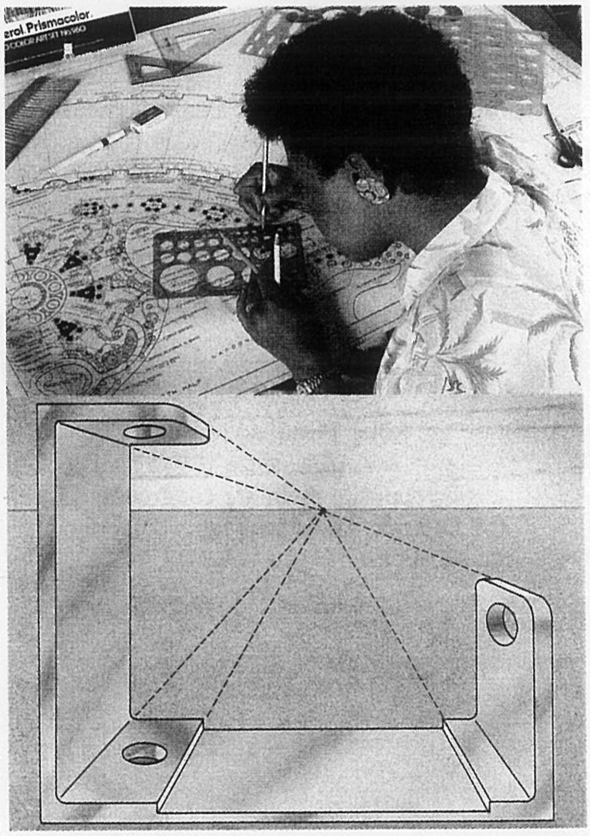
1. If $m\angle 4 = 50$ and $m\angle 5 = 75$, find $m\angle 1$.
2. If $m\angle 6 = 135$ and $m\angle 2 = m\angle 5$, find $m\angle 2$.
3. If $m\angle 3 = 120$ and $m\angle 4$ is twice $m\angle 5$, find $m\angle 4$.
4. If $m\angle 1$ is 160 and $m\angle 5$ is three times $m\angle 4$, find $m\angle 5$.

Determine whether each statement is always, sometimes, or never true. Draw a figure to illustrate your answer.

5. Two adjacent angles form a linear pair.
6. Adjacent angles are congruent.
7. Vertical angles are complementary.
8. An angle and its complement are congruent.
9. Vertical angles are congruent.
10. Two acute angles are supplementary.

Career

Technical Illustrator



The professional artist who prepares drawings that show how machines, tools, and equipment operate is a technical illustrator. Working from blueprints, designs, mockups, and photoprints, he or she may draw an object from different points of view to show its function, its relationship to other equipment, or the assembly sequence of its parts.

One skill technical illustrators develop is drawing in perspective. Using congruent triangles, the artist produces a two-dimensional representation in which parallel lines appear to converge at some distant point. He or she may also draw from schematic, orthographic, or oblique-angle views. Color or shading with ink, crayon, airbrush, or overlays helps emphasize details or eliminate unwanted backgrounds.

The artist may include instructions, comments, or cartoons to help describe how the object works. A technical illustrator uses drafting or optical equipment, photo-offset techniques, and projections transparencies. His or her work appears in reference or technical manuals, brochures, and safety manuals and on posters. Related professions include engineering and production illustration.

OBJECTIVE: *Apply the Exterior Angle Inequality Theorem and theorems involving inequalities in one triangle.*

5-7 Inequalities in One Triangle

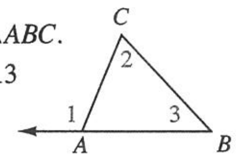
Many theorems concern equality of measures. This lesson deals with inequalities. The first theorem is related to the Exterior Angle Theorem.

◆ **THEOREM 5.20** Exterior Angle Inequality Theorem

The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.

Given: $\angle 1$ is an exterior angle of $\triangle ABC$.

Prove: $m\angle 1 > m\angle 2$, $m\angle 1 > m\angle 3$



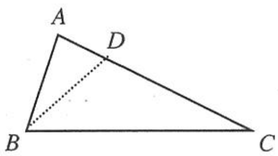
Plan By the Exterior Angle Theorem for Triangles, $m\angle 1 = m\angle 2 + m\angle 3$. It follows that $m\angle 1 > m\angle 2$ and $m\angle 1 > m\angle 3$.

◆ **THEOREM 5.21**

If one side of a triangle is longer than a second side, then the measure of the angle opposite the longer side is greater than the measure of the angle opposite the shorter side.

Given: $BC > AB$

Prove: $m\angle A > m\angle C$



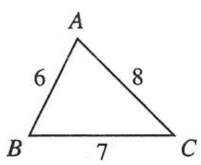
Proof Use the Ruler Postulate to select D on $\overline{AC}$ so $AB = BD$. Then $m\angle A = m\angle ABD$, since if two sides of a triangle are congruent, the angles opposite them are congruent. Therefore $m\angle ADB > m\angle C$ by the Exterior Angle Inequality Theorem. It follows that $m\angle A > m\angle C$ by substitution.

Example 1

Name the largest angle of $\triangle ABC$.

Solution

$\angle B$ *Theorem 5.21*



Try This

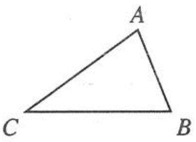
Name the smallest angle of $\triangle ABC$.

◆ THEOREM 5.22

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

Given: $m\angle A > m\angle C$

Prove: $BC > AB$



Proof Assume $BC = AB$. It follows that $m\angle A = m\angle C$. This contradicts the given information. Assume $AB > BC$. By Theorem 5.22, $m\angle C > m\angle A$. This contradicts the given information. Therefore, $BC > AB$.

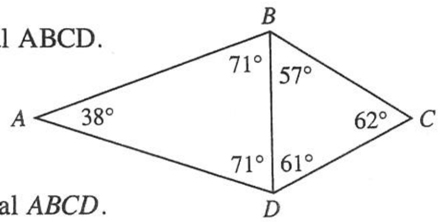
▶ COROLLARY 5.22a

The perpendicular segment from a point to a line is the shortest segment from the point to the line.

You will be asked to prove Corollary 5.22a in Exercise 33.

Example 2

Name the longest side of quadrilateral ABCD.



Solution

$\overline{AB}$ (or $\overline{AD}$) *Theorem 5.22*

Try This

Name the shortest side of quadrilateral *ABCD*.

You will be asked to prove Theorem 5.23 in Exercise 41.

◆ THEOREM 5.23 Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Example 3

Which numbers could represent the lengths of the sides of a triangle?

a. 3, 4, 6 **b.** 10, 11, 21 **c.** 2, 6, 9 **d.** 34, 35, 36

Solution

a and **d** *Triangle Inequality Theorem*

Try This

Two sides of a triangle measure 5 and 13. Which could be the measure of the third side?

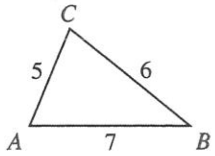
a. 16 **b.** 8 **c.** 9 **d.** 19

Class Exercises

Short Answer

Name the largest angle and the smallest angle for each figure.

1.



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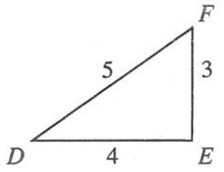
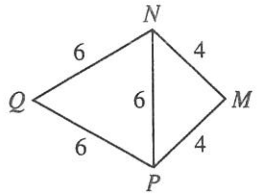
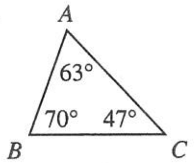


3.

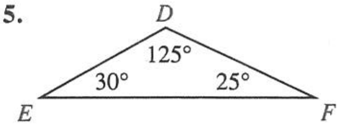


Name the longest side and the shortest side for each figure.

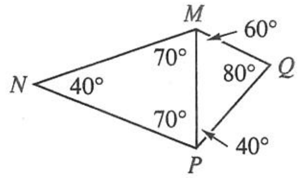
4.



5.

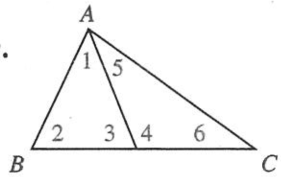


6.



Sample Exercises

Decide which angle is the larger based on the Exterior Angle Inequality Theorem or state that it is not possible to determine which angle is larger.



7. $\angle 3$ and $\angle 6$ **8.** $\angle 4$ and $\angle 1$ **9.** $\angle 3$ and $\angle 1$ **10.** $\angle 2$ and $\angle 4$

11. If the lengths of two sides of a triangle are 6 and 12, what are the possible whole numbers for the length of the third side?

12. If two sides of a parallelogram have lengths 7 and 11, then the lengths of the diagonals must be greater than ___?___ and less than ___?___.

13. In $\triangle ABC$, $m\angle A = 30$, $m\angle B = 50$, and $m\angle C = 100$. List the sides of the triangle from the shortest to the longest.

14. In $\triangle DEF$, $DE = 5$, $EF = 6$, and $DF = 8$. List the angles of the triangle from the largest to the smallest.

Discussion

15. Which side of a right triangle must be the longest side? Explain.

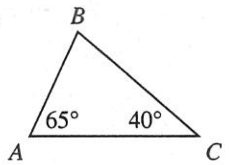
16. Is the base of an isosceles triangle always the shortest side? Explain.

Exercises

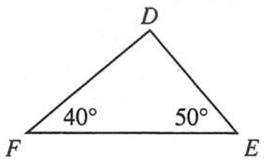
A

List the sides of each figure from the shortest to the longest.

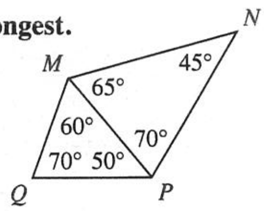
1.



2.



3.



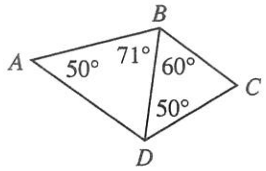
List the sides of $\triangle ABC$ from the shortest to the longest.

4. $m\angle A = 46, m\angle B = 30$ **5.** $m\angle C = 101, m\angle B = 70$
6. $m\angle A = 59, m\angle C = 61$ **7.** $m\angle B = 48, m\angle A = 47$

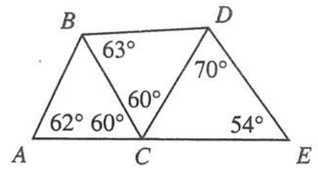
List the angles of $\triangle ABC$ from the smallest to the largest.

8. $AB = 17, BC = 21, AC = 18$ **9.** $AB = 15, AC = 16, BC = 17$

10. List the sides of quadrilateral $ABCD$ from the shortest to the longest.



11. List all the segments in the figure from the shortest to the longest.



Which numbers could represent the lengths of the sides of a triangle?

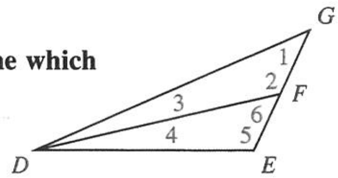
12. 10, 20, 30 **13.** 10, 8, 6 **14.** 5, 14, 7
15. 4, 9, 15 **16.** 6, 6, 11 **17.** 1, 3, 5

18. If the sum of the lengths of two sides of a triangle is 15, what is the largest possible integral value for the third side?

19. If the base of an isosceles triangle is 10, what is the shortest possible integral value for each of the equal sides?

20. If the perimeter of a triangle is 8 and the lengths of the sides are integers, what is the length of the shortest side?

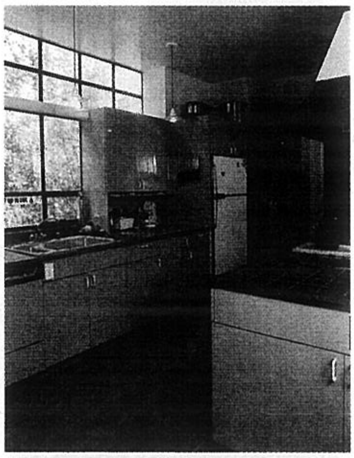
Use the Exterior Angle Inequality Theorem to determine which angle is larger or state *not possible*.



21. $\angle 1$ and $\angle 6$ **22.** $\angle 1$ and $\angle 2$
23. $\angle 2$ and $\angle 4$ **24.** $\angle 5$ and $\angle 2$

25. The three main work centers in the kitchen are the refrigerator, the sink, and the stove. You could picture them as the points of a triangle. It is recommended that the three sides of the kitchen triangle add up to more than 12 ft and less than 22 ft, with the shortest side of the triangle between the sink and the stove. Decide whether each kitchen triangle in the chart below is possible. Then state whether it follows the recommendation.

	a	*b*	*c*	*d*	*e*
stove/sink	5 ft	10 ft	6 ft	3 ft	3 ft
stove/refrigerator	4 ft	11 ft	8 ft	7 ft	8 ft
refrigerator/sink	8 ft	8 ft	7 ft	4 ft	4 ft



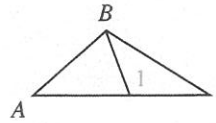
B

26. If the lengths of two sides of a triangle are 6 and p, where p is a whole number, give the possible whole numbers for the length of the third side.

27. If the longer side of a parallelogram has a length x, where x is a whole number, and the longer diagonal has a length of 10, give the possible whole numbers for the length of the shorter side of the parallelogram.

28. **Given:** $AB = BC$

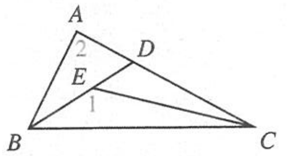
Prove: $m\angle 1 > m\angle C$



29. **Given:** E is in the interior of $\angle BAC$:

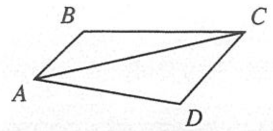
Prove: $m\angle 1 > m\angle 2$

(HINT: Compare $m\angle 1$ and $m\angle 2$ with $m\angle EDC$.)



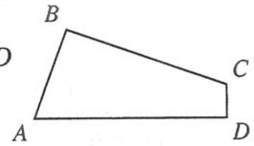
30. **Given:** quadrilateral $ABCD$

Prove: $AB + BC + CD + DA > 2(AC)$

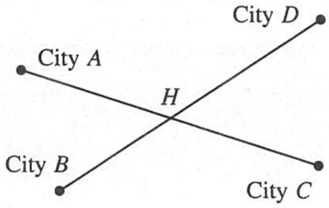


31. **Given:** quadrilateral $ABCD$

Prove: $AB + BC + CD > AD$



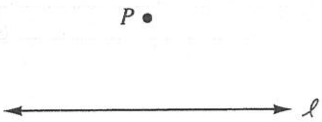
32. A railroad company is to build an engine house to service four cities located at the vertices $ABCD$ of a quadrilateral as shown here. Where should the engine house H be located so that the length, and hence the construction costs, of the roadbed $AH + BH + CH + DH$ is as small as possible?



33. Use the figure to the right to prove Corollary 5.22a.

Given: Line ℓ with P not on ℓ, $\overline{PQ} \perp \ell$

Prove: $\overline{PQ}$ is the shortest segment from P to ℓ.

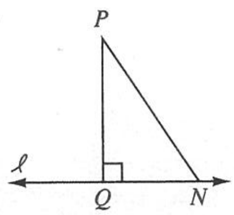


34. Use the figure to the right to prove Theorem 2.9.
Only one perpendicular can be drawn from a point not on a line to a line.

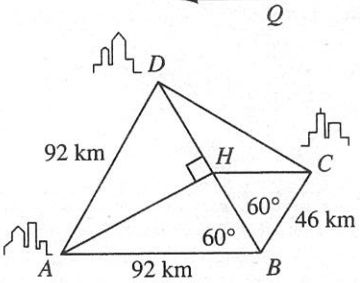
Given: Line ℓ and point P not on ℓ

Prove: **a.** There is one line through P perpendicular to ℓ.

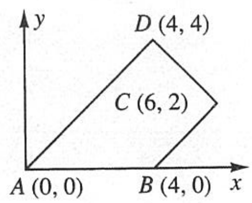
b. There is only one line through P perpendicular to ℓ.



35. Cities are located at A, B, C, and D as shown in the figure to the right. Railroad tracks are to be laid from H to each of the cities. What is the total length of track needed? (HINT: The length of an altitude of an equilateral triangle is $\frac{\sqrt{3}}{2}$ times the length of a side.) Where should H be located so that the total length of the tracks is the least possible?



36. Suppose cities are located at points A, B, C, and D as shown on the coordinate grid. Where is a point H located if $AH + BH + CH + DH$ is a minimum? What is the value of this minimum? (Use the distance formula and a calculator.)



Determine the largest angle of $\triangle ABC$ for the given coordinates of A, B, and C.

37. $A(0, 0)$, $B(5, 0)$, $C(1, 5)$

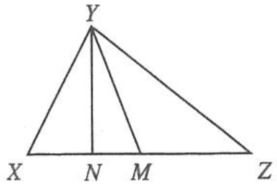
38. $A(-1, 2)$, $B(-2, -1)$, $C(2, 0)$

39. $A(2, -5)$, $B(4, -1)$, $C(7, -4)$

40. Use the figure to the right to prove the statement. The median $\overline{YM}$ from vertex Y of scalene $\triangle XYZ$ is longer than the altitude $\overline{YN}$ from vertex Y.

Given: $\triangle XYZ$ is scalene. $\overline{YN}$ is an altitude. $\overline{YM}$ is a median.

Prove: $YM > YN$

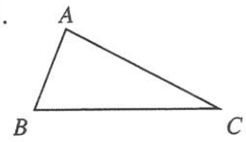


C

41. Use the figure to the right to prove Theorem 5.23.

Given: $\triangle ABC$

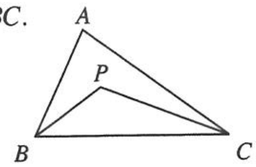
Prove: **a.** $AB + BC > AC$
b. $BC + AC > AB$
c. $AB + AC > BC$



(HINT: Assume one of the sides is the longest. Two of the statements are then easily established. To prove the third statement, drop a perpendicular from the appropriate vertex to the opposite side.)

42. **Given:** P is in the interior of $\triangle ABC$.

Prove: $PB + PC < AB + AC$

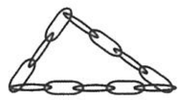


43. Prove that the sum of the lengths of the three altitudes of $\triangle ABC$ is less than the sum of the lengths of the sides of the triangle.

44. Prove that the sum of the lengths of the medians of $\triangle ABC$ is greater than half the perimeter.

Critical Thinking

45. Suppose a triangle is made of 12 chain linkages. List the lengths of the sides of possible triangles. Assume each link is one unit.



Algebra Review

Find all values of x.

1. $\lvert x\rvert = 7$	**2.** $\lvert -x\rvert = -7$	**3.** $\lvert -x\rvert = 7$
4. $x = \lvert 7\rvert$	**5.** $2\lvert x\rvert = 7$	**6.** $x = \lvert -7\rvert$
7. $\lvert x\rvert + 1 = 7$	**8.** $\lvert x\rvert + \lvert -1\rvert = 7$	**9.** $x - \lvert -1\rvert = 7$
10. $-2\lvert x\rvert = -14$	**11.** $\lvert 2x + 1\rvert = 14$	**12.** $\lvert x - 1\rvert - 1 = 7$

Translate to an equation and solve.

13. The product of two consecutive integers is 132. Find the integers

14. The product of two consecutive even integers is 168. Find the integers.

15. The square of a number is six more than the number. Find the number.

16. Twice the square of a number is 10 more than the number. Find the number.

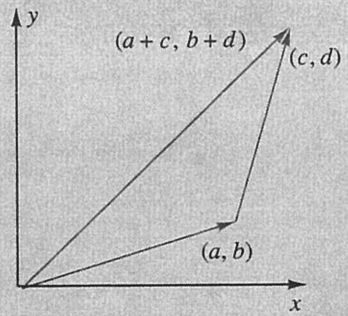
17. Twice the square of a number plus one is 73. Find the number.

18. The square of a number minus twice the number is 48. Find the number.

CONNECTIONS

Geometric and Algebraic Inequalities

The Triangle Inequality Theorem, one of the most famous in geometry, has an important counterpart with the same name in algebra. There the theorem states that for any two real numbers a and b, $|a| + |b| > |a + b|$. The triangle inequality takes many other forms. With vectors, which were introduced in the Connections on page 116, the vector sum of two vectors (a, b) and (c, d) can be defined as the vector $(a + c, b + d)$. Then, as shown by the graph of these three vectors, we can see that the triangle inequality could be written as

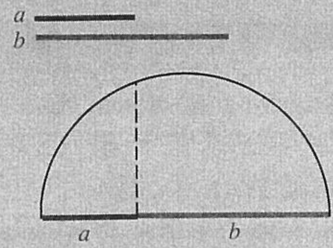


$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} > \sqrt{(a + c)^2 + (b + d)^2}$$

Many other famous inequalities that occur in geometry have algebraic counterparts. Two of the most famous are the arithmetic mean and the geometric mean of numbers. Two numbers a and b have an arithmetic mean $\frac{(a + b)}{2}$ and a geometric mean $\sqrt{ab}$. A famous result provides an inequality relating these two means.

$$\frac{(a + b)}{2} > \sqrt{ab} \text{ for all nonnegative real numbers } a \text{ and } b$$

The arithmetic mean of two numbers is simply their common average. The geometric mean of two nonnegative real numbers a and b can be illustrated by the following process. Find a segment having lengths a and b. Put these segments end to end and draw a semicircle having the two segments as diameter. Construct a segment perpendicular to the two segments at their common endpoint and extend it until it intersects the circle. The length of this segment is the geometric mean of a and b, which is $\sqrt{ab}$.



You will study more about the geometric mean in Chapter 7.

Exercises

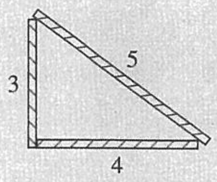
1. For what real numbers a and b is it the case that $|a| + |b| = |a + b|$?
2. Use the fact that the length of the vector (x, y) is $\sqrt{x^2 + y^2}$ and the geometric Triangle Inequality Theorem presented in this section to argue that $\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} > \sqrt{(a + c)^2 + (b + d)^2}$ for the vectors (a, b), (c, d), and their sum, $(a + c, b + d)$.
3. Under what conditions will the arithmetic mean, $\frac{(a + b)}{2}$, of two numbers a and b be equal to the numbers themselves?
4. Under what conditions will the inequality involving the arithmetic mean and geometric mean of two numbers a and b assume the equality $\frac{(a + b)}{2} = \sqrt{ab}$?
5. Give a convincing argument that, for all nonnegative values of a and b, other than those cited in the answer to Exercise 4, $\frac{(a + b)}{2} > \sqrt{ab}$.

5-8 Inequalities in Two Triangles

EXPLORE

Use straws or other suitable objects to make three triangles with the following lengths (in inches): **a.** 3, 4, 2 **b.** 3, 4, 5 **c.** 3, 4, 6. In which triangle is the angle included between the sides of length 3 and 4 the smallest? the largest?

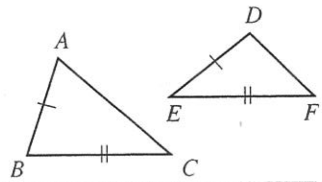
Write a generalization about your findings.



The theorems in this lesson involve two triangles in which two sides of one triangle are congruent to two sides of another triangle.

THEOREM 5.24 SAS Inequality Theorem

If two sides of one triangle are congruent to two sides of a second triangle and the included angle of the first triangle is larger than the included angle of the second triangle, then the third side of the first triangle is longer than the third side of the second triangle.



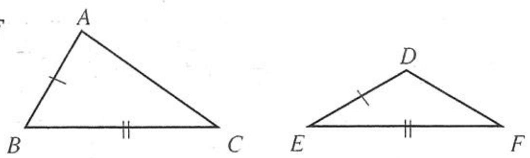
The above theorem says that if $AB = DE$ and $BC = EF$, then the relationship between the lengths of $\overline{AC}$ and $\overline{DF}$ depends on the measures of $\angle B$ and $\angle E$.

You will be asked to prove Theorem 5.24 in Exercise 26.

THEOREM 5.25 SSS Inequality Theorem

If two sides of one triangle are congruent to two sides of a second triangle, and the third side of the first triangle is longer than the third side of the second triangle, then the angle opposite the third side of the first triangle is larger than the angle opposite the third side of the second triangle.

Given: $AB = DE$, $BC = EF$, $AC > DF$
Prove: $m\angle B > m\angle E$

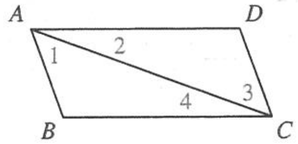


Plan Use an indirect proof. Assume $m\angle B \not> m\angle C$
Case 1: Let $m\angle B = m\angle E$. Show $\triangle ABC \cong \triangle DEF$ by SAS and obtain the contradiction that $AC = DF$.
Case 2: Let $m\angle B < m\angle E$. Obtain the contradiction that $AC < DF$ by the SAS Triangle Inequality Theorem. Conclude that $m\angle B > m\angle E$.

Example 1

What conclusions can be drawn from the given information?

a. $AB = CD$, $m\angle 1 > m\angle 3$
b. $AB = CD$, $m\angle 1 = m\angle 3$
c. $AD = BC$, $AB > CD$



Solution

a. $BC > AD$ **b.** $\triangle ABC \cong \triangle CDA$ **c.** $m\angle 4 > m\angle 2$

Try This

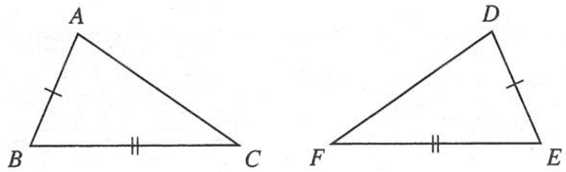
What conclusions can be drawn if $AB = CD$ and $BC > AD$?

Class Exercises

Short Answer

Given $\triangle ABC$ and $\triangle DEF$, $AB = DE$ and $BC = EF$, complete each statement by using $>$, $<$, or $=$.

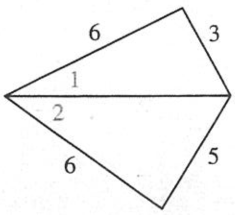
1. If $m\angle B > m\angle E$, then AC __ DF.
2. If $DF < AC$, then $m\angle B$ __ $m\angle E$.
3. If $AC = DF$, then $m\angle E$ __ $m\angle B$.
4. If $m\angle E > m\angle B$, then DF __ AC.
5. If $AC < DF$, then $m\angle E$ __ $m\angle B$.



Sample Exercises

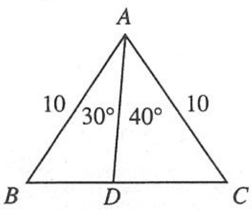
Complete each statement by using $>$, $<$, or $=$.

6.



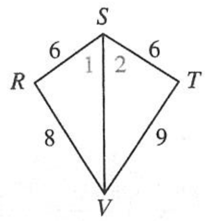
$m\angle 1$ __ $m\angle 2$

7.



CD __ BD

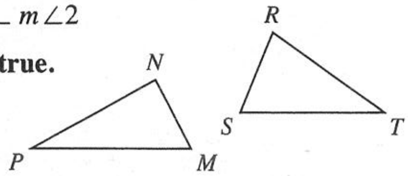
8.



$m\angle 1$ __ $m\angle 2$

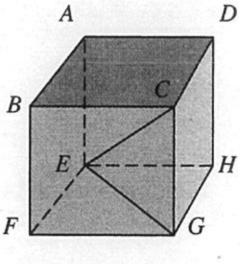
Determine whether the statement is always, sometimes, or never true.

9. If $PN = RT$, $MN = RS$, and $m\angle N < m\angle R$, then $PM > ST$.
10. If $PM = ST$, $MN = RS$ and $PN > RT$, then $m\angle M > m\angle S$.



Discussion

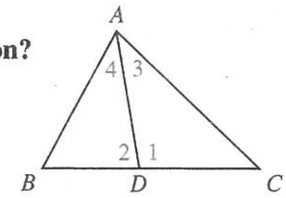
11. Consider the cube to the right. Each edge of the cube has a length of one unit. Consider $\triangle EGC$ and $\triangle EGF$ and use the SAS Inequality Theorem to explain why $EC > FE$.



Exercises

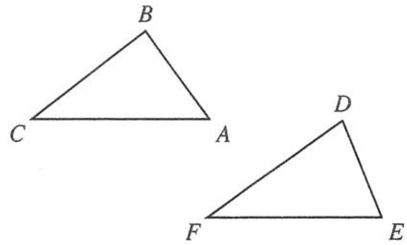
A

What conclusion can be drawn from the given information?

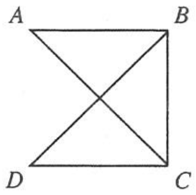


1. $BD = DC$, $m\angle 1 > m\angle 2$
2. $AB = AC$, $m\angle 3 > m\angle 4$
3. $\overline{AD}$ is a median. $AB > AC$
4. $\triangle ABC$ is isosceles with base $\overline{BC}$. $CD > BD$

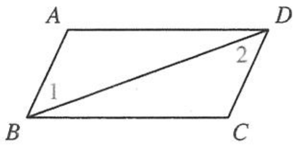
Determine whether each statement is always, sometimes, or never true.



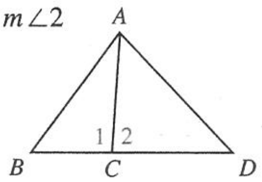
5. If $AB = DE$, $BC = FD$, and $m\angle B > m\angle D$, then $FE > AC$.
6. If $AB = DE$, $AC = FE$ and $BC < FD$, then $m\angle A < m\angle E$.
7. If $m\angle B < m\angle D$, $m\angle C < m\angle F$, then $m\angle A > m\angle E$.
8. If $BC = FD$, $m\angle C > m\angle F$ then $m\angle A > m\angle E$.
9. If $AB = DE$, $BC = FD$, $AC > FE$, then $m\angle D > m\angle B$.
10. If $BC = FD$, $AC = FE$, and $m\angle C < m\angle F$, then $DE > AB$.
11. **Given:** $AB = CD$, $m\angle ABC > m\angle BCD$
    Prove: $AC > BD$



12. **Given:** $AB = CD$, $m\angle 1 > m\angle 2$
    Prove: $ABCD$ is not a ▱.
    (HINT: Use an indirect proof.)



13. **Given:** $\overline{AC}$ is a median. $m\angle 1 > m\angle 2$
    Prove: $AB > AD$



B

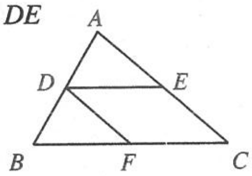
14. In $\triangle MNP$, $\overline{MX}$ is a median, $m\angle MXN = 4x + 6$ and $m\angle MXP = 7x - 2$. Which side is longer, $\overline{MN}$ or $\overline{MP}$? Which angle is larger, $\angle N$ or $\angle P$?
15. Quadrilateral $ABCD$ has $AB = AC = AD$, $m\angle DCA = 50$, and $m\angle ACB = 55$. Which side is longer, $\overline{CD}$ or $\overline{BC}$? Which angle is larger, $\angle B$ or $\angle D$?
16. P is a point above the plane determined by equilateral $\triangle ABC$. If $PC > PB$, which angle is larger, $\angle PAC$ or $\angle PAB$?

Determine whether $m\angle A < m\angle D$, $m\angle A = m\angle D$, or $m\angle A > m\angle D$, given the following coordinates for triangles ABC and DEF.

17. $A(1, 1)$, $B(2, 3)$, $C(4, 0)$,
    $D(1, -2)$, $E(2, 0)$, $F(-2, -3)$
18. $A(-2, 3)$, $B(-3, 0)$, $C(1, 2)$,
    $D(3, 0)$, $E(2, 3)$, $F(0, 1)$

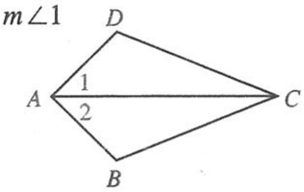
19. **Given:** D is the midpoint of $\overline{AB}$.
$AE = BF$
$AC < BC$

Prove: $DF < DE$



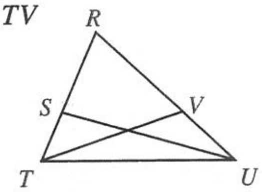
20. **Given:** $AB = AD$
$BC > DC$

Prove: $m\angle 2 > m\angle 1$



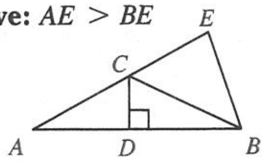
21. **Given:** $RU > RT$
$VU = ST$

Prove: $SU > TV$



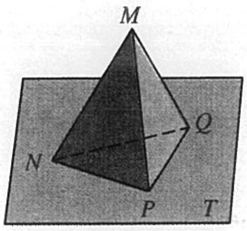
22. **Given:** $\overline{CD}$ ⊥ bisector of $\overline{AB}$
E is on $\overleftrightarrow{AC}$.

Prove: $AE > BE$



23. **Given:** N, P, Q are in plane R.
$m\angle MNQ > m\angle MNP$
$NP = NQ$

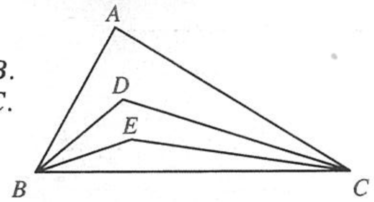
Prove: $m\angle MPQ > m\angle MQP$



C

24. **Given:** $AC > AB$
$\overline{DC}$ and $\overline{EC}$ trisect $\angle ACB$.
$\overline{DB}$ and $\overline{EB}$ trisect $\angle ABC$.

Prove: $EC > BE$

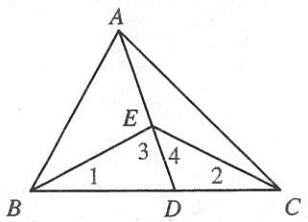


25. **Given:** $m\angle 1 = m\angle 2$
$m\angle 3 > m\angle 4$

Prove: $AC > AB$

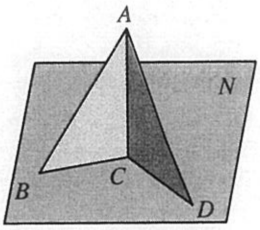
26. **Given:** $m\angle 1 = m\angle 2$
$AB > AC$

Prove: $m\angle 3 < m\angle 4$

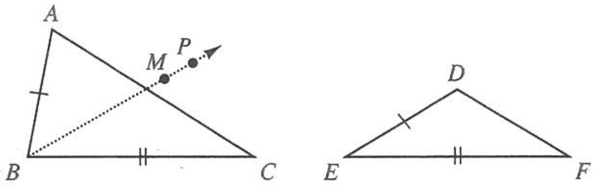


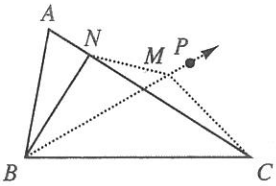
27. **Given:** $\overline{AC}$ ⊥ plane N
$AD > AB$

Prove: $CD > BC$
(HINT: Select point E on $\overleftrightarrow{CD}$ so that $BC = CE$.)



28. Complete the following proof for Theorem 5.24. Draw $\overline{BP}$ so that $\angle PBC \cong \angle E$. Select M so that $BM = DE$. There are two possibilities. Either M is on $\overline{AC}$ or M is not on $\overline{AC}$. Regardless, $\triangle BMC \cong \triangle EDF$ and $MC = DF$. Case 1: M is on $\overline{AC}$. Explain why $AC > DF$. Case 2: M is not on $\overline{AC}$. Let $\overline{BN}$ bisect $\angle ABM$. Draw $\overline{NM}$ and $\overline{MC}$. $AB = BM = DE$. Use the fact that $\triangle ABN \cong \triangle MBN$ and that $AN = NM$ to explain why $AC > DF$.





Critical Thinking

29. $\overline{PA}$ is perpendicular to the plane determined by the equilateral triangle ABC. Which segment is the longest: $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, $\overline{PA}$, $\overline{PB}$, or $\overline{PC}$? Write a paragraph proof to justify your choice.

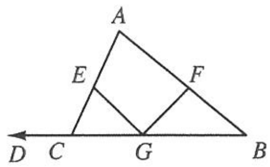
Algebra Review

Solve for x.

1. $\frac{x}{3} = \frac{4}{2}$ 2. $\frac{12}{16} = \frac{6}{x}$ 3. $\frac{16}{x} = \frac{x}{4}$

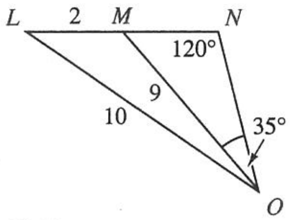
Quiz

E, F, and G are the midpoints of sides $\overline{AC}$, $\overline{AB}$, and $\overline{CB}$ respectively.



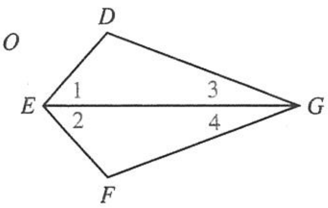
1. If $AC = 24$, find FG.
2. If $EG = 2x + 12$ and $AF = 16$, find EG.
3. If $EG = 7$, $FG = 8$, and $CB = 12$, find the perimeter of $\triangle ABC$.

Refer to the figure to the right.



4. In $\triangle LMO$, name the smallest angle.
5. In $\triangle MNO$, name the shortest side.
6. Which is shorter, $\overline{LO}$ or $\overline{MN}$?
7. Using the Triangle Inequality Theorem, $m\angle NMO > m\underline{\ ?\ }$ and $m\underline{\ ?\ }$.

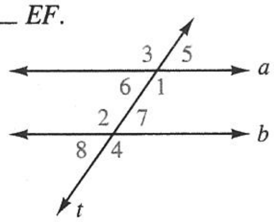
Complete each statement by using $>$, $<$, or $=$.



8. If $DE = EF$ and $DG < FG$, then $m\angle 1$ ___ $m\angle 2$.
9. If $DG = GF$ and $EF = DE$, then $m\angle 1$ ___ $m\angle 2$.
10. If $m\angle 3 > m\angle 4$ and $DG = FG$, then DE ___ EF.
11. Write an indirect proof for the following.

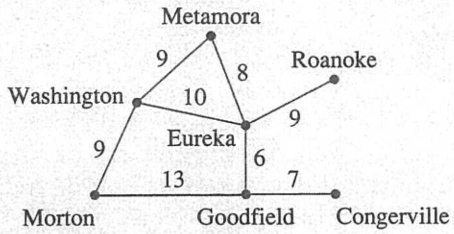
 Given: $\angle 1 \not\cong \angle 2$

 Prove: $\angle 3 \not\cong \angle 4$

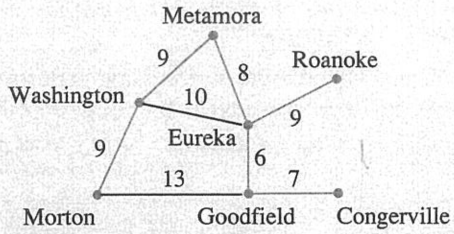


DISCRETE MATH

Finding Shortest Connections



Mathematics is used in planning telephone or other services to find the optimal way of hooking towns, offices, etc. together using the minimal amount of cable or wire. Consider the map for several towns to be served by a cable television network. The problem is to find the minimum amount of cable to connect all of the towns.



The path shown in red illustrates one method of connecting the towns but it uses more than the minimum; as it connects Washington into the network of towns twice, once through Eureka and once through Metamora. Since the link through Eureka is not needed to connect any other town, it can be deleted. This reduces the amount of cable required by 10 mi.

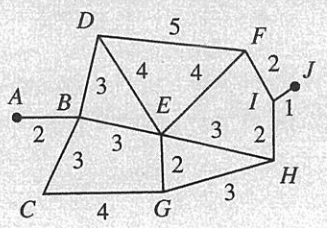
To find the minimum amount for a task such as shown to the right, you start by identifying one of the towns as the beginning point. This choice has no bearing on the minimal amount, as each town must eventually be connected into the network. Start at this town, say Eureka, and look for the closest town. This is Goodfield, so you would connect the two with a cable. Now look for the next closest town to either town connected and add it to the network, in this case, Congerville. Next look for the town not yet connected that is closest to Eureka, Goodfield, or Congerville. This is Metamora. Include it and then continue asking for the next closest town. In this example, you can see that there is a tie between Washington and Roanoke, so either can be chosen at this point. Suppose you choose Washington. You would next add Morton and then Roanoke. What would have happened if you had chosen Roanoke instead of Washington? All of the towns are now connected and the amount of cable needed to complete the cable network is the sum of the lengths shown in blue, 6 + 7 + 8 + 9 + 8 + 9, or 47 mi of cable.

A collection of edges that touch all of the points in a graph, but which allow only one way of traveling from point to point without repeating other points is called a tree. The process of finding this shortest connecting set of links is known as finding a minimal spanning tree. Sometimes more than one minimal spanning tree exists, but the minimal distances for each tree will be the same.

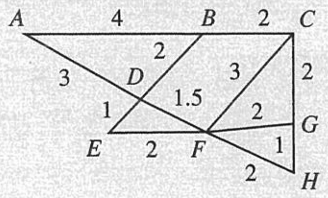
Exercises

Find the minimal spanning tree and the total length for each of the following.

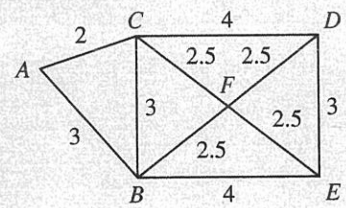
1.



2.



3.



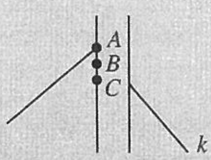
CRITICAL THINKING

Analyzing and Evaluating Evidence

Sometimes it is possible to jump to a conclusion on the basis of evidence that is unreliable or incomplete. Some evidence is **possibly true,** other evidence is **probably true,** and still other evidence is **accepted as true.** Consider the following examples.

Example 1

a. By observation only, decide if line k, when extended, will meet point A, B, C, or none of these points.



b. Read the following silently and then aloud.

A BIRD IN THE
THE HAND IS WORTH
TWO IN THE BUSH

What do you notice? What kind of evidence are you using in these examples?

Solution

In both cases, observational evidence was used. These examples show that we do not always see what we think we see. Often, facts get changed to fit our expectations. These examples show that observational evidence should be treated with caution and classified as possibly or probably true.

Example 2

As Jerry came around a corner, he saw his steady girlfriend Rhonda walking into a theater with another boy. Jerry concluded that Rhonda was dating someone else and was very angry. What kind of evidence was Jerry using in his decision?

Solution

Jerry based his conclusion on circumstantial evidence. The truth was that Rhonda was meeting her aunt at the theater and was only walking through the door next to a boy from school. Circumstantial evidence should be considered as possibly true.

Example 3

Mr. Higami was clocked traveling 48 mph in a 35 mph zone by a policeman's radar gun. Two reputable witnesses, who had checked the radar gun before and after Mr. Higami's ticket was issued, testified that the gun was working properly. Another policeman who was riding in the car with the officer verified the officer's story. What kind of evidence do you have in this case?

Solution

The evidence here is of the type that is accepted to be true. In a case such as this, many different facts corroborate the fact in contention. Even in such cases where the evidence is considered true beyond any reasonable doubt, there is always a small possibility that it could be false.

Exercises

1. After Karen began dating Javier, her report card showed she was getting a lower grade in each of her classes. Her mother checked Karen's study time and found that she spent an average of only four hours a week on homework. Karen seemed to be trying to get better grades and did not want to talk about her lower grades. Decide if each of the following statements of evidence is possibly true, probably true, or accepted as true.
   a. Karen was getting better grades before she started dating Javier.
   b. Karen is spending less time on homework since she started dating Javier.
   c. Karen feels guilty or embarrassed about her lower grades.
2. Plan and rehearse a brief incident that a group of three or four students could act out in class. After a group presents an incident, write a careful "eyewitness report" of exactly what happened. How does your report compare to that of a classmate?
3. Describe a situation in which you drew an incorrect conclusion based on circumstantial evidence.

CHAPTER SUMMARY

Vocabulary

bases of a trapezoid (5-4)
contradiction (5-6)
indirect proof (5-6)
isosceles trapezoid (5-4)
median of a trapezoid (5-4)
midsegment of a triangle (5-5)
negation (5-6)
parallelogram (5-1)
rhombus (5-3)
rectangle (5-3)
square (5-3)
trapezoid (5-4)

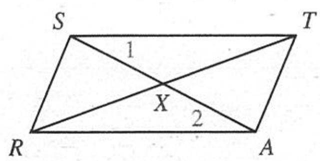
Key Ideas

1. A parallelogram has the following properties.
   a. Opposite sides are parallel.
   b. Opposite sides are congruent.
   c. Opposite angles are congruent.
   d. Consecutive angles are supplementary.
   e. Diagonals bisect each other.
2. Rectangles, rhombuses, and squares are special types of parallelograms.
3. A quadrilateral is a parallelogram if any one of the following is true.
   a. Both pairs of opposite sides are parallel.
   b. Both pairs of opposite sides are congruent.
   c. One pair of sides are parallel and congruent.
   d. Both pairs of opposite angles are congruent.
   e. Consecutive angles are supplementary.
   f. Diagonals bisect each other.
4. The median of a trapezoid is parallel to the bases and its length is half the sum of the lengths of the bases.
5. A segment that joins the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.
6. In an indirect proof, you assume the negation of the *prove* statement and reason logically to reach a contradiction. Then conclude that the assumption is false and the *prove* statement is true.
7. In a triangle, the measure of an exterior angle is greater than the measure of either of the remote interior angles.
8. In a triangle, the longer side is opposite the largest angle. The converse is also true.
9. The sum of any two sides of a triangle is greater than the third side.
10. The SAS and SSS Inequality Theorems are two methods for comparing angles and sides of two triangles.

CHAPTER REVIEW

5-1

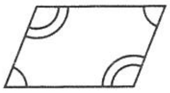
Use parallelogram *STAR* to complete each statement.

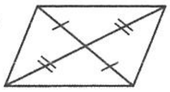


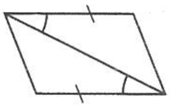
1. If $m\angle 1 = 32$, then $m\angle 2 =$ ___.
2. If $m\angle TSR = 125$, then $m\angle SRA =$ ___ and $m\angle TAR =$ ___.
3. If $ST = 8$, then ___ $= 8$.
4. If $RT = 16$, then $XT =$ ___.
5. If $SR = 5x + 10$ and $TA = 2x + 43$, then $TA =$ ___.

5-2

Determine if the quadrilateral is a parallelogram. If so, state the theorem or definition that justifies the conclusion. If not, write *no conclusion*.

6. 

7. 

8. 

9. 

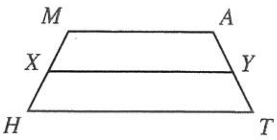
5-3

Classify each statement as always, sometimes, or never true.

10. Opposite sides of a rhombus are parallel.
11. All angles of a rhombus are right angles.
12. Diagonals of a square bisect the opposite angles.
13. All sides of a parallelogram are congruent.

5-4

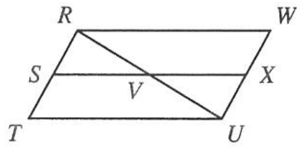
***MATH* is a trapezoid and $\overline{XY}$ is a median. Complete each statement.**



14. $\overline{MA}$ and $\overline{HT}$ are called the ___ of the trapezoid.
15. If $m\angle H = 35$, then $m\angle MXY =$ ___.
16. If $AY = 5$, then $AT =$ ___.
17. If $MA = 10$ and $HT = 24$, then $XY =$ ___.

5-5

Indicate whether each statement is true or false.



18. If $\overline{SV} \parallel \overline{TU}$, then $SV = \frac{1}{2}TU$.
19. If $\overline{RW} \parallel \overline{SX} \parallel \overline{TU}$ and $RS = ST$, then $WX = XU$.
20. A parallelogram results if the midpoints of the consecutive sides of quadrilateral *RWUT* are joined.
21. If *S* is the midpoint of $\overline{RT}$, then $\overline{SV}$ bisects $\overline{RU}$.

5-6

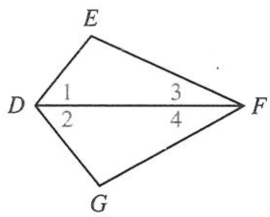
22. Write the negation of the following statement. $\angle ABC$ is an obtuse angle.
23. The statement "$\angle A \cong \angle B$ and $\angle A \not\cong \angle B$" is called a _?_.

5-7

24. In $\triangle RST$, $m\angle R = 35$ and $m\angle S = 40$. List the sides of the triangle from shortest to longest.
25. In $\triangle WXY$, if $WX > WY > XY$, name the largest angle in the triangle.
26. A triangle has two sides of lengths 3 and 7. The length of the third side must be greater than _?_ and less than _?_.

5-8

Complete each statement and state the theorem that supports your conclusion.



27. If $ED = DG$ and $m\angle 1 = m\angle 2$, then ___ = ___.
28. If $EF = FG$ and ___ < ___, then $m\angle 3 < m\angle 4$.
29. If $EF = GD$ and $m\angle 3 > m\angle 2$, then ___ > ___.

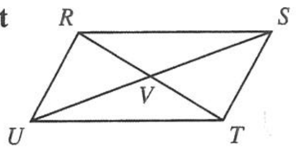
CHAPTER TEST

List the letters of all the figures that have each of the following properties.

1. Diagonals are congruent.
2. Both pairs of opposite sides are congruent.
3. Diagonals are perpendicular.
4. Exactly one pair of opposite sides are parallel.
5. Diagonals bisect each other.
6. Opposite sides are parallel.

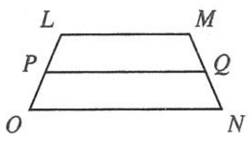
a. quadrilateral
b. trapezoid
c. parallelogram
d. rectangle
e. rhombus
f. square

Use the given information to write the theorem or definition that supports the conclusion that *RSTU* is a parallelogram.



7. $RS = UT$ and $RU = ST$
8. $\overline{RU} \parallel \overline{ST}$ and $RU = ST$
9. $RV = VT$ and $UV = VS$
10. $\overline{RS} \parallel \overline{UT}$ and $\overline{RU} \parallel \overline{ST}$

Refer to trapezoid *LMNO* and median $\overline{PQ}$.

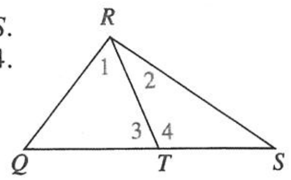


11. If $LM = 7$ and $ON = 13$, find PQ.
12. If $PQ = 2x - 6$, $LM = 7$, and $ON = 5x - 25$, find ON.
13. If $\overline{LM} \parallel \overline{PQ}$ and $LP = PO$, then MQ __?__ QN.

14. Find the perimeter of the triangle that is formed by joining the midpoints of the sides of a triangle with sides of lengths 8 cm, 12 cm, and 14 cm.

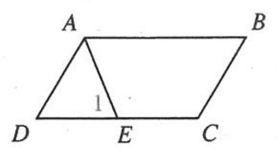
Classify each statement as true or false.

15. A triangle can have sides of lengths 7 in., 13 in., and 15 in.
16. In $\triangle ABC$, if $AB > AC$, then $m\angle B > m\angle C$.
17. In $\triangle RST$, if $m\angle R < m\angle S$, then $ST < RT$.
18. If a line is parallel to one side of a triangle, then it contains the midpoints of the other two sides.
19. If $RQ = TS$ and $m\angle 1 > m\angle 4$, then $QT > RS$.
20. If $RQ = TS$ and $QT > RS$, then $m\angle 1 < m\angle 4$.
21. The negation of $QR = RT$ is $\overline{QR} \parallel \overline{RT}$.

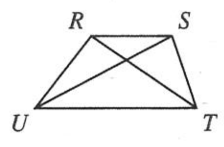


22. Write the negation you would use to begin an indirect proof.
    Prove: $\triangle ABC$ is not an isosceles triangle.

23. Write a two-column proof.
    Given: ▱ *ABCD*
    $m\angle 1 > m\angle B$
    Prove: $BC > AE$



24. Write an indirect proof.
    Given: trapezoid *RSTU*
    $\overline{RT} \not\cong \overline{US}$
    Prove: *RSTU* is not isosceles.



PROBLEM SOLVING

Work Backward

The strategy Work Backward is sometimes helpful in solving problems. Ask yourself, "What information do I need in order to reach the conclusion I want?" Consider statements **a–f** in the following example.

Example

Suppose you have one pail that holds 4 L of water and another that holds 9 L. There are no markings on either pail to indicate how many liters it contains. How can you measure out exactly 6 L of water using these two pails?

Solution

a. I want to end with 6 L. I can do this if I can pour 3 L from the 9 L pail.
b. I can pour 3 L from the 9 L pail if there is space for only 3 L in the 4 L pail.
c. There would be space for only 3 L in the 4 L pail if I could first put 1 L in it.
d. I can put 1 L in the 4 L pail if I can use the 9 L to measure 1 L.
e. I can measure 1 L in the 9 L pail if I can pour out 8 L.
f. I can pour 8 L from the 9 L pail by filling the 4 L pail twice.
Now work backward and use statements **a–f** to write a solution to the problem.

Problem-Solving Strategies	
Draw a Diagram	Find a Pattern
Make a Table	Work Backward

The problem-solving strategies that have been introduced up to this point in the book are presented in the chart.

Problems

Solve each problem by using one of the strategies presented so far.

1. How can you measure out exactly 4 L if you have only an unmarked 5 L pail and an unmarked 7 L pail?
2. A goat owner wanted to make a rectangular pen with as much grazing area as possible. He had only 72 m of fence and wanted the length and width of his pen to be whole numbers. What should the dimensions of his pen be?
3. Suppose there are eight balls that look alike. All are the same weight except one, which is heavier than all the rest. How could you use a balance scale only twice to find the heavier ball? (HINT: Start with the second use of the balance scale. Figure out how you could decide which of two (or three) balls is heavier in only one balancing and work backward.)
4. What is the minimum number of pieces of pizza than can be obtained if a circular pizza is cut eight times with straight cuts.
5. A rhombus-shaped flower bed is to be built in the center of a circular patio. The plans show the rhombus inside a dotted rectangle. $EF = 4$ m and $FG = 6$ m. The builder thinks it will be difficult to calculate the length of the side of the flower bed. Her assistant thinks it will be very easy to find the length with almost no calculation at all. What do you think? Find the length.