CHAPTER 1

PRELIMINARIES

Number was born in superstition and reared in mystery, . . . numbers were once made the foundation of religion and philosophy, and the tricks of figures have had a marvellous effect on a credulous people.

F. W. Parker

1.1 MATHEMATICAL INDUCTION

The theory of numbers is concerned, at least in its elementary aspects, with properties of the integers and more particularly with the positive integers 1, 2, 3, . . . (also known as the *natural numbers*). The origin of this misnomer harks back to the early Greeks for whom the word *number* meant positive integer, and nothing else. The natural numbers have been known to us for so long that the mathematician Leopold Kronecker once remarked, “God created the natural numbers, and all the rest is the work of man.” Far from being a gift from Heaven, number theory has had a long and sometimes painful evolution, a story that is told in the ensuing pages.

We shall make no attempt to construct the integers axiomatically, assuming instead that they are already given and that any reader of this book is familiar with many elementary facts about them. Among these is the Well-Ordering Principle, stated here to refresh the memory.

Well-Ordering Principle. Every nonempty set S of nonnegative integers contains a least element; that is, there is some integer a in S such that $a \leq b$ for all b's belonging to S.

Because this principle plays a critical role in the proofs here and in subsequent chapters, let us use it to show that the set of positive integers has what is known as the Archimedean property.

Theorem 1.1 Archimedean property. If a and b are any positive integers, then there exists a positive integer n such that $na \geq b$.

Proof. Assume that the statement of the theorem is not true, so that for some a and b, $na < b$ for every positive integer n. Then the set

$$S = \{b - na \mid n \text{ a positive integer}\}$$

consists entirely of positive integers. By the Well-Ordering Principle, S will possess a least element, say, $b - ma$. Notice that $b - (m + 1)a$ also lies in S, because S contains all integers of this form. Furthermore, we have

$$b - (m + 1)a = (b - ma) - a < b - ma$$

contrary to the choice of $b - ma$ as the smallest integer in S. This contradiction arose out of our original assumption that the Archimedean property did not hold; hence, this property is proven true.

With the Well-Ordering Principle available, it is an easy matter to derive the First Principle of Finite Induction, which provides a basis for a method of proof called *mathematical induction*. Loosely speaking, the First Principle of Finite Induction asserts that if a set of positive integers has two specific properties, then it is the set of all positive integers. To be less cryptic, we state this principle in Theorem 1.2.

Theorem 1.2 First Principle of Finite Induction. Let S be a set of positive integers with the following properties:

(a) The integer 1 belongs to S.
(b) Whenever the integer k is in S, the next integer $k + 1$ must also be in S.

Then S is the set of all positive integers.

Proof. Let T be the set of all positive integers not in S, and assume that T is nonempty. The Well-Ordering Principle tells us that T possesses a least element, which we denote by a. Because 1 is in S, certainly $a > 1$, and so $0 < a - 1 < a$. The choice of a as the smallest positive integer in T implies that $a - 1$ is not a member of T, or equivalently that $a - 1$ belongs to S. By hypothesis, S must also contain $(a - 1) + 1 = a$, which contradicts the fact that a lies in T. We conclude that the set T is empty and in consequence that S contains all the positive integers.

Here is a typical formula that can be established by mathematical induction:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n + 1)(n + 1)}{6} \tag{1}$$

for $n = 1, 2, 3, \ldots$. In anticipation of using Theorem 1.2, let S denote the set of all positive integers n for which Eq. (1) is true. We observe that when $n = 1$, the formula becomes

$$1^2 = \frac{1(2+1)(1+1)}{6} = 1$$

This means that 1 is in S. Next, assume that k belongs to S (where k is a fixed but unspecified integer) so that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(2k+1)(k+1)}{6} \tag{2}$$

To obtain the sum of the first $k+1$ squares, we merely add the next one, $(k+1)^2$, to both sides of Eq. (2). This gives

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(2k+1)(k+1)}{6} + (k+1)^2$$

After some algebraic manipulation, the right-hand side becomes

$$\begin{aligned}(k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right] &= (k+1)\left[\frac{2k^2+7k+6}{6}\right] \\ &= \frac{(k+1)(2k+3)(k+2)}{6}\end{aligned}$$

which is precisely the right-hand member of Eq. (1) when $n = k+1$. Our reasoning shows that the set S contains the integer $k+1$ whenever it contains the integer k. By Theorem 1.2, S must be all the positive integers; that is, the given formula is true for $n = 1, 2, 3, \ldots$.

Although mathematical induction provides a standard technique for attempting to prove a statement about the positive integers, one disadvantage is that it gives no aid in formulating such statements. Of course, if we can make an "educated guess" at a property that we believe might hold in general, then its validity can often be tested by the induction principle. Consider, for instance, the list of equalities

$$\begin{aligned}&1 = 1\\ &1+2 = 3\\ &1+2+2^2 = 7\\ &1+2+2^2+2^3 = 15\\ &1+2+2^2+2^3+2^4 = 31\\ &1+2+2^2+2^3+2^4+2^5 = 63\end{aligned}$$

We seek a rule that gives the integers on the right-hand side. After a little reflection, the reader might notice that

$$\begin{array}{ccc} 1 = 2-1 & 3 = 2^2-1 & 7 = 2^3-1 \\ 15 = 2^4-1 & 31 = 2^5-1 & 63 = 2^6-1 \end{array}$$

(How one arrives at this observation is hard to say, but experience helps.) The pattern emerging from these few cases suggests a formula for obtaining the value of the expression $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$; namely,

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1 \tag{3}$$

for every positive integer n.

To confirm that our guess is correct, let S be the set of positive integers n for which Eq. (3) holds. For $n = 1$, Eq. (3) is certainly true, whence 1 belongs to the set S. We assume that Eq. (3) is true for a fixed integer k, so that for this k

$$1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$$

and we attempt to prove the validity of the formula for $k + 1$. Addition of the term 2^k to both sides of the last-written equation leads to

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k &= 2^k - 1 + 2^k \\ &= 2 \cdot 2^k - 1 = 2^{k+1} - 1 \end{aligned}$$

But this says that Eq. (3) holds when $n = k + 1$, putting the integer $k + 1$ in S so that $k + 1$ is in S whenever k is in S. According to the induction principle, S must be the set of all positive integers.

Remark. When giving induction proofs, we shall usually shorten the argument by eliminating all reference to the set S, and proceed to show simply that the result in question is true for the integer 1, and if true for the integer k is then also true for $k + 1$.

We should inject a word of caution at this point, to wit, that one must be careful to establish both conditions of Theorem 1.2 before drawing any conclusions; neither is sufficient alone. The proof of condition (a) is usually called the *basis for the induction*, and the proof of (b) is called the *induction step*. The assumptions made in carrying out the induction step are known as the *induction hypotheses*. The induction situation has been likened to an infinite row of dominoes all standing on edge and arranged in such a way that when one falls it knocks down the next in line. If either no domino is pushed over (that is, there is no basis for the induction) or if the spacing is too large (that is, the induction step fails), then the complete line will not fall.

The validity of the induction step does not necessarily depend on the truth of the statement that one is endeavoring to prove. Let us look at the false formula

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 + 3 \tag{4}$$

Assume that this holds for $n = k$; in other words,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2 + 3$$

Knowing this, we then obtain

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= k^2 + 3 + 2k + 1 \\ &= (k + 1)^2 + 3 \end{aligned}$$

which is precisely the form that Eq. (4) should take when $n = k + 1$. Thus, if Eq. (4) holds for a given integer, then it also holds for the succeeding integer. It is not possible, however, to find a value of n for which the formula is true.

There is a variant of the induction principle that is often used when Theorem 1.2 alone seems ineffective. As with the first version, this Second Principle of Finite Induction gives two conditions that guarantee a certain set of positive integers actually consists of all positive integers. This is what happens: we retain requirement (a), but (b) is replaced by

(b′) If k is a positive integer such that $1, 2, \ldots, k$ belong to S, then $k + 1$ must also be in S.

The proof that S consists of all positive integers has the same flavor as that of Theorem 1.2. Again, let T represent the set of positive integers not in S. Assuming that T is nonempty, we choose n to be the smallest integer in T. Then $n > 1$, by supposition (a). The minimal nature of n allows us to conclude that none of the integers $1, 2, \ldots, n - 1$ lies in T, or, if we prefer a positive assertion, $1, 2, \ldots, n - 1$ all belong to S. Property (b′) then puts $n = (n - 1) + 1$ in S, which is an obvious contradiction. The result of all this is to make T empty.

The First Principle of Finite Induction is used more often than is the Second; however, there are occasions when the Second is favored and the reader should be familiar with both versions. It sometimes happens that in attempting to show that $k + 1$ is a member of S, we require proof of the fact that not only k, but all positive integers that precede k, lie in S. Our formulation of these induction principles has been for the case in which the induction begins with 1. Each form can be generalized to start with any positive integer n_0. In this circumstance, the conclusion reads as "Then S is the set of all positive integers $n \geq n_0$."

Mathematical induction is often used as a method of definition as well as a method of proof. For example, a common way of introducing the symbol $n!$ (pronounced "n factorial") is by means of the inductive definition

(a) $1! = 1$,

(b) $n! = n \cdot (n - 1)!$ for $n > 1$.

This pair of conditions provides a rule whereby the meaning of $n!$ is specified for each positive integer n. Thus, by (a), $1! = 1$; (a) and (b) yield

$$2! = 2 \cdot 1! = 2 \cdot 1$$

while by (b), again,

$$3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1$$

Continuing in this manner, using condition (b) repeatedly, the numbers $1!, 2!, 3!, \ldots, n!$ are defined in succession up to any chosen n. In fact,

$$n! = n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1$$

Induction enters in showing that $n!$, as a function on the positive integers, exists and is unique; however, we shall make no attempt to give the argument.

It will be convenient to extend the definition of $n!$ to the case in which $n = 0$ by stipulating that $0! = 1$.

Example 1.1. To illustrate a proof that requires the Second Principle of Finite Induction, consider the so-called *Lucas sequence*:

$$1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$$

Except for the first two terms, each term of this sequence is the sum of the preceding two, so that the sequence may be defined inductively by

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 \\ a_n &= a_{n-1} + a_{n-2} \qquad \text{for all } n \geq 3 \end{aligned}$$

We contend that the inequality

$$a_n < (7/4)^n$$

holds for every positive integer n. The argument used is interesting because in the inductive step, it is necessary to know the truth of this inequality for two successive values of n to establish its truth for the following value.

First of all, for $n = 1$ and 2, we have

$$a_1 = 1 < (7/4)^1 = 7/4 \qquad \text{and} \qquad a_2 = 3 < (7/4)^2 = 49/16$$

whence the inequality in question holds in these two cases. This provides a basis for the induction. For the induction step, choose an integer $k \geq 3$ and assume that the inequality is valid for $n = 1, 2, \ldots, k - 1$. Then, in particular,

$$a_{k-1} < (7/4)^{k-1} \qquad \text{and} \qquad a_{k-2} < (7/4)^{k-2}$$

By the way in which the Lucas sequence is formed, it follows that

$$\begin{aligned} a_k = a_{k-1} + a_{k-2} &< (7/4)^{k-1} + (7/4)^{k-2} \\ &= (7/4)^{k-2}(7/4 + 1) \\ &= (7/4)^{k-2}(11/4) \\ &< (7/4)^{k-2}(7/4)^2 = (7/4)^k \end{aligned}$$

Because the inequality is true for $n = k$ whenever it is true for the integers $1, 2, \ldots, k - 1$, we conclude by the second induction principle that $a_n < (7/4)^n$ for all $n \geq 1$.

Among other things, this example suggests that if objects are defined inductively, then mathematical induction is an important tool for establishing the properties of these objects.

PROBLEMS 1.1

1. Establish the formulas below by mathematical induction:
   (a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \geq 1$.
   (b) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.
   (c) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$ for all $n \geq 1$.

(d) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \geq 1$.

(e) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$ for all $n \geq 1$.

2. If $r \neq 1$, show that for any positive integer n,

$$a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1}-1)}{r-1}$$

3. Use the Second Principle of Finite Induction to establish that for all $n \geq 1$,

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1)$$

[*Hint:* $a^{n+1} - 1 = (a+1)(a^n - 1) - a(a^{n-1} - 1)$.]

4. Prove that the cube of any integer can be written as the difference of two squares. [*Hint:* Notice that

$$n^3 = (1^3 + 2^3 + \cdots + n^3) - (1^3 + 2^3 + \cdots + (n-1)^3).]$$

5. (a) Find the values of $n \leq 7$ for which $n! + 1$ is a perfect square (it is unknown whether $n! + 1$ is a square for any $n > 7$).

(b) True or false? For positive integers m and n, $(mn)! = m!n!$ and $(m+n)! = m! + n!$.

6. Prove that $n! > n^2$ for every integer $n \geq 4$, whereas $n! > n^3$ for every integer $n \geq 6$.

7. Use mathematical induction to derive the following formula for all $n \geq 1$:

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$$

8. (a) Verify that for all $n \geq 1$,

$$2 \cdot 6 \cdot 10 \cdot 14 \cdot \cdots \cdot (4n-2) = \frac{(2n)!}{n!}$$

(b) Use part (a) to obtain the inequality $2^n(n!)^2 \leq (2n)!$ for all $n \geq 1$.

9. Establish the Bernoulli inequality: If $1 + a > 0$, then

$$(1+a)^n \geq 1 + na$$

for all $n \geq 1$.

10. For all $n \geq 1$, prove the following by mathematical induction:

(a) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$.

(b) $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \dfrac{n+2}{2^n}$.

11. Show that the expression $(2n)!/2^n n!$ is an integer for all $n \geq 0$.

12. Consider the function defined by

$$T(n) = \begin{cases} \dfrac{3n+1}{2} & \text{for } n \text{ odd} \\ \dfrac{n}{2} & \text{for } n \text{ even} \end{cases}$$

The $3n + 1$ conjecture is the claim that starting from any integer $n > 1$, the sequence of iterates $T(n)$, $T(T(n))$, $T(T(T(n)))$, ..., eventually reaches the integer 1 and subsequently runs through the values 1 and 2. This has been verified for all $n \leq 10^{16}$. Confirm the conjecture in the cases $n = 21$ and $n = 23$.

13. Suppose that the numbers a_n are defined inductively by $a_1 = 1, a_2 = 2, a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Use the Second Principle of Finite Induction to show that $a_n < 2^n$ for every positive integer n.

14. If the numbers a_n are defined by $a_1 = 11, a_2 = 21$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$, prove that

$$a_n = 5 \cdot 2^n + 1 \qquad n \geq 1$$

1.2 THE BINOMIAL THEOREM

Closely connected with the factorial notation are the *binomial coefficients* $\binom{n}{k}$. For any positive integer n and any integer k satisfying $0 \leq k \leq n$, these are defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

By canceling out either $k!$ or $(n-k)!$, $\binom{n}{k}$ can be written as

$$\binom{n}{k} = \frac{n(n-1)\cdots(k+1)}{(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

For example, with $n = 8$ and $k = 3$, we have

$$\binom{8}{3} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5!} = \frac{8 \cdot 7 \cdot 6}{3!} = 56$$

Also observe that if $k = 0$ or $k = n$, the quantity $0!$ appears on the right-hand side of the definition of $\binom{n}{k}$; because we have taken $0!$ as 1, these special values of k give

$$\binom{n}{0} = \binom{n}{n} = 1$$

There are numerous useful identities connecting binomial coefficients. One that we require here is *Pascal's rule*:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \qquad 1 \leq k \leq n$$

Its proof consists of multiplying the identity

$$\frac{1}{k} + \frac{1}{n-k+1} = \frac{n+1}{k(n-k+1)}$$

by $n!/(k-1)!(n-k)!$ to obtain

$$\frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)(n-k)!}$$
$$= \frac{(n+1)n!}{k(k-1)!(n-k+1)(n-k)!}$$

Falling back on the definition of the factorial function, this says that

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!}$$

from which Pascal's rule follows.

This relation gives rise to a configuration, known as *Pascal's triangle*, in which the binomial coefficient $\binom{n}{k}$ appears as the $(k+1)$th number in the nth row:

$$\begin{array}{ccccccccccccc} & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ & & & & & & \cdots & & & & & & \end{array}$$

The rule of formation should be clear. The borders of the triangle are composed of 1's; a number not on the border is the sum of the two numbers nearest it in the row immediately above.

The so-called *binomial theorem* is in reality a formula for the complete expansion of $(a+b)^n$, $n \geq 1$, into a sum of powers of a and b. This expression appears with great frequency in all phases of number theory, and it is well worth our time to look at it now. By direct multiplication, it is easy to verify that

$$\begin{aligned} (a+b)^1 &= a + b \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \text{ etc.} \end{aligned}$$

The question is how to predict the coefficients. A clue lies in the observation that the coefficients of these first few expansions form the successive rows of Pascal's triangle. This leads us to suspect that the general binomial expansion takes the form

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

or, written more compactly,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

Mathematical induction provides the best means for confirming this guess. When $n = 1$, the conjectured formula reduces to

$$(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^{1-k}b^k = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 = a + b$$

which is certainly correct. Assuming that the formula holds for some fixed integer m, we go on to show that it also must hold for $m+1$. The starting point is to notice that

$$(a+b)^{m+1} = a(a+b)^m + b(a+b)^m$$

Under the induction hypothesis,

$$\begin{aligned} a(a+b)^m &= \sum_{k=0}^{m} \binom{m}{k} a^{m-k+1} b^k \\ &= a^{m+1} + \sum_{k=1}^{m} \binom{m}{k} a^{m+1-k} b^k \end{aligned}$$

and

$$\begin{aligned} b(a+b)^m &= \sum_{j=0}^{m} \binom{m}{j} a^{m-j} b^{j+1} \\ &= \sum_{k=1}^{m} \binom{m}{k-1} a^{m+1-k} b^k + b^{m+1} \end{aligned}$$

Upon adding these expressions, we obtain

$$\begin{aligned} (a+b)^{m+1} &= a^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] a^{m+1-k} b^k + b^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} a^{m+1-k} b^k \end{aligned}$$

which is the formula in the case $n = m+1$. This establishes the binomial theorem by induction.

Before abandoning these ideas, we might remark that the first acceptable formulation of the method of mathematical induction appears in the treatise *Traité du Triangle Arithmetiqué*, by the 17th century French mathematician and philosopher Blaise Pascal. This short work was written in 1653, but not printed until 1665 because Pascal had withdrawn from mathematics (at the age of 25) to dedicate his talents to religion. His careful analysis of the properties of the binomial coefficients helped lay the foundations of probability theory.

PROBLEMS 1.2

1. (a) Derive Newton's identity

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r} \qquad n \geq k \geq r \geq 0$$

(b) Use part (a) to express $\binom{n}{k}$ in terms of its predecessor:

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1} \qquad n \geq k \geq 1$$

2. If $2 \leq k \leq n-2$, show that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} \qquad n \geq 4$$

3. For $n \geq 1$, derive each of the identities below:

(a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

[*Hint:* Let $a = b = 1$ in the binomial theorem.]

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

(c) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$.

[*Hint:* After expanding $n(1+b)^{n-1}$ by the binomial theorem, let $b = 1$; note also that

$$n\binom{n-1}{k} = (k+1)\binom{n}{k+1}.]$$

(d) $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n} = 3^n$.

(e) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots$

$= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$.

[*Hint:* Use parts (a) and (b).]

(f) $\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \cdots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}$.

[*Hint:* The left-hand side equals

$$\frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \cdots + (-1)^n\binom{n+1}{n+1}\right].]$$

4. Prove the following for $n \geq 1$:

(a) $\binom{n}{r} < \binom{n}{r+1}$ if and only if $0 \leq r < \frac{1}{2}(n-1)$.

(b) $\binom{n}{r} > \binom{n}{r+1}$ if and only if $n - 1 \geq r > \frac{1}{2}(n-1)$.

(c) $\binom{n}{r} = \binom{n}{r+1}$ if and only if n is an odd integer, and $r = \frac{1}{2}(n-1)$.

5. (a) For $n \geq 2$, prove that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$$

[*Hint:* Use induction and Pascal's rule.]

(b) From part (a), and the relation $m^2 = 2\binom{m}{2} + m$ for $m \geq 2$, deduce the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(c) Apply the formula in part (a) to obtain a proof that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

[*Hint:* Observe that $(m-1)m = 2\binom{m}{2}$.]

6. Derive the binomial identity

$$\binom{2}{2} + \binom{4}{2} + \binom{6}{2} + \cdots + \binom{2n}{2} = \frac{n(n+1)(4n-1)}{6} \qquad n \geq 2$$

[*Hint:* For $m \geq 2$, $\binom{2m}{2} = 2\binom{m}{2} + m^2$.]

7. For $n \geq 1$, verify that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \binom{2n+1}{3}$$

8. Show that, for $n \geq 1$,

$$\binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} 2^{2n}$$

9. Establish the inequality $2^n < \binom{2n}{n} < 2^{2n}$, for $n > 1$.
[*Hint:* Put $x = 2 \cdot 4 \cdot 6 \cdots (2n)$, $y = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, and $z = 1 \cdot 2 \cdot 3 \cdots n$; show that $x > y > z$, hence $x^2 > xy > xz$.]

10. The *Catalan numbers*, defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} \qquad n = 0, 1, 2, \ldots$$

form the sequence $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$. They first appeared in 1838 when Eugène Catalan (1814–1894) showed that there are C_n ways of parenthesizing a nonassociative product of $n+1$ factors. [For instance, when $n = 3$ there are five ways: $((ab)c)d$, $(a(bc))d$, $a((bc)d)$, $a(b(cd))$, $(ab)(ac)$.] For $n \geq 1$, prove that C_n can be given inductively by

$$C_n = \frac{2(2n-1)}{n+1} C_{n-1}$$