3 Properties of Axiomatic Systems

Now that the reader has been introduced to several particular systems of axioms he should be able to understand an abstract discussion of axiomatic systems in general. There are three important concepts usually associated with any axiomatic system: consistency, independence, and completeness. A discussion of these topics, using the systems introduced in Chapter 2 as examples, will provide answers to the questions raised in that chapter.

3.1 Consistency

If the purpose of language is to communicate, it is self-defeating if something is said and then "un-said." This is what is done when one says "P and not-P." In order to express this more precisely, let us first review what has previously been said about sentences.

First, recall that not all sentences are true or false. Even when such nonsense as "All triangles are courageous" and such exclamations as "Shut the door!" are excluded, a very important type of sentence remains, one to which we cannot apply the words true or false. For example, when one says, "He has red hair" or "X is an even number," such sentences, strictly speaking, cannot be said to be either true or false. In today's parlance, they are called *open sentences*. That is, they are sentences that become *statements*, become true or false, when the "variables" in the sentence are assigned some definite referent. Thus, "George Brown has red hair" and "Seven is an even number" are no longer open sentences; they are *statements*, or *propositions* (we use the two words synonymously); they are sentences that are either true or false. All sentences in mathematics are (or should be) either open sentences or statements.

With this agreement, the classical laws of logic hold:

The Law of Contradiction: *No statement can be both true and false.*

The Law of Excluded Middle: *Every statement is either true or false.*

Now let us return to our first point. If P is a statement, then when we say "P and not-P" we are not merely making a false statement; we are saying something and retracting it. We are in effect saying nothing or, to put it another way, we are breaking our rules of language and thus speaking nonsense. Therefore, in a system of axioms one may permit any number of statements to be false if one chooses but one must never allow any two statements to be of the type "P" and "not-P." This can be said more succinctly if we introduce the following definition:

Definition. An axiom system is *consistent* iff there do not exist in the system any two axioms, any axiom and theorem, or any two theorems of the form "P" and "not-P."

Using this definition we can now summarize: **it is absolutely essential for an axiomatic system to be consistent.**

Very well, but how does one determine this? What is being said, in effect, is that for an axiomatic system to be consistent it must be impossible to ever prove a theorem contradicting another theorem or axiom. Unless one has reason to believe that one has already proved every single theorem that it is possible to prove from a given set of axioms, there is no way of knowing whether or not a contradiction will be discovered just ahead. And even if one knew one had derived every theorem, might there not be so many of them, or might they not be so complex or so subtle, that a contradiction might rest inextricably hidden among them? How can one be sure? This is a question to which there is no definitive answer.

There is, however, a pragmatic test for consistency that mathematicians have been using for years. To explain precisely how the test works, it will be helpful to introduce a few definitions and then to use the axiomatic systems of the preceding chapter to illustrate the test.

The systems introduced as Axiom Set 1 and Axiom Set 2 are "abstract systems" as long as the terms "point" and "line" are taken as undefined. As long as these terms remain undefined the axioms are open sentences. It is not until some meaning is given to the undefined terms that one may legitimately ask whether the axioms are true or false.

Definition. By an *interpretation* of an axiomatic system we mean: the assignment of meanings to the undefined technical terms in such a way that the axioms become either true or false.

Definition. An interpretation that makes an axiom *true* is said to *satisfy* that axiom. If there exists an interpretation in which every axiom in a set becomes true, then the set is said to be *satisfiable.*

Definition. If a set is satisfiable, then such an interpretation is called a *model.*

Test for Consistency: *If there exists a model for a set of axioms, the set is consistent.*

The existence of a model as a test for consistency may be justified as follows. If one can find such a model, then all axioms in the system become true statements. Hence all statements implied by the axioms—that is, all theorems—must become true statements because it is impossible for a true statement to imply a false statement. If each axiom is true, then the conjunction of all axioms is a true statement and anything implied by this conjunction of statements must itself be true by the laws of logic. Some examples should clarify how the model test for consistency works.

3.2 Models for Consistency

Suppose that a college offers the following challenge to its best students. The top three members of each class, sophomore, junior, and senior, will be given an all-expense-paid trip to nine countries whose political attitudes toward the United States are either neutral or negative. On their return they must debate and discuss what they have learned.

To make the project challenging, the three discussion teams will be made up solely of classmates. To make it fair, but not too time-consuming, the teams are divided into sets of three, with one member of each class on a team and each of the three teams to spend one week in a country. They will regroup in such a way that no two men are ever on the same team; the new teams will then spend one week in another country. For clarity, the students make up the following chart:

Sophomores: Alan, Bob, Charley
Juniors: Dick, Ernie, Fred
Seniors: George, Herm, Irving
Algeria: Alan, Dick, George
Bulgaria: Bob, Ernie, Irving
Cambodia: Charley, Fred, Herm
Dominican Republic: Alan, Ernie, Herm
Egypt: Bob, Fred, George
Finland: Charley, Dick, Irving

Ghana: Alan, Fred, Irving
Hungary: Bob, Dick, Herm
India: Charley, Ernie, George

The interpretation should be obvious: "point" = student; "line" = team; "belonging to" = is a member of.

This interpretation will be seen to be a model for Axiom Set 1. The reader should check carefully to be convinced that each of the axioms of the system is satisfied by this interpretation. The same holds for the following three models, which are listed for future reference:

Model II (9 points, 12 lines)

1	4	1	1	1	2	2	2	3	3	3	7
2	5	4	5	6	4	5	6	4	5	6	8
3	6	9	7	8	7	8	9	8	9	7	9

In this model "points" are numbers and "lines" are columns of numbers.

Model III (16 points, 20 lines)

1	5	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	9	10
2	6	5	6	7	8	5	6	7	8	5	6	7	8	5	6	7	8	11	12
3	7	9	10	11	12	10	11	12	9	11	15	9	13	14	9	13	10	14	13
4	8	13	14	15	16	16	13	14	15	12	16	10	14	15	12	16	11	16	15

Model IV (Infinite points, lines)
Ordinary Euclidean geometry

Even if it is now apparent that these are indeed models and that therefore all theorems provable in the system must be true, doubts remain as to the reliability of using such criteria. This is especially so when the model has an infinite number of elements, as in Model IV. How does one know that it really satisfies Axioms 1a and 1b and the others? It would take an infinite number of checks to find out. How does one know that the very model being used to guarantee consistency is not itself inconsistent? In such cases only a relative type of test for consistency has been found. All that can be said is that if Euclidean geometry is consistent, then so is that system for which it is a model—and in several thousand years no one has discovered an inconsistency in Euclidean geometry; therefore, it is probably consistent. Unfortunately, such relative tests will always be required when one is dealing with any system with an infinite number of elements. In the case of finite models such as the first three given above, it is not too difficult to check for hidden problems because one can check every case. Such models have sometimes been said to be "absolute" tests for consistency.

Let us look at a few more models, this time for Axiom Set 2.

Suppose that a mathematics department has seven faculty members and suppose they decide to form seven committees to study and determine the best way to teach specific topics in axiomatic geometry. It is agreed, for the sake of fairness, that each member will be the chairman of one committee and serve on exactly two others and that each committee will have exactly three members. They draw up the following chart:

Committee to Study	*Chairman*	*Members*	
Axioms	Appolonius	Bolyai	Ceva
Betweenness	Bolyai	Desargues	Fano
Congruence	Ceva	Desargues	Euclid
Dissection theory	Desargues	Appolonius	Gauss
Equivalence relations	Euclid	Appolonius	Fano
Finite systems	Fano	Gauss	Ceva
Geometries, infinite	Gauss	Euclid	Bolyai

With the interpretation that "points" are the mathematicians and "lines" the committees as listed in each row, an examination of this chart will show that every axiom of Set 2 is satisfied. Switching the members in the second and third columns may be of help in checking but it is not necessary. Call this Model V.

As another interpretation consider the following. A manufacturer is interested in making a trinket made up of seven beads, each of a different color. He wishes to join them to seven wires in such a way that there are three beads on each wire and three wires through each bead. He makes up the following list: amber, yellow, red, blue, green, orange, and violet. The trinket looks like Figure 3.1.

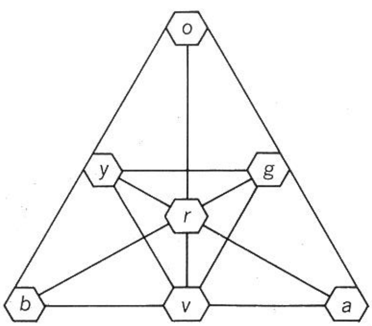

FIGURE 3.1

With the interpretation that "point" = bead, "line" = wire, and the wires and beads arranged as in Figure 3.1, they can be listed:

$W_1 = \{b, y, o\}$ $W_2 = \{a, g, o\}$ $W_3 = \{a, y, r\}$ $W_4 = \{b, g, r\}$
$W_5 = \{a, b, v\}$ $W_6 = \{y, g, v\}$ $W_7 = \{o, r, v\}$

One finds on examination that this is another model for System 2. Call it Model VI.

For another interpretation, take the following array of numbers:

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

in which a "point" = a number and "line" = a column. Since this interpretation also makes all the axioms true, it may be considered Model VII.

For a final interpretation:

A	B	C	D	E	F	G	H	I	J	K	L	M
B	C	D	E	F	G	H	I	J	K	L	M	A
D	E	F	G	H	I	J	K	L	M	A	B	C
J	K	L	M	A	B	C	D	E	F	G	H	I

where once again "point" = letter of the alphabet and "line" = column. This interpretation also makes each of the axioms a true statement; call it Model VIII.

We have now listed eight models, four models for each of our two axiom sets. Each model is sufficient to show the consistency of its respective system. In addition, it should be apparent that all eight interpretations are models for Axiom Set 3. It is not necessary to gather so much evidence; all one needs to show consistency is a single model. However, in later discussions, we shall find other uses for some of the models listed in this section.

3.3 Independence

After a set of axioms has been chosen and it has been determined that they are consistent, the next question that arises is whether each statement is truly primitive—that it cannot be derived from the other members of the set of axioms. Stated another way, how can it be known that one of the axioms is not a theorem? When phrased in this way it might lead one to counter with another question, namely, "What if it is?" If it is, nothing is seriously wrong; at worst it might be difficult to prove and is simply left as an axiom. On the other hand, a proof might be supplied. In either case the system suffers no irreparable damage.

Does this mean that independence, as defined in the next definition, is an unnecessary property in an axiom system? Evidently. But there are many mathematicians who, for aesthetic and logical reasons, try to reduce an axiom set to a set of independent axioms. In fact, such an attempt plays a significant role in the history of mathematics, as will be apparent later.

Let us consider a new problem. Suppose one wishes to choose an independent set of axioms. How does one go about such a task? We might start with the following definition:

Definition. A statement is said to be *independent* in a set of statements if it is impossible to derive it from the other members of the set.

This is not a very practical definition, for how can one tell whether or not a statement is provable? If it has not been proven, might it still not be provable? As in the case of consistency, there is a test for independence.

Test: *If an axiom set is consistent and if, when the statement being tested is replaced by its denial there exists a model for the new set, then the statement being tested is independent.*

That is, if $A_1, \cdots, A_i, \cdots, A_n$ is consistent and if $A_1, \cdots,$ not-$A_i, \cdots, A_n$ is consistent, then A_i is independent. For if the original system is consistent and A_i is a theorem, then it is implied by the other axioms, and the contradiction of A_i together with the other axioms could not possibly be consistent; that is, there would exist no model for such a set of statements.

Let us illustrate this in Axiom Set 1. We shall find models showing the independence of each of the axioms in the set.

The following interpretation shows the independence of

Axiom 1a:

$$l = \{P_1, P_2, P_3\}$$
$$m = \{P_4, P_5, P_6\}$$

The "interpretation" is precisely what is given; namely, the two sets as "lines" and the six elements as "points." For the interpretation to test the independence of Axiom 1a it must satisfy the *negation* of Axiom 1a *and* must satisfy each of the other axioms. It certainly satisfies the negation of the first axiom because the points P_2 and P_4, for example, do not have a line containing them. A careful check will show that it satisfies the other axioms.

It should be pointed out that in negating an axiom one does not need to limit the negation to the true contradictory of the given statement; a contrary may be used as well. The difference between them, one will recall from an earlier discussion, is that of two contradictories one is true and the other is false; whereas of two contrary statements both cannot be true but it is possible for both to be false. Because a contrary will suffice to negate an axiom, one can deny Axiom 1b by finding an interpretation with two lines, three lines, four lines, and so forth, containing two given points.

Now let us find a model showing the independence of Axiom 1b. Taking the easy axioms first, we know the following: the model must have a line, the line must have at least three points, and there must exist a point not on the line. Now, to deny Axiom 1b, there must exist at least two lines containing the same two points; and, in order for the lines to be distinct, they must contain a third distinct point. So one might start with:

4	1
5	4
6	5

As before, the numbers are "points" and the columns "lines." In this interpretation, Axiom 1b is negated; Axioms 2, 3, and 4 are satisfied. But what of each of the others, which must also be satisfied? To satisfy 5a and 5b there must exist a unique parallel through any point not on a given line, and to satisfy 1a there must exist lines containing points 1 and 6 and any other points that happen to be generated in the process of satisfying the rest of the axioms. So one arrives at the independence model for

Axiom 1b:

1	4	1	2	2	1	2	1	3	1
2	5	4	3	4	3	4	3	4	2
3	6	5	6	5	6	6	5	5	6

(Note: We can add {1, 3, 4}; {2, 5, 6}; {1, 5, 6}; {2, 3, 4}; {2, 3, 5}; {1, 4, 6}; {3, 5, 6}; {1, 2, 4} and still satisfy the system.)

To complete the test for the independence of the axioms in Axiom Set 1 we cite the following models. In each instance the interpretation should be checked to verify that it is indeed a model for the independence of the axiom in question.

Axiom 2:

1	3	1	2	2	1
2	4	4	4	3	3

Axiom 3: $l = \{1, 2, 3\}$

Axiom 4: a single point; no line

Axiom 5a: Any one of the models for the consistency of the Axiom Set 2. Thus we may take Models V, VI, VII, or VIII of the preceding section.

In checking to see that the interpretation is indeed a model showing the independence of Axiom 4, the question arises as to how this interpretation satisfies the other axioms. The answer lies in the fact that all of the other axioms are of the form "If . . . then . . ."; in order for them to be false, there must *exist* a line such that . . . or there must *exist* two points such that . . . . If lines do not exist, if two points do not exist, the statements cannot be false; hence they are true. This is sometimes called *vacuous* satisfaction.

Axiom 5b: Finally, for Axiom 5b we present the following rather complicated model:

1	5	5	1	1	1	1	1	2	2	2	2	2	3	3
2	6	19	5	16	7	8	12	5	6	7	8	11	5	6
3	7	15	9	6	19	10	15	16	10	12	9	13	12	19
4	8	10	13	11	14	18	17	18	14		15	19	14	9
									17					

3	3	3	4	4	4	4	4	6	7	8	9	13	17
7	8	10	5	6	7	8	9	12	9	11	10	14	18
11	17	16	11	15	10	12	18	18	16	14	11	15	19
15	13		17		13	16	14	13	17		12	16	
18						19							

It can be seen that this model has 19 points and 29 lines and that not every line has the same number of points.

It is instructive to attempt to find an independence model for Axiom 5b that has the same number of points on a line. While the proof of Theorem 11 uses Axiom 5b, is it necessary to use it? If it is, it would seem that without Axiom 5b we cannot prove that every line must have the same number of points. On the other hand, the exercises following the proof of Theorem 11 indicate that the same theorem holds in Axiom System 2, which has no Axiom 5b. Furthermore, if the proof of the independence of Axiom 5b were incompatible with all lines having the same number of points, what then of the consistency of the System of Young where we stipulate that all lines have three points? An answer to this problem might be found in the solution to Exercise 3.3.III.

In conclusion, it should be pointed out once again that independence, unlike consistency, is not essential to a system. While it is true that for aesthetic or logical reasons one might attempt to adopt independent statements as axioms, it is often not done. If a theorem is difficult to prove and is assumed as an axiom the system will not itself be ruined.

EXERCISES 3.3

I. From the following list select models showing the independence of each axiom in Axiom Set 2.

Interpretations:

1.

1	4	1	1	1	2	2	2	3	3	3	7
2	5	4	5	6	4	5	6	4	5	6	8.
3	6	9	7	8	7	8	9	8	9	7	9

2. A single point; no line.

3. $l = \{P_1, P_2, P_3\}$; $m = \{P_1, P_4, P_5\}$.

4.

1	3	1	2	2	1
2	4	4	4	3	3.

5. Three points; no line.

6. $l = \{P_1, P_2\}$; $m = \{P_2, P_3\}$; $k = \{P_3, P_1\}$.

7.

1	4	1	2	2	1	2	1	3	1
2	5	4	3	4	3	4	3	4	2.
3	6	5	6	5	6	6	5	5	6

8. $k = \{P_1\}$; $l = \{P_1, P_2\}$; $m = \{P_1, P_3\}$; $n = \{P_2, P_3\}$.

9. $k = \{P_2, P_3, P_4\}$; $l = \{P_1, P_2, P_3\}$; $m = \{P_1, P_2, P_4\}$.

10. $l = \{P_1, P_3, P_2\}$.

11.

1	2	3	4	5	6	7
2	3	4	5	6	7	1.
4	5	6	7	1	2	3

12. $l = \{P_1, P_2, P_3\}$; $m = \{P_4, P_5, P_6\}$.

II. Consider the following axiom system:

1. If l and m are any two distinct lines, they have at least one point in common.
2. If P_1 and P_2 are any two distinct points, they have *at least one* line through them.
3. If P_1 and P_2 are any two distinct points, they have *at most one* line through them.
4. Not all points are on the same line.
5. There exist exactly three distinct points.

(a) By trying to find independence models for each axiom, one should be able to determine that one of the axioms is not independent.

(b) Prove it as a theorem.

III. The following two models are possible models for the independence of Axiom 5b of Axiom Set 1. Test each to see if it is.

1.

1	1	2	3	1	2	3	4	1	2	3	5	6	7	1	2	3	5
2	4	4	4	6	7	5	8	10	8	8	8	8	8	9	9	9	9
3	5	6	7	7	5	6	9	8	11	12	13	14	15	11	12	13	14

6	7	2	3	4	5	6	3	4	6	7	1	5	7	1	4	2
9	9	10	10	10	10	10	11	11	11	11	12	12	12	13	13	14
15	10	13	14	15	11	12	15	12	13	14	14	15	13	15	14	15

2.

1	1	1	1	1	1	1	1	1
2	4	5	6	12	13	15	17	10
3	7	8	9	14	16	18	19	11

2	2	2	2	2	2	2	2
4	5	6	10	11	14	15	16
8	9	7	12	13	18	19	17

3	3	3	3	3	3	3	3
4	5	6	10	11	12	16	17
9	7	8	13	14	15	19	18

4	4	4	4	4	4
5	10	11	12	13	18
6	14	15	16	17	19

5	5	5	5	5
10	11	12	13	14
15	16	17	18	19

6	6	6	6	6
10	11	12	13	14
16	17	18	19	15

7	7	7	7	7	7
8	10	11	12	13	15
9	17	18	19	14	16

8	8	8	8	8
10	11	12	14	15
18	19	13	16	17

9	9	9	9	9
10	11	13	14	16
19	12	15	17	18

3.4 Completeness

If a system contains a statement which is expressible in the technical terms and relations of an axiom system and which cannot be proved to be true or false, the system obviously lacks a statement of axiom status, an independent statement. This was apparent in Axiom Set 3, when we found that it was not possible to prove *or* disprove certain statements. Recall that from Axiom Set 3 we could not prove either of the following: there exist at least eight points; there exist at most seven points. Yet one is true if and only if the other is false; hence, one must be true. This is not to suggest that one of these two statements should be adopted as an axiom but, rather, that when there are such statements that can be neither proved or disproved, the system lacks an independent statement and is incomplete according to the following definition:

Definition. An axiom system is *incomplete* if it is possible to add an independent axiom (phraseable in the system's technical vocabulary). If it is impossible to add such a statement, the system is *complete.*

We shall always assume that the independent axiom can be phrased in the system's technical vocabulary and that we need not repeat this requirement except for emphasis. If it were ignored, one would get such trivial situations as in Axiom Set 1, where the statement "All redheaded truck drivers are six feet tall," if added to that system, is an independent statement. We wish to avoid this.

We must now question whether it is in fact possible to find all of the independent statements of a system. Is it possible to prove or disprove every statement expressible within the vocabulary of the system? Is it possible to know when a system is complete? The definition is of little use, for how does one determine when the conditions of the definition have been met, namely, that all the possible independent statements phraseable in the system's terms have been discovered?

This problem is more tractable when approached in a slightly different way. Instead of looking at the system as an abstraction, one should consider some interpretation of it. Suppose, for example, one wishes to set up the axioms of Euclidean geometry. The problem is then specific: if one regards points and lines as undefined things, can one state a set of axioms from which the theorems implied will be those of Euclidean geometry alone and essentially different from any other geometry? Stated differently, can an abstract system be so completely characterized that it applies to essentially one and only one concrete interpretation, has, that is, an essentially unique model? If so, the system is said to be *categorical.*

In order to see how this approach supplies an answer to the problem, it is necessary to have a more precise definition of "categorical." To give such a definition, several new concepts must be introduced, concepts which in and of themselves are very useful in mathematics: *one-one correspondence* and *isomorphism.*

If the elements of two sets can be paired off in such a way that each element of one set occurs exactly once, matched with exactly one element of the other set, there is said to exist a *one-one correspondence* between the two sets.

If there exists a correspondence between two sets S_1 and S_2 such that every statement which is true when made about elements of S_1 is also true when made about the corresponding elements of S_2, the correspondence is said to *preserve relations.*

Every isomorphic is one to one correspon- which - preserve.

Definition. Two models of an axiomatic system are said to be *isomorphic with respect to that system* if there exists at least one one-one correspondence between the elements of the system which preserves relations.

This concept is important enough to cover in more detail, rather than digress now, however, we shall postpone it until the next section, where we shall attempt to clarify it with several illustrations.

We are now able to state:

Definition. Whenever an axiomatic system is such that any two models are isomorphic, the system is said to be *categorical.*

And finally we can state the test:

Test: *If a system is categorical, then it is complete.*

To prove the statement in the test, suppose an axiomatic system is categorical but not complete. If not complete, then there exists a statement "A_n" such that it and "not-A_n" are consistent with the given set of axioms. Thus there exist models for the system $\{A_1, A_2, \cdots A_n\}$ and for $\{A_1, A_2, \cdots \text{not-}A_n\}$, hence showing that A_n is independent in the system. If it is now further supposed that the system is categorical, these two models must then be isomorphic; hence, corresponding statements in the two systems are either both true or both false. But this is impossible by the assumption that "A_n" is true in one and "not-A_n" is true in the other. This assumption must therefore be false; hence, if a system is categorical, it is complete.

In a preceding section it was seen that Axiom Set 3 is satisfied by every one of the models that satisfies Sets 1 and 2. As axioms were added the variety of interpretations decreased until, finally, when we included Axiom 6, we arrived at the System of Young, which is satisfied only by a model with nine points and twelve lines. Similarly, when we add Axiom 6 to Axiom Set 2 we get a system satisfied only by a model with seven points and seven lines. In these latter systems every two models satisfying one system are essentially the same; they are merely different symbols for the same things; they are, in a word, isomorphic. Hence, these last two systems are complete.

In conclusion, it should be pointed out that completeness is not only unessential but generally undesirable. One of the great advantages of an abstract axiomatic system is that in proving one theorem we are in effect proving many theorems. For every interpretation that satisfies the system, any theorem proved for the uninterpreted system becomes true in the interpreted system; therefore, the greater variety of models one can find for a system, the greater range of application it has. On the other hand, the "more" complete a system is, the fewer essentially different models one can find, the narrower is its range of applications. If one wishes to study one particular system intensively, completeness is useful; otherwise, it is not.

3.5 Examples of Isomorphisms

Consider Models I and II of Section 3.2

l: Sophomores:	Alan, Bob, Charley
m: Juniors:	Dick, Ernie, Fred
n: Seniors:	George, Herm, Irving
o: Algeria:	Alan, Dick, George
p: Bulgaria:	Bob, Ernie, Irving
q: Cambodia:	Charley, Fred, Herm
r: Dominican Republic:	Alan, Ernie, Herm
s: Egypt:	Bob, Fred, George
t: Finland:	Charley, Dick, Irving
u: Ghana:	Alan, Fred, Irving
v: Hungary:	Bob, Dick, Herm
w: India:	Charley, Ernie, George

l'	m'	n'	o'	p'	q'	r'	s'	t'	u'	v'	w'
1	4	1	1	1	2	2	2	3	3	3	7
2	5	4	5	6	4	5	6	4	5	6	8
3	6	9	7	8	7	8	9	8	9	7	9

As we know, the students are the "points" in one system and the numbers the "points" in the other. There are literally hundreds of ways in which a one-one correspondence between these two sets can be set up. Suppose we choose the following two:

(A)		(B)	
Alan	$\leftrightarrow$ 1	Alan	$\leftrightarrow$ 1
Bob	$\leftrightarrow$ 2	Bob	$\leftrightarrow$ 2
Charley	$\leftrightarrow$ 3	Charley	$\leftrightarrow$ 3
Dick	$\leftrightarrow$ 4	Dick	$\leftrightarrow$ 4
Ernie	$\leftrightarrow$ 5	Ernie	$\leftrightarrow$ 5
Fred	$\leftrightarrow$ 6	Fred	$\leftrightarrow$ 6
George	$\leftrightarrow$ 9	George	$\leftrightarrow$ 7
Herm	$\leftrightarrow$ 7	Herm	$\leftrightarrow$ 8
Irving	$\leftrightarrow$ 8	Irving	$\leftrightarrow$ 9

In order now to display an isomorphism, the "lines" must be corresponded in such a way that relations are preserved. For example, since Alan, Bob, and Charley belong to l, one must first of all be sure that there exists a line containing their "corresponding points," that is, that there exists a line containing 1, 2, and 3. These lines must then be made to correspond. Thus correspondence (A) is accompanied by the following:

(A′)

$l \leftrightarrow l'$	$r \leftrightarrow o'$
$m \leftrightarrow m'$	$s \leftrightarrow s'$
$n \leftrightarrow w'$	$t \leftrightarrow t'$
$o \leftrightarrow n'$	$u \leftrightarrow p'$
$p \leftrightarrow r'$	$v \leftrightarrow q'$
$q \leftrightarrow v'$	$w \leftrightarrow u'$

Once the correspondence (A) has been determined, any change in (A′) would cause the relation-preserving property to be lost. There is another way in which one can fail to preserve relations. Consider the correspondence given by (B). Because o contains Alan, Dick, and George, then by (B) there should exist a line containing 1, 4, and 7. But there is no such line in Model II; hence (B) fails to begin with. Of all the possible correspondences that can be set up between two models, there are usually many that preserve (all) relations (if the models are isomorphic), many that preserve some relations and, usually, some correspondences in which no relations are preserved. To show that there is an isomorphism, however, only one correspondence that preserves relations need be found.

EXERCISES 3.5

1. Suppose that the following is a model for Axiom Set 1 and set up a relation preserving one-one correspondence between it and Model II:

1	1	1	1	2	2	2	3	3	3	4	5
2	4	6	8	4	5	7	4	5	6	7	6
3	5	7	9	6	8	9	9	7	8	8	9

2. Set up a relation preserving one-one correspondence between:

(a) Model V and Model VI in Section 3.2.

(b) Model I and itself

(c) Model VII and itself

3. Can you find a one-one correspondence preserving *no* relations between:

(a) Model V and Model VI

(b) Model I and Model II

REVIEW EXERCISES

Answer true or false and explain or justify your answer.

1. A good test for consistency of a system is to derive all the theorems possible and, if no contradictions are uncovered, the system is consistent.
2. If two systems have the same number of elements they are isomorphic.
3. It is possible for a theorem to be implied by one axiomatic system and its contradictory by another axiomatic system.
4. In an axiomatic system completeness is always desirable.
5. In an axiomatic system independence is always desirable.
6. An inconsistent axiomatic system might imply a statement and its contradictory.
7. If an axiomatic system is satisfiable its model must be finite.
8. The test for independence involves consistency.
9. Completeness, independence, and consistency—as determined by the tests—all depend on the concept of satisfiability.