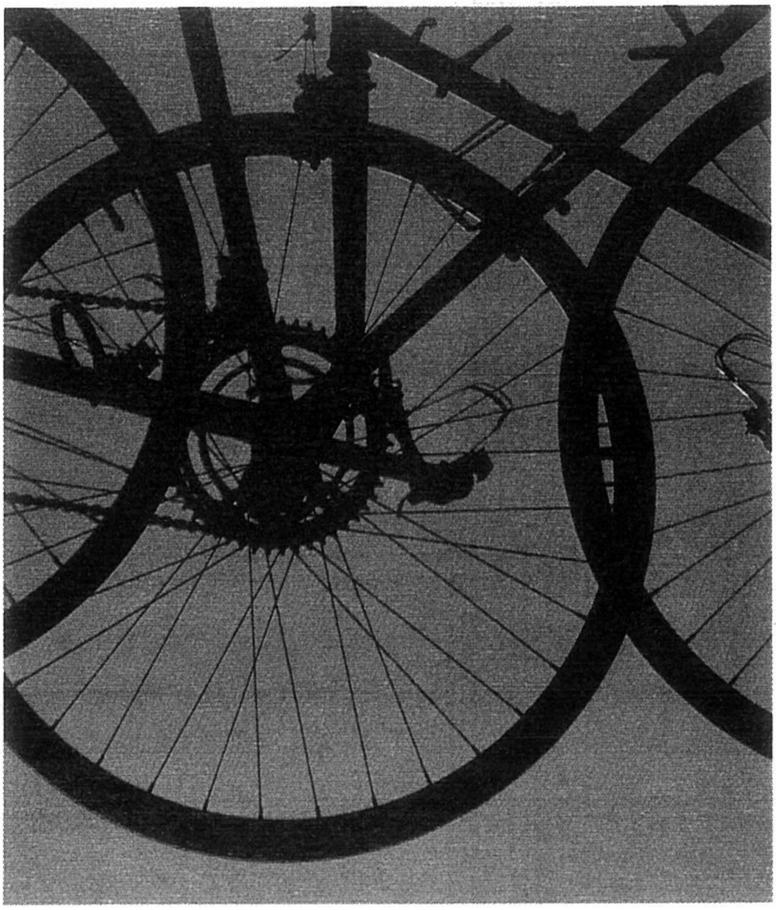
8

Circles

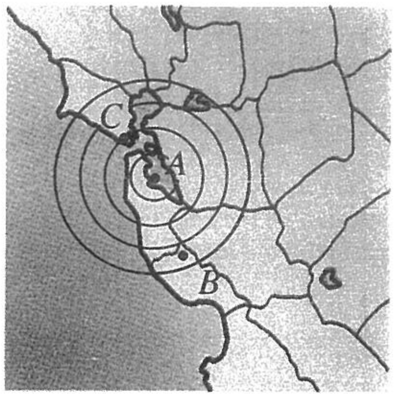


A bicycle wheel has a radius of 14 in. Find the diameter and circumference of the wheel.

OBJECTIVE: *Apply basic definitions and concepts related to circles.*

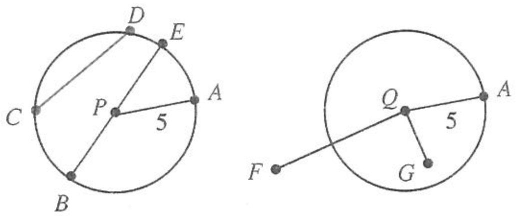
8-1 Basic Terms

A **circle** is the set of all points in a plane that are a given distance from a given point in the plane called the **center.** A **radius** of the circle is a segment with one endpoint the center and the other endpoint on the circle. The length of the radius is called *the radius* of the circle. A circle is named by its center. The circle below may be referred to as circle P ($\odot P$). In $\odot P$, $\overline{PA}$ is a radius and the radius is 5. All radii of a circle are congruent.

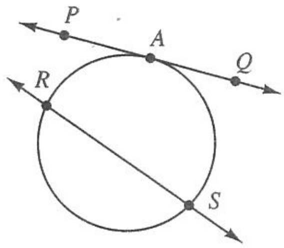


The distance from the airport is shown on this map by a set of circles with center A. If the radius of each successive circle represents an additional 10 mi, about how far are B and C from the airport at A?

Congruent circles are circles that have congruent radii. So $\odot P$ and $\odot Q$ below are congruent.

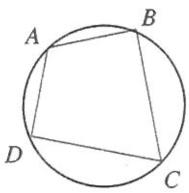


A **chord** is a segment that joins two points on the circle. A **diameter** is a chord through the center of the circle. For $\odot P$, $\overline{CD}$ is a chord and $\overline{BE}$ is a diameter. Notice that a diameter is twice as long as a radius. Point F is *on the exterior of* the circle so $QF > 5$; point G is *on the interior of* the circle so $QG < 5$.

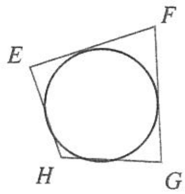


A line that contains a chord of the circle is a **secant.** A line in the plane of the circle that intersects the circle in exactly one point is a tangent to the circle. That point is the **point of tangency.** In the figure to the right, $\overleftrightarrow{RS}$ is a secant and $\overleftrightarrow{PQ}$ is a tangent with A the point of tangency for $\overleftrightarrow{PQ}$.

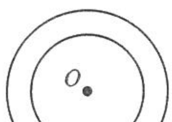
A polygon is **inscribed** in a circle if all its vertices are on the circle. A polygon is **circumscribed** about a circle if each of its sides is tangent to the circle. Two circles in the same plane with the same center are called **concentric** circles.



ABCD is an inscribed polygon.

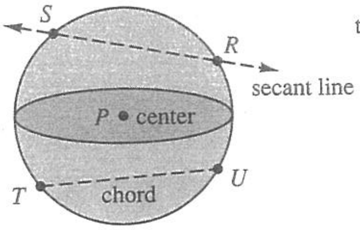


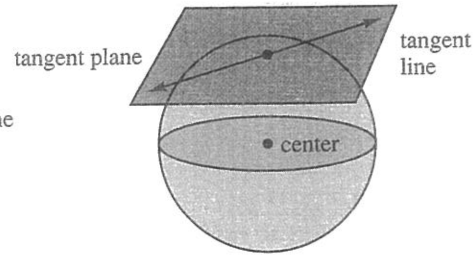
EFGH is a circumscribed polygon.



concentric circles with center O

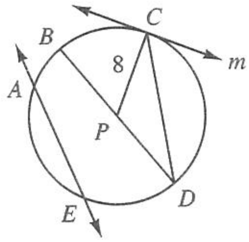
A **sphere** is the set of all points in space a given distance from a point called the *center* of the sphere. The given distance is called the radius of the sphere.





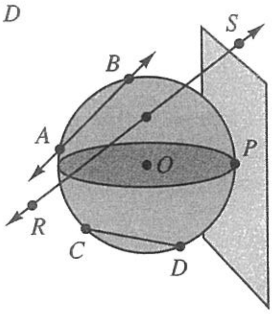
Chords, secants, and tangents of spheres are defined for spheres in a way similar to the way they are for circles. A plane is *tangent to the sphere* and is called a *tangent plane* if it contains exactly one point of the sphere.

Class Exercises



Short Answer

1. Find the length of the radius of ⊙P.
2. Name two chords of ⊙P that are not diameters.
3. Name a diameter of ⊙P.
4. Find the length of a diameter of ⊙P.
5. Name a secant of ⊙P.
6. Name a tangent of ⊙P.
7. Name the point of tangency for the tangent plane to sphere O.
8. Name a secant of sphere O.
9. Name a chord of sphere O.
10. Name a line that is tangent to sphere O.



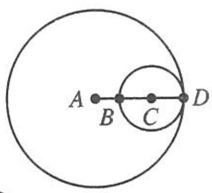
Sample Exercises

Complete each statement.

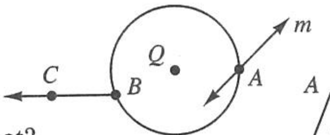
11. The diameter of a circle
    a. with radius 12 is ___. **b.** with radius $\sqrt{5}$ is ___. **c.** with radius $\frac{\sqrt{3}}{2}$ is ___.
12. The radius of a circle
    a. with diameter 24 is ___. **b.** with diameter 15 is ___. **c.** with diameter $\frac{5}{2}$ is ___.

The radius of ⊙C is 5 cm, the radius of ⊙A is 13 cm. Complete each statement.

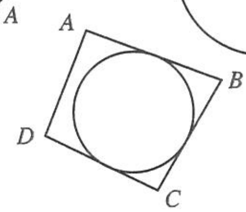
13. AB = ___ 14. AC = ___



Discussion



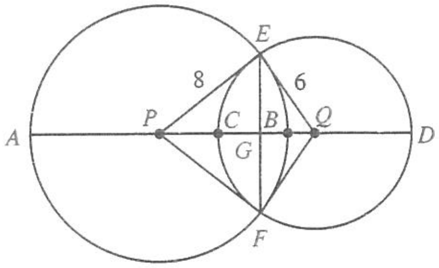
15. Is line m a tangent line? Why or why not?
16. Is point A a point of tangency? Why or why not?
17. Is line m a secant? Why or why not?
18. Is ray BC a tangent? Why or why not?
19. Explain why $ABCD$ is not a circumscribed polygon for the circle shown.



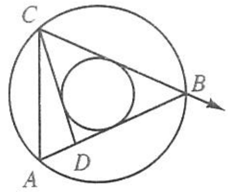
Exercises

A

1. Name a radius of $\odot P$. Find its length.
2. Find the length of a radius of $\odot Q$.
3. BQ = _?_
4. CP = _?_
5. Name a segment that is a chord of both circles.
6. What kind of triangle is $\triangle PEF$?
7. Explain why $\triangle PEQ \cong \triangle PFQ$.
8. Explain why $\triangle GEQ \cong \triangle GFQ$.



9. Name a triangle that is inscribed in a circle.
10. Name a triangle that is circumscribed about a circle.
11. Name a secant line and the circle for which it is a secant.
12. Name a tangent and the circle for which it is a tangent.



Point *P* lies outside $\odot O$.

13. How many lines through P are tangents to $\odot O$?
14. How many lines through P are secants for $\odot O$?

Point *Q* lies inside $\odot O$.

15. How many lines through Q are tangents of $\odot O$?
16. How many lines through Q contain a diameter of $\odot O$?

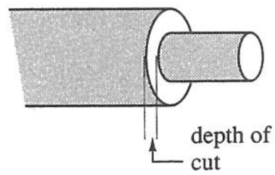
B

Draw a circle and then complete the figure as indicated.

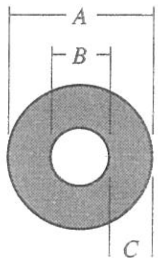
17. Draw a segment in the interior of the circle that is not a chord.
18. Draw a segment that intersects the circle in two points, but is not a diameter, or chord.
19. Draw a line that is neither a secant nor a tangent line.
20. Draw a ray that intersects the circle in only one point but is not a tangent.
21. Draw a segment that has one endpoint on the circle but is not a chord.

Determine whether each statement is always, sometimes, or never true.

22. A chord is a diameter.
23. A secant to a circle is a tangent to the same circle.
24. A segment with one endpoint the center of a circle is a radius of the circle.
25. A line that passes through a point inside a circle is a secant for that circle.
26. A line that lies in a plane that is tangent to a sphere is a tangent line of the sphere.
27. A handle 0.41 cm in diameter is reduced to a diameter of 0.34 cm as shown to the right. Find the depth of cut.
28. Give a convincing argument that if a plane and a sphere intersect, the intersection is a single point, or a circle.



29. A washer with a listed size of $\frac{9}{16}$ in. has an inside diameter B of 0.562 in. and an outside diameter A of 1.156 in. Find the width C of the washer.



C

Use always, sometimes, or never to complete the statements below about a pair of concentric circles.

30. There ___ exists a line that is a tangent to one circle and a secant to the other circle.
31. There ___ exists a line that is a tangent to both circles.
32. A line that is a secant to one circle is ___ a secant for the other circle.
33. A line that intersects the inside of the small circle is ___ a secant for both circles.

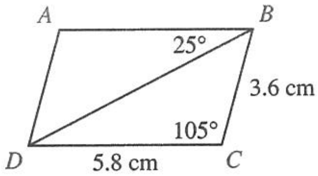
Critical Thinking

Complete each statement to form a correct generalization.

34. A line that contains an interior point of a circle and is in the same plane as the circle is a ___ line.
35. Two circles can intersect each other in at most ___ points.

Mixed Review

***ABCD* is a parallelogram. Complete each statement.**



1. $m\angle A =$ ___
2. $m\angle ABC =$ ___
3. $m\angle ABD =$ ___
4. $m\angle ADB =$ ___
5. $m\angle DBC =$ ___
6. $m\angle ADC =$ ___
7. $AD =$ ___
8. $AB =$ ___

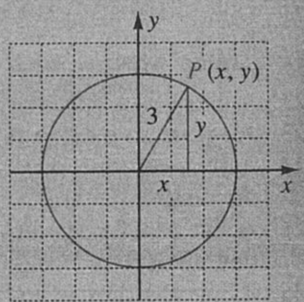
CONNECTIONS

Algebra and Geometry

A rectangular coordinate system of the plane allows you to connect geometry and algebra. See how a circle can be described using algebra.

The circle shown has radius 3 with center at the origin of a rectangular coordinate system. Point $P(x, y)$ is on the circle.
Point P and the origin are vertices of a right triangle with sides of length x, y, and 3. You can use the Pythagorean Theorem to obtain the equation $x^2 + y^2 = 3^2$.

Since the coordinates of each point on the circle satisfy this equation, it is called an equation of the circle.



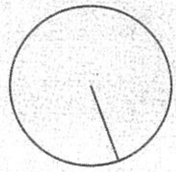
Exercises

1. Write the equation of the circle with center (0, 0) and radius 4.
2. Write the equation of the circle with center (0, 0) and radius 8.

OBJECTIVE: *State and apply theorems regarding circles and tangents.*

8-2 Tangent Lines

Use a compass and straightedge to construct a line ℓ perpendicular to a radius of a circle. What kind of line is line ℓ?



In the previous lesson, you learned that a line is tangent to a circle if it lies in the plane of the circle and intersects the circle in only one point. Theorem 8.1 and its converse Theorem 8.2 are about tangents to a circle.

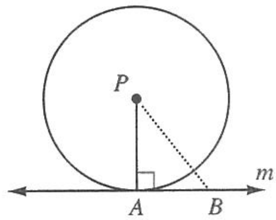
THEOREM 8.1

If a line is perpendicular to a radius of a circle at a point on the circle, then the line is tangent to the circle.

Given: Line m is $\perp$ PA at A.

Prove: Line m is a tangent to $\odot P$.

Proof Let B be any point on m other than A. Since $PA \perp m$, $\triangle PAB$ is a right triangle with hypotenuse PB. This means that $PB > PA$ and B must be on the exterior of the circle. Therefore B cannot lie on the circle and point A is the only point of m that is on the circle. It follows that m is tangent to the circle.



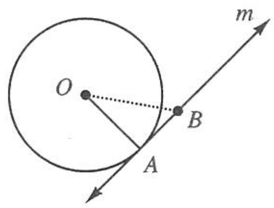
THEOREM 8.2

If a line is tangent to a circle, then it is perpendicular to the radius at the point of tangency.

Given: Line m is tangent at point A.

Prove: Line m is $\perp$ $\overline{OA}$.

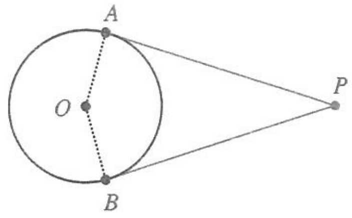
Proof Use an indirect proof. So assume that the desired conclusion is not true. That is, suppose $\overline{OA}$ is not $\perp$ m. Then there exists a segment $\overline{OB}$ which is $\perp$ m. If so, $OB < OA$. But B lies exterior to the circle since m is a tangent line. This means that $OB > OA$. Statements $OB < OA$ and $OB > OA$ are contradictory. Therefore the supposition is false and m is $\perp$ $\overline{OA}$.



Many theorems in geometry have interesting relationships with one another. In Exercises 18 and 19 you may decide to apply Theorem 8.2 to prove Theorem 8.3 and its corollary.

THEOREM 8.3

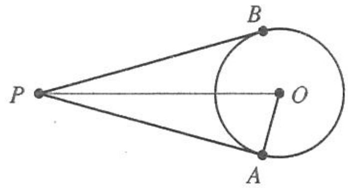
The two tangent segments from an exterior point of a circle are congruent.



COROLLARY 8.3a

The line through an external point and the center of a circle bisects the angle formed by the two tangents from the external point.

Example 1



$\overline{PA}$ and $\overline{PB}$ are tangent segments from P.
$PO = 17$, $OA = 8$, $m\angle APO = 28$.
Find **a.** $m\angle POA$ **b.** PB

Solution

Let $PA = x$

a. $m\angle POA = 90 - 28$ *Since PA is tangent, conclude*
$= 62$ *from Theorem 8.2 that $\overline{PA} \perp \overline{AO}$.*

b. $8^2 + x^2 = 17^2$ *Use the Pythagorean Theorem.*
$x^2 = 17^2 - 8^2$
$x = 15$
$PA = 15$

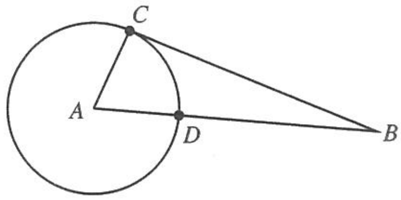
Since $PB = PA$, $PB = 15$.

Try This

Find PO if $OA = 5$ and $PB = 12$.

Example 2

BC is tangent to $\odot A$ from B. $BC = 12$, $AC = 5$. Find BD.



Solution

$AB^2 = 5^2 + 12^2$ *Apply the Pythagorean Theorem.*
$AB^2 = 25 + 144 = 169$ *From Theorem 8.2 we know that $\triangle ACB$ is a right triangle.*

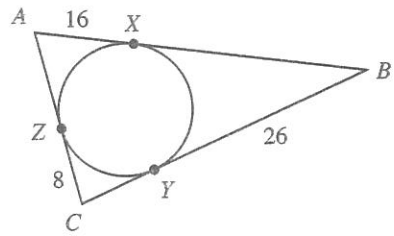
$AB = 13$, $BD = 13 - 5$ or $BD = 8$.

Try This

$BD = 8$ and the radius of the circle is 7. Find BC.

Example 3

$\triangle ABC$ is a circumscribed triangle. Find $AB + BC + AC$.



Solution

$AX = AZ = 16$
$BX = BY = 26$
$CY = CZ = 8$
These are tangent segments from exterior points.
$AB + BC + AC = (16 + 26) + (26 + 8) + (8 + 16)$
$= 100$

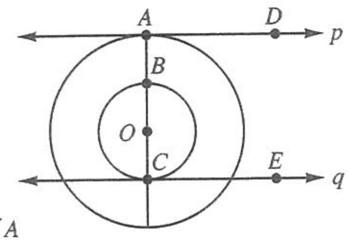
Try This

Find $AB + BC + AC$ if $BX = 21$, $CZ = 7$, and $AC = 22$.

Class Exercises

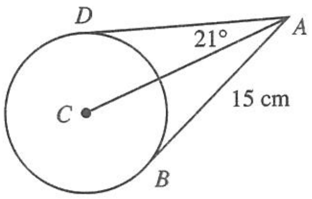
Short Answer

Consider the pair of concentric circles with center *O* and complete each statement.



1. Suppose that p is tangent to the circle. Then $m\angle OAD =$ ___.
2. Suppose that $m\angle OCE = 90$. Then q is ___ to the circle at C.
3. Suppose that q is tangent to the circle at C and that $p \parallel q$. Then $m\angle CAD =$ ___.

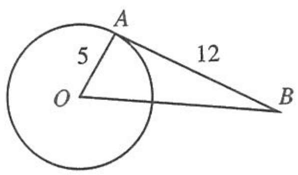
***AB* and *AD* are tangent segments from *A*. Complete each statement.**



4. $AD =$ ___
5. $m\angle BAC =$ ___

Sample Exercises

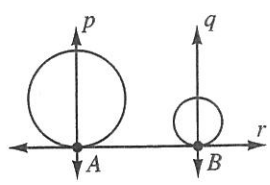
6. If the radius of the circle to the right is 8 cm then $AC =$?.



***AB* is a tangent segment from *B*. Complete each statement.**

7. $m\angle OAB =$ ___
8. If $AB = 12$ and $OA = 5$, then $OB =$ ___.
9. If $m\angle ABO = 25$, then $m\angle AOB =$ ___.

Discussion

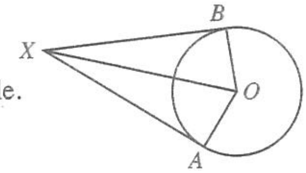


10. If r is tangent to both circles and if lines p and q pass through the centers of the circles and the points of tangency, give a convincing argument that p is parallel to q.
11. In the figure suppose that lines p and q pass through the centers of the circles and are perpendicular to line r at points A and B which are points on the circles. Give a convincing argument that r is tangent to both circles.

Exercises

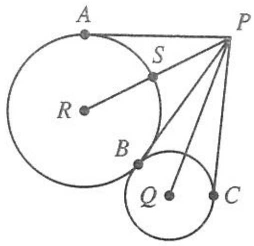
A

$\overline{XA}$ and $\overline{XB}$ are tangent from the external point X.



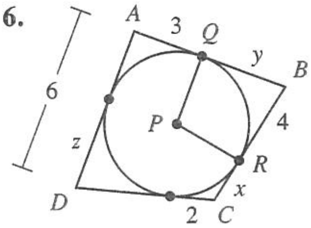
1. If $OA = 7$ and $XO = 25$, find AX.
2. If $XA = 24$ and $XO = 26$, find the radius of the circle.
3. $m\angle AXO = 32$ Find $m\angle AXB$.
4. $m\angle AOX = 48$ Find $m\angle AXB$.
5. $m\angle AXB = 38$ Find $m\angle AOB$.

$\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ are tangents to $\odot Q$ and $\odot R$ from an external point P.



6. If $PA = 10$, find PC.
7. If $PA = 15$ and $AR = 8$, find PS.
8. If $m\angle RPQ = 35$, find $m\angle APC$.

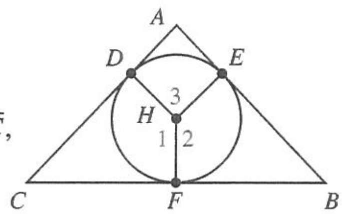
$ABCD$ is a circumscribed quadrilateral with $AD = 6$.



9. Find x.
10. Find y.
11. Find z.
12. Find CD.
13. Find the perimeter of $ABCD$.
14. **Given:** $m\angle QPR = 90$

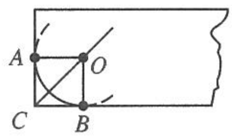
    Prove: $PQBR$ is a square.

$\triangle ABC$ is a circumscribed triangle.



15. Prove that if $\angle 1 \cong \angle 2 \cong \angle 3$, then $\triangle ABC$ is an equilateral triangle.
16. Prove that $\triangle ABC$ is an isosceles triangle with base $\overline{BC}$, if and only if $\angle 1 \cong \angle 2$.

17. Suppose that a corner of a board is to be rounded off. An angle bisector $\overline{CO}$ is drawn, and A and B are points on the edge of the board so that $\overline{OA}$ and $\overline{OB}$ are perpendicular to the edges of the board. If a circle is drawn centered at O with radius $\overline{OA}$, why do you know that the two edges of the board are tangent to the drawn circle?

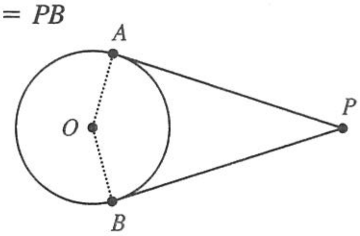


B

✓ 18. Prove Theorem 8.3.

Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$.

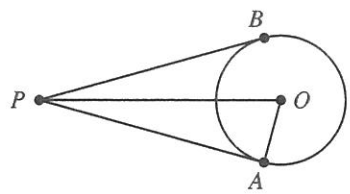
Prove: $PA = PB$



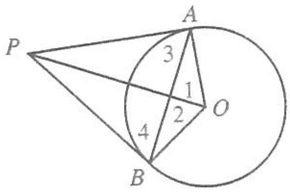
✓ 19. Prove Corollary 8.3a.

Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$.

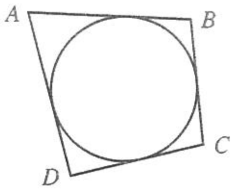
Prove: OP bisects $\angle BPA$.



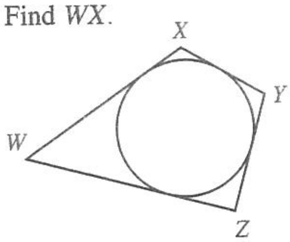
20. Construct a circle and a line tangent to it.
21. Construct two circles with a common tangent.
22. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents from external point P.
Prove: $\angle 1 \cong \angle 2$
23. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents from external point P.
Prove: $\angle 3 \cong \angle 4$



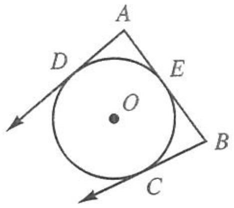
24. Given: circumscribed polygon $ABCD$
$AB = 12$, $BC = 15$, $CD = 25$
Find AD.



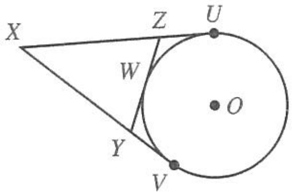
25. Given: circumscribed polygon $WXYZ$
$XY = 8$, $YZ = 10$, $WZ = 21$
Find WX.



26. **Given:** $\overline{AD}$, $\overline{AB}$, and $\overline{BC}$ are tangents to $\odot O$.
Prove: $AD + BC = AB$

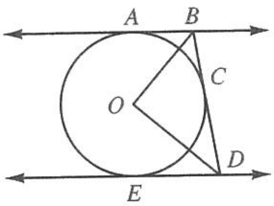


27. Given: $\overline{XU}$ and $\overline{XV}$ are tangents to $\odot O$.
$XO = 17$, $OU = 8$, $ZW = WY$
Find $XZ + YZ + XY$.



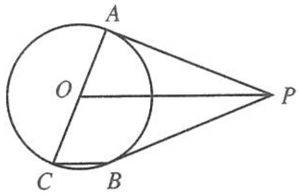
$\overline{AB}$, $\overline{BD}$, and $\overline{DE}$ are tangents to $\odot O$.

28. **Given:** $\overline{BO} \perp \overline{DO}$
Prove: $\overline{AB} \parallel \overline{DE}$
29. **Given:** $\overline{AB} \parallel \overline{DE}$
Prove: $\overline{BO} \perp \overline{DO}$

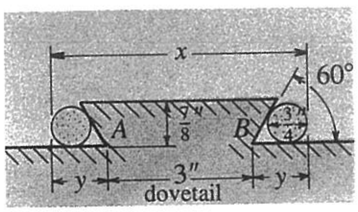


C

30. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents to $\odot O$.
$\overline{AC}$ is a diameter.
Prove: $\overline{BC} \parallel \overline{PO}$

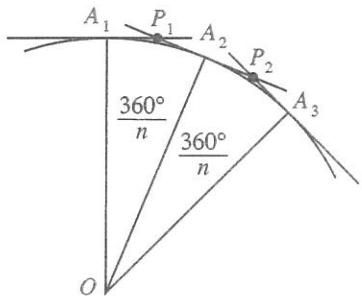


31. Suppose that quadrilateral $WXYZ$ is circumscribed about a circle. Prove that $XY + WZ = WX + YZ$.
32. Quality control in the manufacturing of machine parts often requires unusual methods of measurement. For example, in order to check that the angles A and B are correct on a part called a dovetail, circular plugs are inserted as shown. Then the distance x is measured with a micrometer. For the dovetail shown here, what should this distance equal?



Critical Thinking

33. Suppose points $A_1, A_2, \ldots, A_n$ are arranged around a circle so that for each i, $i = 1, 2, \ldots, n$, $m\angle A_iOA_{i+1} = \frac{360}{n}$ for all n. Suppose tangent lines are drawn at each point A_i intersecting at points $P_1, P_2, \ldots, P_n$. State a generalization about the kind of polygon $P_1P_2 \ldots P_n$ is. Give a convincing argument that your answer is correct.



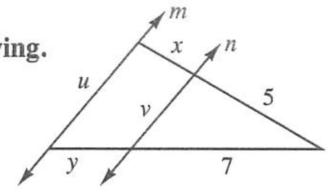
Mixed Review

$m \parallel n$ **Write a proportion involving each of the following.**

1. u and v **2.** x and u **3.** x and y

Find y.

4. $x = 2$ **5.** $x = 3$ **6.** $x = 1.8$

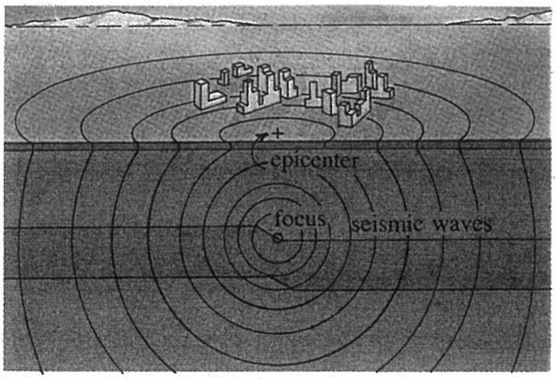


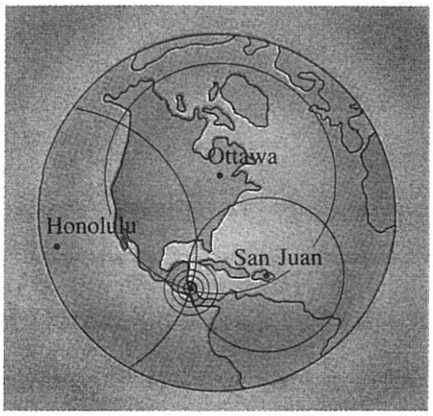
Career

Seismologist

An earthquake causes motions of the earth's surface, the site of which is the focus of the earthquake. The focus is centered in rocks that have broken and shifted position and most of the energy released in an earthquake travels from the focus in seismic waves.

A seismologist studies and interprets data about earthquakes in an effort to predict when and where they will take place. Seismologists locate earthquakes by studying the time intervals between which the seismic waves reach different seismographic stations. Circles are drawn on the map to show the distance of the earthquake from each station; the epicenter of the earthquake is located where the circles intersect.

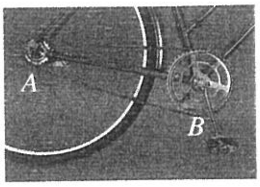




OBJECTIVE: *Calculate lengths related to common tangents to circles.*

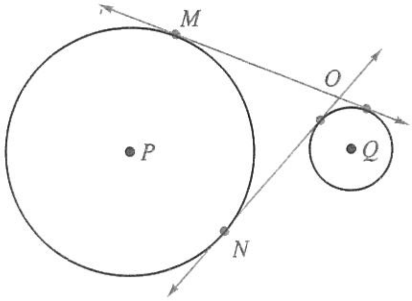
8-3 Common Tangents and Tangent Circles

A line that is tangent to two coplanar circles is called a **common tangent** of the two circles. Common tangents are classified into two types. A common **external tangent** does not intersect the segment joining the centers of the two circles. A common **internal tangent** does intersect the segment joining the centers of the two circles.



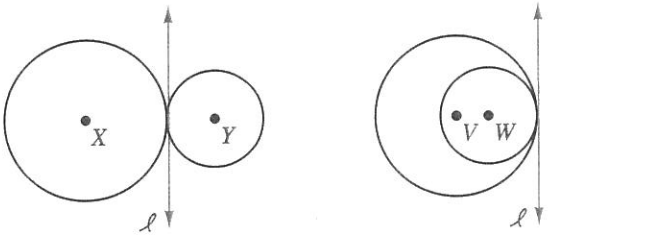
A belt wrapped around two wheels transfers power from one wheel to the other. In this lesson you will learn how to calculate the length AB on such a belt linkage system.

In the figure to the right, line MO is a common external tangent to circles P and Q. Line NO is a common internal tangent to circles P and Q.



Two circles are called tangent circles if they intersect in exactly one point and they may be externally tangent or internally tangent.

In the figure below you can see that two tangent circles are called externally tangent if each lies in the exterior of the other. They are called internally tangent if one lies in the interior of the other.



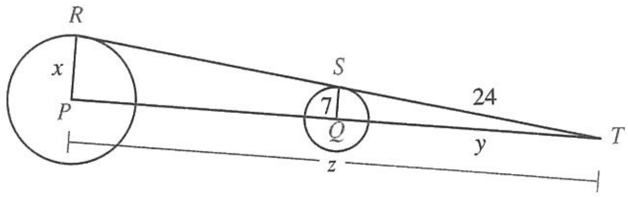
Externally tangent circles　　　Internally tangent circles

The following example illustrates how similar triangles can be used to find unknown lengths in problems that involve common tangents.

Example 1

$\overline{RT}$ is a common tangent to $\odot P$ and $\odot Q$.
$QS = 7$, $ST = 24$, $RT = 48$

a. $x = PR =$ ___?___　**b.** $y = QT =$ ___?___



Solution

$\triangle PRT \sim \triangle QST$　*Both are right triangles with a common acute angle and so are similar by AA Similarity.*

a. $\frac{x}{48} = \frac{7}{24}$ or $x = PR = 14$　*Corresponding sides of similar triangles are proportional.*

b. $y^2 = 7^2 + 24^2$ or $y = QT = 25$　*Apply the Pythagorean Theorem.*

Try This

$z = PT =$ ___?___

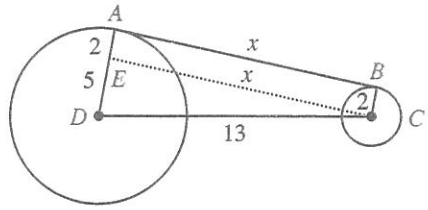
Example 2

$\overline{AB}$ is a common external tangent to circles D and C and $DC = 13$. Find AB.

Solution

$13^2 = 5^2 + x^2$ *Draw $\overline{CE}$ to complete rectangle ABCE. Apply the Pythagorean Theorem.*

$x = AB = 12$



Try This

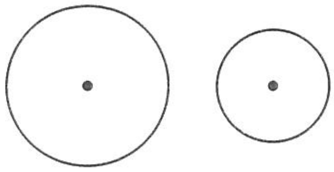
Suppose $\odot C$ is moved so that $DC = 15$. Find AB.

Class Exercises

Short Answer

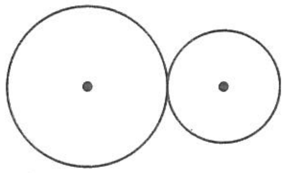
State the number of external tangents and the number of internal tangents that exist for each pair of circles.

1\.



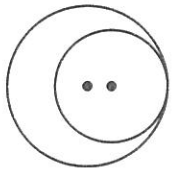
Disjoint circles

2\.



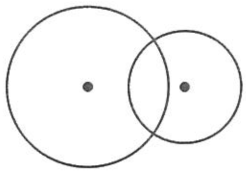
Externally tangent circles

3\.



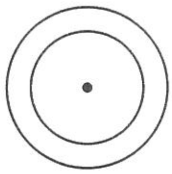
Internally tangent circles

4\.



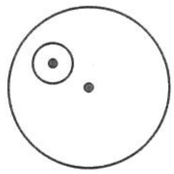
Intersecting circles

5\.



Concentric circles

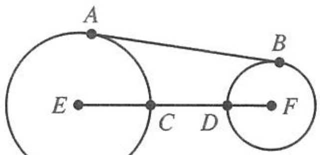
6\.



One inside the other

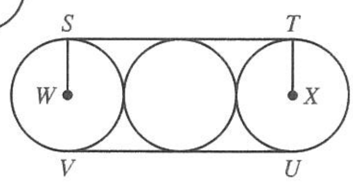
Sample Exercises

$\overline{AB}$ is a common tangent to $\odot E$ and $\odot F$.



7. $EA = 10$, $FB = 3$, $CD = 12$ Find EF.
8. $EA = 10$, $FB = 3$, $CD = 12$ Find AB.
9. $EA = 10$, $FB = 2$, $CD = 5$ Find AB.

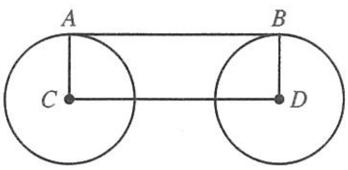
Complete each statement.



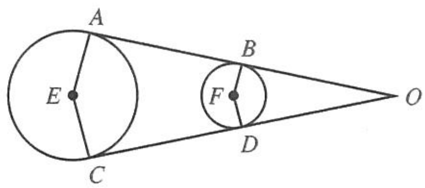
10. $ST =$ ___ **11.** $UV =$ ___

Discussion

12. $\overline{AB}$ is a common tangent between circles of equal radii. Give a convincing argument that $ABDC$ is a rectangle.



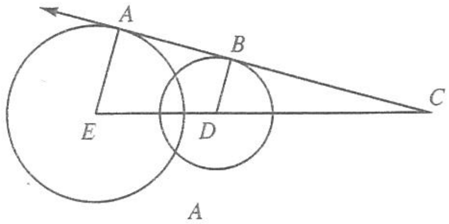
13. $\overline{AB}$ and $\overline{CD}$ are common tangents. Give a convincing argument that $\angle AEC \cong \angle BFD$.



Exercises

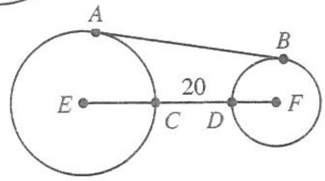
A

$\overline{AB}$ is a common tangent. $AE = 10$, $BD = 5$, $BC = 12$



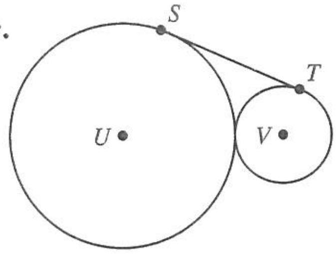
1. Find AC. 2. Find AB.
3. Find CD. 4. Find ED.
5. If $AE = 9$, $BD = 3$, and $BC = 4$ find ED. Are the two circles tangent to each other?

$\overline{AB}$ is a common tangent. $AE = 15$, $BF = 6$, $CD = 20$.



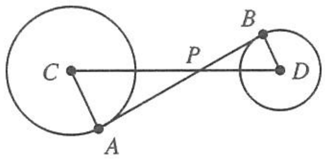
6. Find EF.
7. Find AB.
8. Find $AB + BF + EF + AE$.

$\overline{ST}$ is a common tangent and the circles are tangent to each other.



9. If the radii are 9 and 4, find ST.
10. If the radii are 8 and 5, find ST.
11. If the radii are 9 and 3, find ST.
12. If $ST = 15$, $UV = 17$, and $VT = \frac{9}{2}$, find US.
13. If $ST = 24$, $UV = 25$, and $US = 16$, find VT.

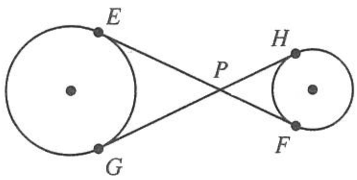
$\overline{AB}$ is a common tangent. $CA = 8$, $CP = 17$, $AP = 15$, $BD = 5$



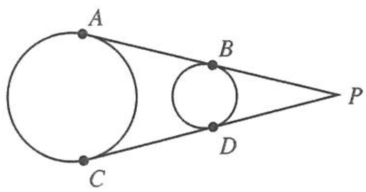
14. Find PB.
15. Find PD.
16. Find AB.
17. Prove that $\angle PCA \cong \angle PDB$.
18. The centers of two circles with radii 4 and 9 are 17 units apart. Find the length of a common external tangent.
19. The centers of two circles with radii 5 and 7 are 14 units apart. Find the length of a common external tangent.

B

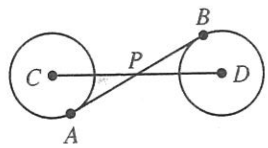
20. **Given:** $\overline{EF}$ and $\overline{GH}$ are common internal tangents.
Prove: $\overline{EF} \cong \overline{GH}$



21. **Given:** $\overline{AB}$ and $\overline{CD}$ are common external tangents that intersect each other at P.
Prove: $\overline{AB} \cong \overline{CD}$

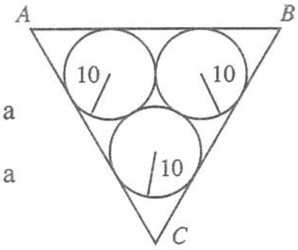


22. Prove that in the figure for Exercise 21, $ABDC$ is a trapezoid.
23. **Given:** $\overline{AB}$ is a common internal tangent of two disjoint congruent circles.
Prove: $\overline{AB}$ and $\overline{CD}$ bisect each other.



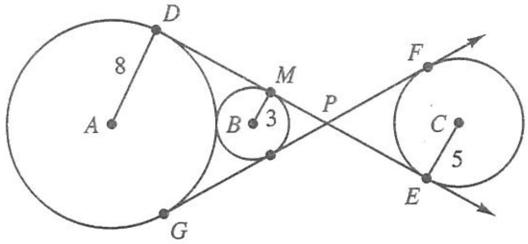
24. Three metal disks each with a radius of 10 cm are tangent to each other. The disks are enclosed by a metal frame that forms an equilateral triangle. What is the length of one side of the frame?

25. Suppose that four disks, each with radius 10 cm and arranged to form a square, are enclosed by a square metal frame similar to the figure. How does the length of a side of this square compare to the length of a side of the triangle in Exercise 24?



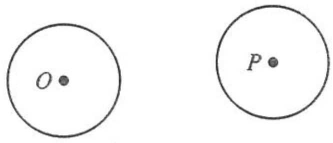
C

26. $\overline{DE}$ and $\overline{FG}$ are common tangents for the three circles. If the radii of the three circles are 8, 3, and 5 as shown, find BC.



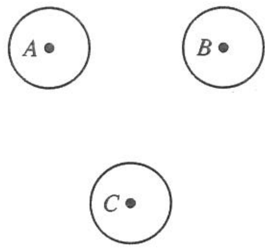
27. Construct $\odot O$ and $\odot P$ each with a radius of 2 such that $OP = 6$. Construct two circles that are tangent to both $\odot O$ and $\odot P$.

28. After completing the constructions in Exercise 27, construct as many additional circles as you can that are tangent to both $\odot O$ and $\odot P$.



Critical Thinking

29. Construct $\odot A$, $\odot B$, and $\odot C$ with equal radii such that $\triangle ABC$ is an equilateral triangle as shown. Next construct at least four circles that are tangent to all three circles. How many other circles that are tangent to all three circles do you think exist?



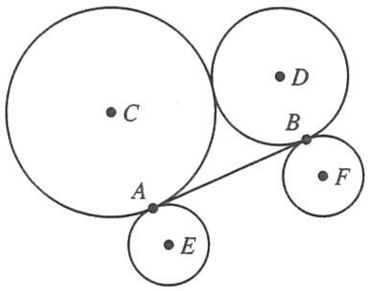
Algebra Review

Multiply.

1. $5(x - 2)$
2. $4(x - 8)$
3. $-6(y - 4)$
4. $-12(y - 4)$
5. $6(2x - 3y - 8z)$
6. $-9(3x - 2y - 5z)$
7. $8(-2x - 6y + 7)$
8. $15(2a + 4b - 8)$

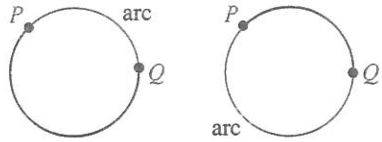
Math Contest Problem

$\overline{AB}$ is a common tangent of $\odot C$, $\odot D$, $\odot E$, and $\odot F$. Suppose that $AC = 12$, $DB = 9$, and $AE = BF = 3$. If the midpoint of $\overline{EF}$ is the center of a circle that is tangent to both $\odot E$ and $\odot F$, what are the possible values for its radius?



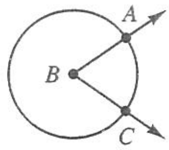
OBJECTIVE: *State and apply theorems about arcs and central angles.*

8-4 Arcs and Their Measure

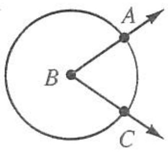


An **arc** of a circle consists of two points and a continuous part of a circle between them. Notice that any two given points on a circle determine two different arcs.

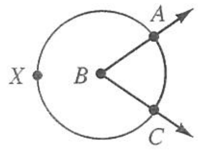
A **central angle** of a circle is an angle whose vertex is the center of the circle. $\angle ABC$ is a central angle of $\odot B$ in the figure shown.



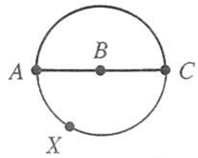
The sides of a central angle separate the circle into two different arcs. The endpoints and all the points on both the circle and the interior of the central angle are called a **minor arc.** The arc in the exterior of the central angle is called a **major arc.** Note that three letters are usually used to name a major arc. If the endpoints are the endpoints of a diameter then the circle is divided into two arcs called **semicircles.**



minor arc $\overset{\frown}{AC}$

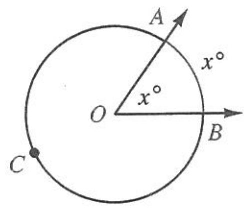


major arc $\overset{\frown}{AXC}$



semicircle $\overset{\frown}{AXC}$

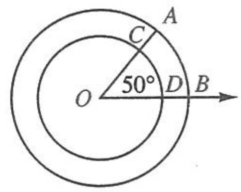
An arc, like an angle, is measured in degrees.



The measure of a minor arc is the measure of its central angle. $m\overset{\frown}{AB} = x$
The measure of a major arc is $360 - m$, where m is the measure of the central angle of its minor arc. $m\overset{\frown}{ACB} = 360 - x$

The measure of a semicircle is 180.

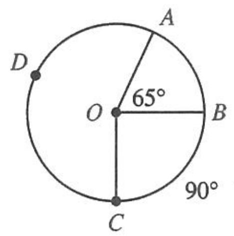
Congruent arcs are arcs with equal measure that lie in the same circle or in congruent circles. In the figure to the right $m\overset{\frown}{AB} = m\overset{\frown}{CD} = 50$. So two arcs with the same measure are not congruent unless they lie in congruent circles.



Example 1

Complete each statement.

a. $m\overset{\frown}{AB} =$ ___ **b.** $m\overset{\frown}{ACB} =$ ___



Solution

a. $m\overset{\frown}{AB} = 65$ **b.** $m\overset{\frown}{ACB} = 360 - 65 = 295$

Try This

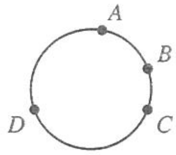
Complete each statement.

a. $m\angle BOC =$ ___ **b.** $m\overset{\frown}{ADC} =$ ___

● POSTULATE 20 Arc Addition Postulate

The measure of adjacent nonoverlapping arcs is the sum of the measures of the two arcs. That is, if C is on $\overset{\frown}{AB}$, then $m\overset{\frown}{AC} + m\overset{\frown}{CB} = m\overset{\frown}{AB}$.

This postulate applies to both minor arcs and major arcs. For example,
$m\overset{\frown}{AC} = m\overset{\frown}{AB} + m\overset{\frown}{BC}$, and
$m\overset{\frown}{DAC} = m\overset{\frown}{DAB} + m\overset{\frown}{AC}$
A point B on $\overset{\frown}{AC}$ is the *midpoint* of $\overset{\frown}{AC}$, if $m\overset{\frown}{AB} = m\overset{\frown}{BC}$.



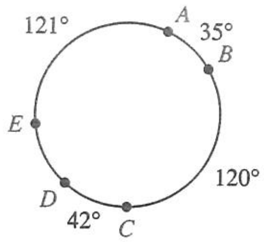
Example 2

Find each measure. **a.** $m\overset{\frown}{AC}$ **b.** $m\overset{\frown}{AD}$

Solution

a. $m\overset{\frown}{AC} = 35 + 120$
$= 155$

b. $m\overset{\frown}{AD} = 360 - (35 + 120 + 42)$
$= 163$



Try This

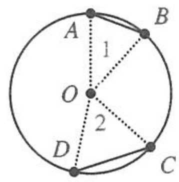
a. $m\overset{\frown}{BD} =$ __?__ **b.** Show that D is the midpoint of $\overset{\frown}{CE}$.

Recall that a chord is a segment that joins two points on a circle. Theorem 8.4 and its converse, Theorem 8.5, describe a relationship between congruent chords and congruent minor arcs.

◆ THEOREM 8.4

In a circle or in congruent circles congruent chords have congruent minor arcs.

Given: $\overline{AB} \cong \overline{CD}$
Prove: $\overset{\frown}{AB} \cong \overset{\frown}{CD}$



Proof

Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$	1. Given
2. Draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$.	2. Two points determine a segment.
3. $\overline{OA} \cong \overline{OB} \cong \overline{OC} \cong \overline{OD}$	3. Radii of a circle are congruent.
4. $\triangle AOB \cong \triangle COD$	4. SSS Congruence Postulate
5. $\angle 1 \cong \angle 2$	5. Corr. parts of $\cong$ $\triangle$'s are $\cong$.
6. $m\angle 1 = m\overset{\frown}{AB}$	6. Definition of measure of minor arc
7. $m\angle 2 = m\overset{\frown}{CD}$	7. Definition of measure of minor arc
8. $m\overset{\frown}{AB} = m\overset{\frown}{CD}$	8. Substitution

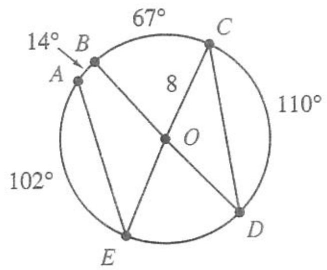
◆ THEOREM 8.5

In a circle or in congruent circles congruent minor arcs have congruent chords and congruent central angles.

Class Exercises

Short Answer

1. What is the radius of $\odot O$?
2. Name a minor arc.
3. Name a major arc.
4. Name a semicircle.
5. Name two congruent arcs.
6. Name the central angle of $\overset{\frown}{CD}$.
7. Name two arcs with endpoint E.

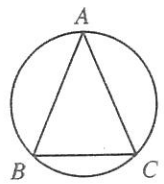


Sample Exercises

8. Find $m\overset{\frown}{BC}$.
9. Find $m\overset{\frown}{CD}$.
10. Find $m\overset{\frown}{AC}$.
11. Find $m\overset{\frown}{ACD}$.
12. Find $m\overset{\frown}{AEC}$.
13. Find $m\overset{\frown}{ADB}$.
14. Explain why $\overline{BD}$ is not a diameter.

Discussion Exercises

15. Suppose that A, B, and C are points on a circle. Give a convincing argument that $m\overset{\frown}{ACB} + m\overset{\frown}{AB} = 360$.
16. Suppose that $m\overset{\frown}{AB} = m\overset{\frown}{AC}$. Give a convincing argument that $\triangle ABC$ is isosceles.

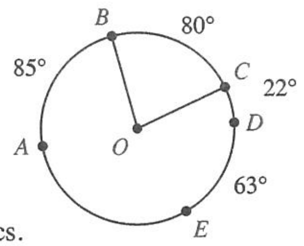


Exercises

A

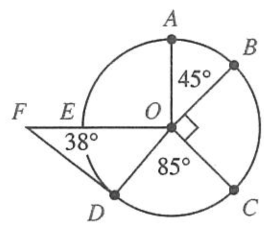
Find each measure.

1. $m\overset{\frown}{BC}$
2. $m\overset{\frown}{AC}$
3. $m\overset{\frown}{BAD}$
4. $m\overset{\frown}{ADC}$
5. $m\angle BOC$
6. $m\overset{\frown}{AE}$
7. Name a pair of congruent arcs.



$\overline{DF}$ is a tangent to $\odot O$ from the external point F and $m\angle OFD = 38$. Find each measure.

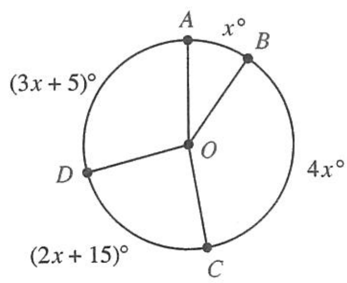
8. $m\overset{\frown}{AB}$
9. $m\overset{\frown}{AD}$
10. $m\overset{\frown}{AC}$
11. $m\overset{\frown}{BC}$
12. $m\overset{\frown}{ADC}$
13. $m\overset{\frown}{ACD}$
14. $m\angle DOF$
15. $m\overset{\frown}{ED}$
16. $m\overset{\frown}{AE}$



B

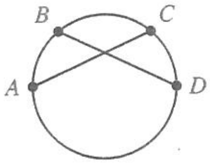
Find each measure.

17. $m\angle AOB$
18. $m\angle BOC$
19. $m\angle COD$
20. $m\angle AOD$



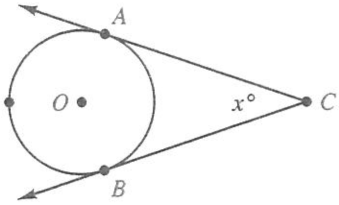
21. **Given:** $\overset{\frown}{AB} \cong \overset{\frown}{CD}$
Prove: $\overset{\frown}{AC} \cong \overset{\frown}{BD}$

22. **Given:** $\overset{\frown}{AC} \cong \overset{\frown}{BD}$
Prove: $\overset{\frown}{AB} \cong \overset{\frown}{CD}$

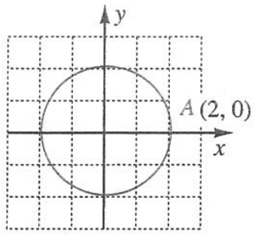


23. **Given:** $\odot O$ with tangents $\overline{CA}$ and $\overline{CB}$
$m\angle ACB = x$
Prove: $m\overset{\frown}{AB} = 180 - x$

24. **Given:** $\odot O$ with tangents $\overline{CA}$ and $\overline{CB}$
$m\angle ACB = x$
Prove: $m\overset{\frown}{ADB} = 180 + x$

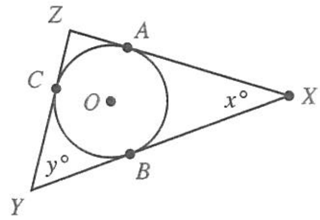


25. Find the coordinates of a point B above the x axis such that $m\overset{\frown}{AB} = 90$.
26. Find the coordinates of a point C below the x axis such that $m\overset{\frown}{AC} = 90$.
27. Find the coordinates of a point D in the quadrant I such that $m\overset{\frown}{AD} = 45$.
28. Find the coordinates of a point E in the quadrant III such that $m\overset{\frown}{AE} = 135$.

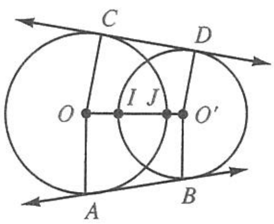


C

29. **Given:** $\triangle XYZ$ is a circumscribed triangle.
$m\angle X = x$
$m\angle Y = y$
Prove: $m\overset{\frown}{AC} = x + y$



30. **Given:** $\overleftrightarrow{AB}$ is a tangent of $\odot O$ and $\odot O'$.
I is the midpoint of $\overset{\frown}{BID}$.
J is the midpoint of $\overset{\frown}{AC}$.
$\overleftrightarrow{CD}$ is the tangent of $\odot O'$.
Prove: $\overleftrightarrow{CD}$ is tangent to $\odot O$.



Critical Thinking

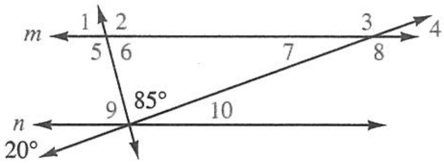
31. Complete the table below and discover a formula for the number of chords and arcs for n points on a circle.

number of points	2	3	4	5	. . .	n
number of chords	1	3	6	___	. . .	___
number of arcs	2	6	12	___	. . .	___

Mixed Review

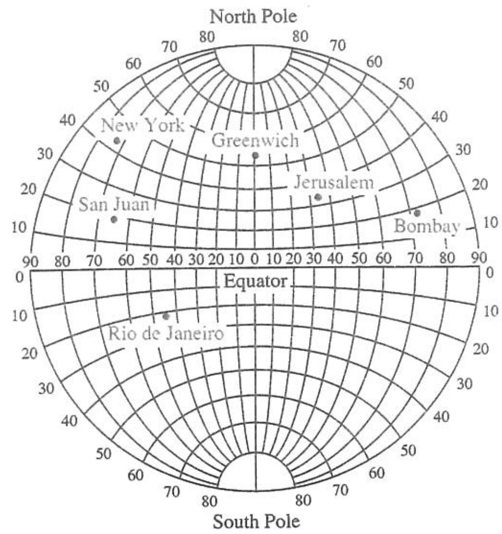
Lines m and n are parallel. Find each measure.

1. $m\angle 10$
2. $m\angle 9$
3. $m\angle 2$
4. $m\angle 6$
5. $m\angle 7$
6. $m\angle 8$



Enrichment

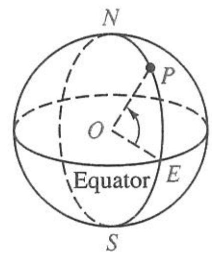
The Earth as a Sphere



As you know, the earth is considered a sphere. This may seem strange since there are mountains on the earth's surface several miles high. However, if a scale model of the earth were made 8 ft in diameter it would be smoother than a finely polished bowling ball. The highest mountain would be represented by a projection of only 0.007 in. above the surface. The slight difference between the polar and equatorial diameters would also be negligible. Thus, for most purposes, the earth may be considered a sphere.

To locate a point on the earth, reference lines are needed (much like the grid used for locating points on a plane). Since the earth revolves from east to west on an axis through its poles, a series of circular arcs is drawn through the poles. These are called meridians.

The meridian through Greenwich, England, is arbitrarily chosen as the zero meridian. The other meridians are then defined in degrees both east and west from this meridian. New York City is near the meridian 74° W; Jerusalem is near the meridian 35° E. The degree measure associated with a point on the earth, along with the east or west designation, is called the longitude of the point.



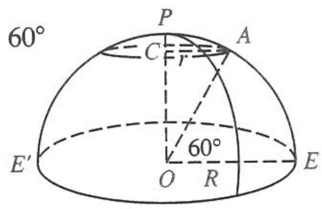
To completely determine a point on the earth, another reference line is needed. This is the equator. A point is located by means of the arc, from the point to the equator, that is intercepted on the meridian passing through the point. Thus, if $m\widehat{PE} = 60$ ($m\angle POE = 60$) the point P is said to have latitude 60° N. The latitude of New York City is about 41° N as shown in the figure. The equator has latitude 0° and the north pole has latitude 90° N.

Longitude and latitude are used in many fields including astronomy, map making, and sailing. For example, to determine the location of a ship at sea, navigators first use an instrument called a sextant to calculate their latitude.

Exercises

Use the above figure to give the approximate latitude and longitude of each city.

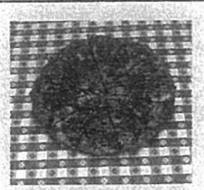
1. Bombay
2. Rio de Janeiro
3. San Juan
4. Show that the radius of the circle of latitude 60° is half the radius of the earth.



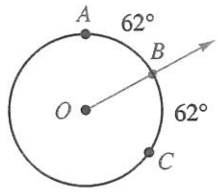
8-5 Chords of Circles

EXPLORE

Get a cardboard pizza platter from a pizza shop. Experiment and discover a way to find the center of the platter.



The midpoint of an arc bisects the arc. Any line or ray containing this midpoint is also said to bisect the arc. In this figure, point B and $\overrightarrow{OB}$ bisect $\overset{\frown}{AC}$.



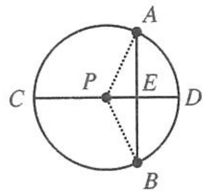
THEOREM 8.6

If a diameter is perpendicular to a chord, then it bisects the chord and its minor and major arcs.

Given: $\odot P$ with diameter $\overline{CD}$ and chord $\overline{AB}$
$\overline{AB} \perp \overline{CD}$

Prove: $\overline{AE} \cong \overline{BE}$, $\overset{\frown}{AD} \cong \overset{\frown}{BD}$, $\overset{\frown}{AC} \cong \overset{\frown}{BC}$

Plan Draw $\overline{PA}$ and $\overline{PB}$ and use HL congruence to conclude that $\triangle APE \cong \triangle BPE$. Then conclude that $\overline{AE} \cong \overline{BE}$ and $\angle APE \cong \angle BPE$. Consequently $\overset{\frown}{AD} \cong \overset{\frown}{BD}$ and $\overset{\frown}{AC} \cong \overset{\frown}{BC}$.



Compare the hypotheses of Theorems 8.6 and 8.7. The hypothesis of each is part of the conclusion of the other.

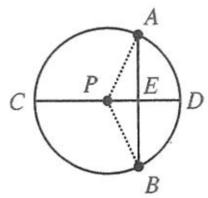
THEOREM 8.7

If a diameter bisects a chord that is not a diameter, then it is perpendicular to the chord and bisects its major and minor arcs.

Given: $\odot P$ has diameter $\overline{CD}$ bisecting chord $\overline{AB}$.

Prove: $\overline{CD} \perp \overline{AB}$, $\overset{\frown}{AD} \cong \overset{\frown}{BD}$, $\overset{\frown}{AC} \cong \overset{\frown}{BC}$

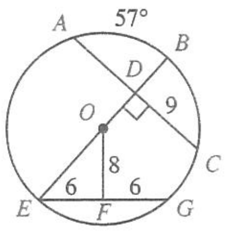
Plan Use SSS congruence to prove $\triangle APE \cong \triangle BPE$. Then use corresponding parts of congruent triangles are congruent to draw the final conclusion.



Example 1

Refer to the figure to complete each statement.

a. $AD =$ ___ **b.** $OE =$ ___



Solution

a. $AD = 9$	*EB ⊥ AC means that EB bisects AC.*
b. $OE^2 = 6^2 + 8^2$	*OF bisects EG and hence ⊥ it.*
$= 100$	*Use the Pythagorean Theorem.*
$OE = 10$	

Try This

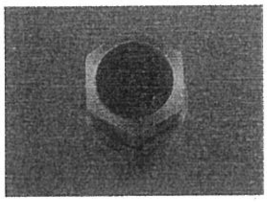
Find $m\widehat{AEC}$ in the figure to the right.

The proofs of the following theorems are completed in Exercises 30–32.

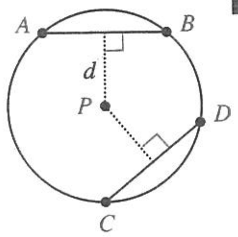
◆ THEOREM 8.8

The perpendicular bisector of a chord contains the center of the circle.

An electrician's drill bit has a shank end with six flat surfaces as shown to the right. These flat surfaces must be an equal distance from the rotating axis to ensure that the drill runs smoothly. How can a machinist determine that this is the case if the center of the drill is not known? To answer this question consider the distance from the center of a circle to a chord of the circle.

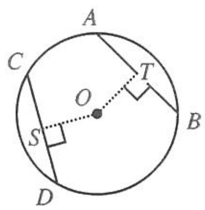


Recall that the distance from a point to a line is defined as the perpendicular distance from the point to the line. In this figure d is the perpendicular distance to $\overline{AB}$ from center P.



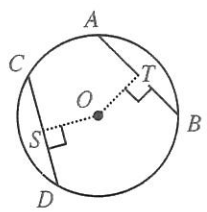
◆ THEOREM 8.9

In the same circle or in congruent circles congruent chords are equidistant from the center.



◆ THEOREM 8.10

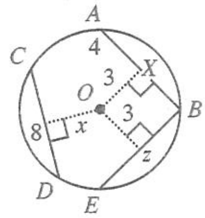
In the same circle or in congruent circles, chords equidistant from the center are congruent.



Example 2

Refer to the figure to complete each statement:

a. $AB =$ ___ **b.** $x =$ ___



Solution

$AB = 8$ *$\overline{OX} \perp \overline{AB}$ means that X is the midpoint of $\overline{AB}$.*

$x = 3$ *Since $\overline{AB}$ and $\overline{CD}$ are the same length, they are the same distance from the center.*

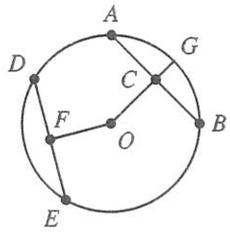
Try This

Find z. Explain your answer.

Class Exercises

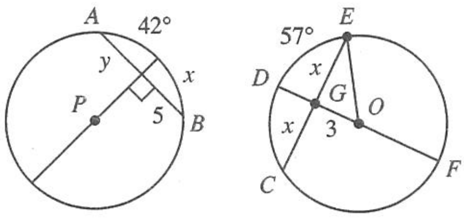
Short Answer

Complete each statement with a valid conclusion.



1. If $\overline{OC}$ is the perpendicular bisector of $\overline{AB}$, then ___.
2. If $AB = DE$, then ___.
3. If C is the midpoint of $\overline{AB}$ and G bisects $\widehat{AB}$, then ___.
4. If $\overline{OF} \perp \overline{DE}$, then ___.
5. If $\overline{OC} \perp \overline{AB}$, $\overline{OF} \perp \overline{DF}$, and $OC = OF$, then ___.

Sample Exercises



6. Find x.
7. Find y.
8. Find $m\angle EGO$.
9. Find $m\angle GEO$.
10. Find $m\widehat{CD}$.

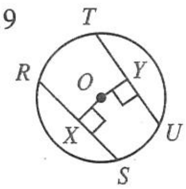
Discussion

11. Explain how you can find the center of this circle in order to redraw a complete circle with a compass.

12. Would the following be the start of a proof of Theorem 8.9 or Theorem 8.10?

    Given: $OX = OY$

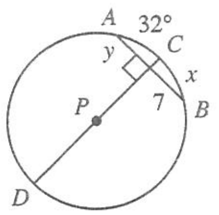
    Prove: $RS = TU$

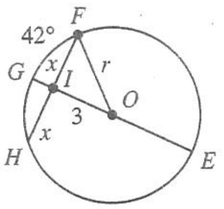


13. Give a convincing argument that Theorem 8.10 is true.

Exercises

A

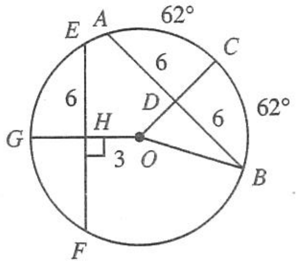




Complete each of the following.

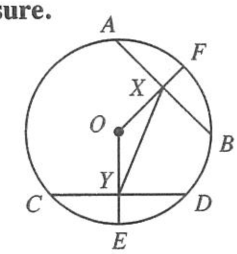
1. $y =$ ___
2. $x =$ ___
3. $m\widehat{AD} =$ ___
4. $m\widehat{ABD} =$ ___
5. $m\angle FIO =$ ___
6. $m\widehat{GH} =$ ___
7. $m\widehat{HE} =$ ___
8. $m\widehat{FEH} =$ ___

9. Find FH.
10. Find $m\widehat{GF}$.
11. Find OD.
12. Find $m\angle ODB$.
13. Find $m\angle OBD$.
14. Find $m\widehat{FE}$.
15. If $m\widehat{GC} = 137$, then $m\widehat{AE} =$ ___.

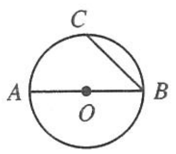


$m\widehat{AB} = m\widehat{CD} = 100$, $m\widehat{BD} = 40$, $\overline{OX} \perp \overline{AB}$, $\overline{OY} \perp \overline{CD}$
Find each measure.

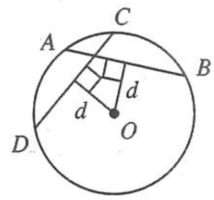
16. $m\widehat{EB}$
17. $m\angle YOB$
18. $m\angle OYX$
19. $m\angle YXB$
20. $m\angle AOC$



21. If $\overline{AB}$ is a diameter, $AB = 8$, and $m\angle ABC = 45$, how far is $\overline{BC}$ from the center of O?

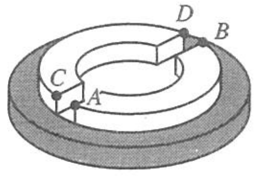


B

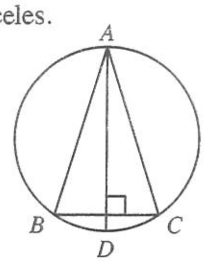


22. Chords $\overline{PQ}$ and $\overline{RS}$ in a circle are equal in length. If $\overline{PQ}$ is distance $4x$ and $\overline{RS}$ is distance x^2 from the center, how far is each chord from the center?
23. $\overline{AB}$ and $\overline{CD}$ are both d units from the center of the circle. $AB = 2x + 7$ and $CD = 3x - 5$. What is the length of each chord?

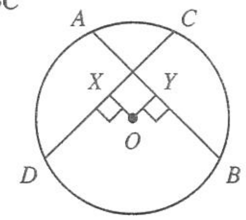
24. Often machine parts like the one shown here will work properly only if the groove is "centered." Why does measuring $\overline{AB}$ and $\overline{CD}$ tell you whether or not the groove is centered?



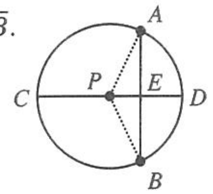
25. **Given:** $\overline{AD} \perp \overline{BC}$, $\overline{AD}$ is a diameter.
Prove: $\triangle ABC$ is isosceles.



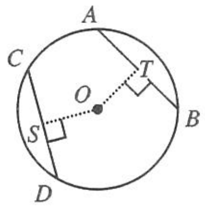
26. **Given:** $\overline{OY} \perp \overline{AB}$, $\overline{OX} \perp \overline{CD}$, $OX = OY$
Prove: $\widehat{AD} \cong \widehat{BC}$



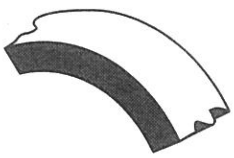
27. A chord is located 5 in. from the center of a circle with radius 13 in. Find the length of the chord.
28. An equilateral triangle with sides of length 10 is inscribed in a circle. Find the distance from the center of the circle to the side of the triangle.
29. Prove Theorem 8.7.
Given: $\odot P$ has diameter $\overline{CD}$ bisecting chord $\overline{AB}$.
Prove: $\overline{CD} \perp \overline{AB}$, $\widehat{DA} \cong \widehat{DB}$, $\widehat{AC} \cong \widehat{BC}$



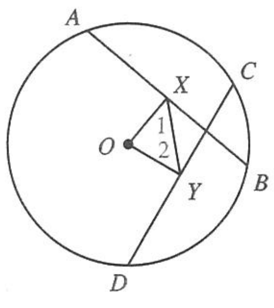
30. Prove Theorem 8.8.
Given: $\odot P$ with chord $\overline{AB}$
$\overline{CD}$ is the $\perp$ bisector of $\overline{AB}$.
Prove: $\overline{CD}$ contains the center of $\odot P$.
31. Prove Theorem 8.9.
Given: $\overline{AB} \cong \overline{CD}$
Prove: $OS = OT$



32. Prove Theorem 8.10.
Given: $OS = OT$
Prove: $\overline{AB} \cong \overline{CD}$
33. A part of an old wheel is found by an archeologist in a dig. The archeologist would like to determine how large the complete wheel was. How can she determine the radius of the complete wheel?



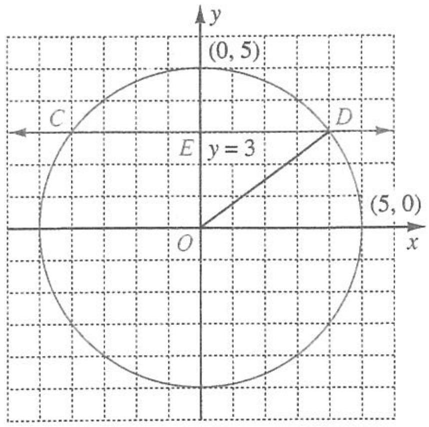
34. **Given:** $\overline{OX} \perp \overline{AB}$
$\overline{OY} \perp \overline{CD}$
$m\angle 1 = m\angle 2$
Prove: $AB = CD$



35. **Given:** $\overline{OX} \perp \overline{AB}$
$\overline{OY} \perp \overline{CD}$
$AB = CD$
Prove: $m\angle 1 = m\angle 2$

C

Circle O pictured is the set of all points in the coordinate plane whose coordinates satisfy the equation $x^2 + y^2 = 25$. The horizontal line consists of all points whose y-coordinate is 3.



36. Use the Pythageorean Theorem to find the length ED. Then find the length CD.
37. Use the results of Exercise 36 to find the coordinates of points C and D.
38. Use the distance formula to find the length of CD.
39. Solve the system of equations $x^2 + y^2 = 25$ and $y = 3$ to find the coordinates of points C and D.

Critical Thinking

Complete each statement with the word *always*, *sometimes*, or *never*. Give a convincing argument to support each answer.

40. If AB and CD are parallel chords, then $ABCD$ is ___ a rectangle.
41. A line through the midpoint of a chord and its intercepted arc ___ passes through the center of the circle.
42. A line that is perpendicular to a chord ___ includes the center of the circle.

Algebra Review

Solve.

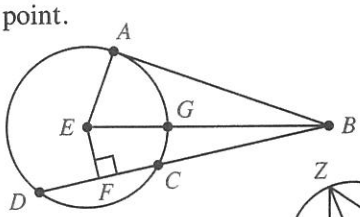
1. $4 + 3x = 28$
2. $5 + 4y = 37$
3. $5y - 9 = 21$
4. $4 - 3y = 13$
5. $10y - 7 = -12$
6. $6 - 8x = 18$
7. $8 - 2y = 22$
8. $3 - 6y = 15$

Quiz

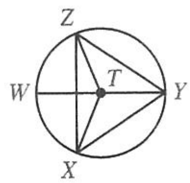
Complete each statement with the most appropriate word.

1. A chord that contains the center of a circle is a ___.
2. A ___ is a line that intersects the circle in exactly one point.

$\overline{AB}$ is tangent to $\odot E$.



3. If $AB = 8$ and $EB = 12$, find AE.
4. If $DC = 10$ and $AE = 13$, find EF.
5. If $m\angle ABE = 28$, find $m\widehat{AG}$.
6. Draw a pair of circles that have two common external tangents and no common internal tangents.



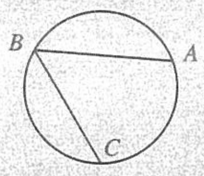
7. If $\overline{XY} \cong \overline{YZ}$ and $m\widehat{XY} = 105$, find $m\angle XTZ$.
8. Find $m\widehat{WX}$.
9. Find $m\angle XYZ$.

OBJECTIVE: *State and apply theorems concerning measures of inscribed angles.*

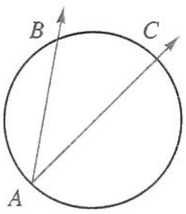
8-6 Inscribed Angles

EXPLORE

Construct a circle and an inscribed angle. Use a protractor to measure $\overset{\frown}{AC}$ and $\angle ABC$. What relationship do you discover?



An **inscribed angle** is an angle with vertex on a circle and sides that contain chords of the circle. In the figure, $\angle BAC$ is an inscribed angle and $\overset{\frown}{BC}$ is its **intercepted arc.**

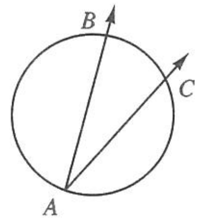


THEOREM 8.11 Inscribed Angle Measure

The measure of an inscribed angle is half the measure of its intercepted arc.

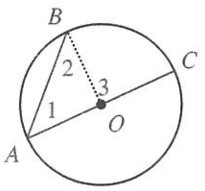
Given: inscribed $\angle BAC$

Prove: $m\angle BAC = \frac{1}{2} m\overset{\frown}{BC}$



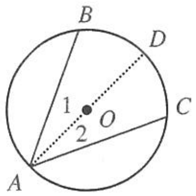
Plan Consider these three cases. In Case 1 the center of the circle is on one side of the inscribed angle. In Case 2 the center is in the interior of the angle and in Case 3 the center of the circle is in the exterior of the angle.

Case 1



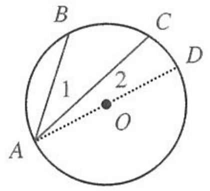
$m\overset{\frown}{BC} = m\angle 3$
$= m\angle 1 + m\angle 2$ or
$m\overset{\frown}{BC} = 2m\angle 1$

Case 2



Use Case 1 for $\angle 1$ and $\angle 2$. Add angles and arcs.

Case 3



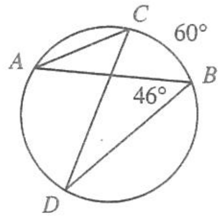
Use Case 1 for $\angle BAD$ and $\angle 2$. Subtract angles and arcs.

In Case 1, the conclusion follows from the fact that $\angle 3$ is an exterior angle of $\triangle ABO$. This proof will be completed in Exercises 25, 26, and 27.

Example 1

Find each measure.

a. $m\angle CAB$
b. $m\angle CDB$



Solution

a. $m\angle CAB = 30$ *The measure of $\angle CAB$ is $\frac{1}{2}$ its intercepted arc BC.*
b. $m\angle CDB = 30$ *$\angle CDB$ has the same intercepted arc as $\angle CAB$.*

Try This

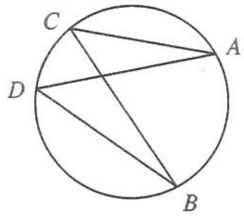
Find each measure.

a. $m\overset{\frown}{AD}$
b. $m\angle ACD$

Several corollaries to Theorem 8.11 are presented below. Their proofs follow directly from the theorem and you will be asked to prove them in Exercises 28, 29, and 30.

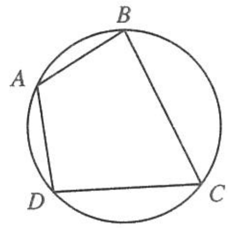
COROLLARY 8.11a

If two inscribed angles intercept the same arc, then the angles are congruent.



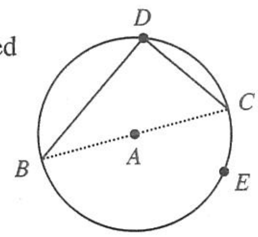
COROLLARY 8.11b

If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

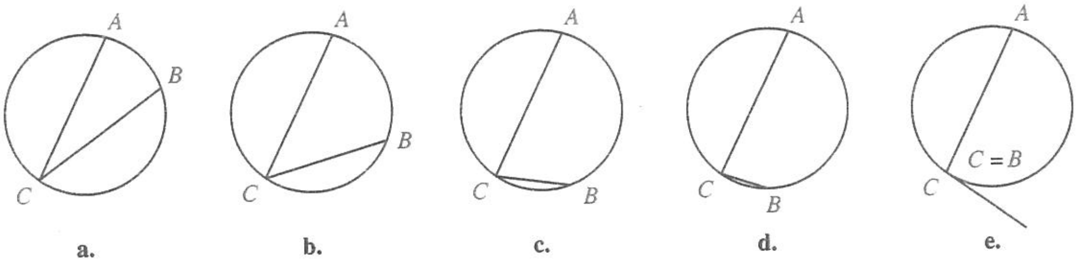


COROLLARY 8.11c

An angle inscribed in a semicircle is a right angle. Also, if an inscribed angle is a right angle its intercepted arc is a semicircle.



Think about point B moving along the circle toward point C. The inscribed angle measure theorem continues to be true as $\overline{CB}$ moves to form a tangent at C. A **tangent-chord angle** has its vertex on a circle with one side tangent to the circle at the vertex and the other side containing a chord. Figure **e** shows a tangent-chord angle with intercepted arc AC.

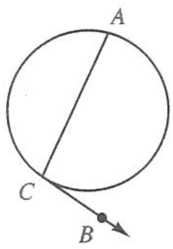


In all five cases, $m\angle C = \frac{1}{2}m\widehat{AB}$.

This sequence of figures provides a convincing argument for the next theorem.

THEOREM 8.12

The measure of a tangent-chord angle is half the measure of its intercepted arc.

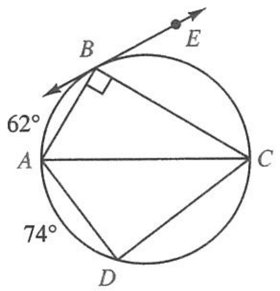


Example 2

$\overleftrightarrow{BE}$ **is tangent at** B**. Find each measure.**

a. $m\angle ADC$

b. $m\angle BCD$



Solution

a. $m\angle ADC = 90$ *Opposite angles of an inscribed quadrilateral are supplementary.*

b. $m\angle BCD = m\angle DCA + m\angle BCA$

$= (\frac{1}{2})74 + (\frac{1}{2})62$

$= 37 + 31$

$= 68$

Try This

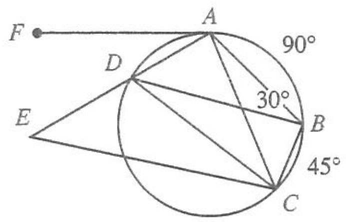
Find each measure.

a. $m\angle CBE$

b. $m\angle DAB$

Class Exercises

Short Answer



1. Name two congruent inscribed angles.
2. Name two inscribed angles that are not congruent.
3. Name an angle that is not an inscribed angle.
4. Name a tangent-chord angle.
5. Name two inscribed angles that have measure 45.
6. Name two inscribed angles whose intercepted arc is $\overset{\frown}{AD}$.

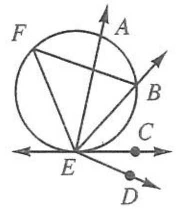
Sample Exercises

Find each measure.

7. $m\angle BDC$ 8. $m\angle BAC$ 9. $m\overset{\frown}{ADC}$
10. $m\angle ABC$ 11. $m\angle ADC$ 12. $m\angle FAD$

Discussion

$\overline{EC}$ is tangent at E.

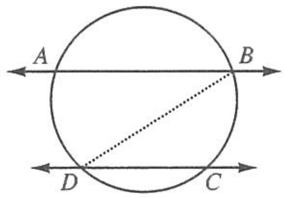


13. Explain why $\angle BED$ is not an inscribed angle.
14. Explain why $\angle CED$ is not a tangent-chord angle.
15. Explain why $\angle BFE \cong \angle BEC$.

16. Complete the proof. State a theorem (in if-then form) that you have proven.

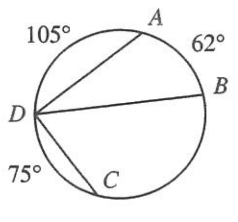
Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. (Draw auxiliary line BD.)

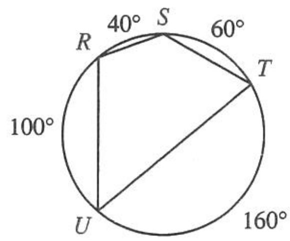
Statements	Reasons
1. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	1. ___
2. $\angle ABD \cong \angle BDC$	2. ___
3. ___	3. ___



Exercises

A



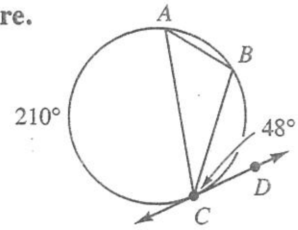


Find each measure.

1. $m\angle ADB$ 2. $m\overset{\frown}{BC}$ 3. $m\angle BDC$ 4. $m\overset{\frown}{AC}$
5. $m\angle R$ 6. $m\angle S$ 7. $m\angle T$ 8. $m\angle U$

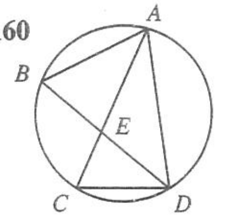
$\overleftrightarrow{CD}$ is tangent at C. Find each measure.

9. $m\angle CAB$
10. $m\angle ABC$
11. $m\angle ACB$

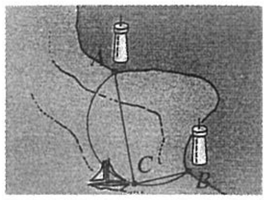


$\overrightarrow{AC}$ bisects $\angle BAD$. $m\overset{\frown}{CD} = 80$, $m\overset{\frown}{AD} = 160$
Find each measure.

12. $m\angle BAC$
13. $m\angle BDC$
14. $m\angle AEB$
15. $m\angle ADB$

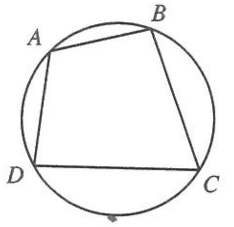


16. Suppose that a harbor is too shallow for ships to enter inside the circular arc ACB, also known as the "danger circle." The navigator of a ship at location C measured $\angle ACB$ and discovered it was equal to a published "danger angle." How did the ship's navigator know that the ship was on the danger circle?

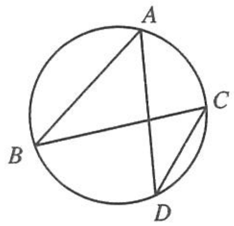


B

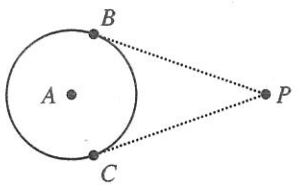
17. Suppose that $ABCD$ is a quadrilateral inscribed in a circle and that $\overline{AC}$ is a diameter of the circle. If $m\angle A$ is three times $m\angle C$, what are the measures of all four angles?



18. If $m\angle A = 4x + 35$, $m\angle B = 3x + 35$, and $m\angle D = 7x + 15$, find the measure of all four angles of quadrilateral $ABCD$.
19. If $m\angle ABC = 3x + 5$ and $m\angle ADC = 5x - 21$, find the measure of these two angles.
20. If $m\angle BAD = 3x + 50$, $m\angle ABC = 4x + 25$, and $m\angle BCD = 7x + 30$, find $m\angle ADC$.



21. Prove that a trapezoid inscribed in a circle is an isosceles trapezoid.
22. Given $\odot A$ and a point P exterior to the circle, construct the circle with diameter $\overline{AP}$ and let B and C be the points at which the two circles intersect. Prove that $\overline{PB}$ and $\overline{PC}$ are tangent segments from the exterior point P.

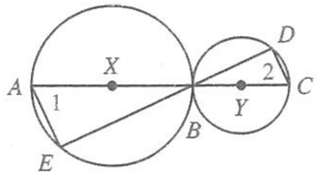


23. Given: $\overline{AB}$ and $\overline{BC}$ are diameters. $\odot X$ and $\odot Y$ are tangent at B.

Prove: $\angle 1 \cong \angle 2$

24. Given: $\overline{AB}$ and $\overline{BC}$ are diameters. $\odot X$ and $\odot Y$ are tangent at B.

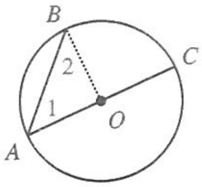
Prove: $\frac{AE}{EB} = \frac{CD}{DB}$



25. Prove Case 1 of Theorem 8.11.

Given: inscribed $\angle BAC$

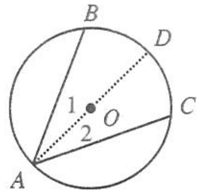
Prove: $m\angle BAC = \frac{1}{2} m\widehat{BC}$



26. Use Case 1 to prove Case 2 of Theorem 8.11.

Given: inscribed $\angle BAC$

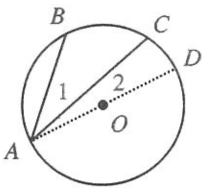
Prove: $m\angle BAC = \frac{1}{2} m\widehat{BC}$



27. Use Case 1 to prove Case 3 of Theorem 8.11.

Given: inscribed $\angle BAC$

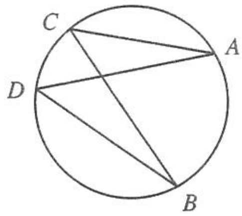
Prove: $m\angle BAC = \frac{1}{2} m\widehat{BC}$



28. Prove Corollary 8.11a.

Given: inscribed $\angle$s ACB and ADB

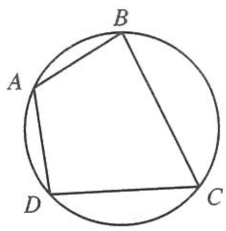
Prove: $\angle ACB \cong \angle ADB$



29. Prove Corollary 8.11b.

Given: inscribed quadrilateral $ABCD$

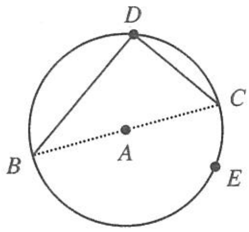
Prove: $\angle A$ and $\angle C$ are supplementary.



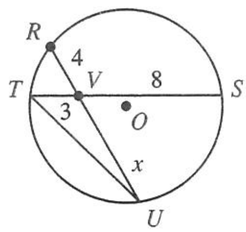
30. Prove Corollary 8.11c.

Given: $\angle BDC$ is inscribed in semicircle $\widehat{BDC}$.

Prove: $\angle D$ is a right angle.



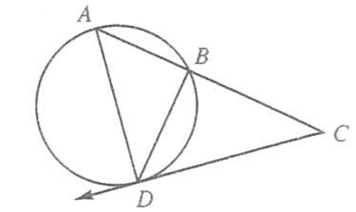
31. If $TV = 3$, $VS = 8$, $RV = 4$, find x.



C

32. **Given:** $\overline{CD}$ is tangent to the circle at D and $\overline{AD}$ contains the center of the circle.

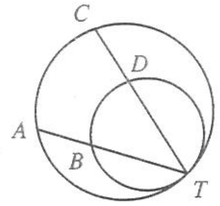
Prove: $\frac{AD}{AC} = \frac{DB}{DC}$



33. Suppose that a triangle is inscribed in each of two congruent circles. If a side and an adjacent angle of one is congruent to a side and an adjacent angle of the other prove that the two triangles are congruent.

Assume the two circles are tangent at *T*.

34. Prove that $\overline{AC} \parallel \overline{BD}$. (HINT: Draw the tangent line at point T.)
35. If $AB = 5$, $DT = 8$, and $CD = 6$, find BT.



Critical Thinking

Complete each statement with the word *always*, *sometimes*, or *never*. Give a convincing argument that your answer is correct.

36. A quadrilateral that is inscribed in a circle with at least one diagonal a diameter is ___ a parallelogram.
37. A quadrilateral that is inscribed in a circle with both diagonals a diameter is ___ a rectangle.
38. If a triangle is inscribed in a circle to which a tangent line to the circle at one of the vertices forms a 60° tangent-chord angle with one side of the triangle, then the triangle is ___ an equilateral triangle.

Algebra Review

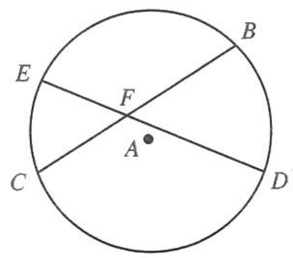
Find the LCM.

1. a^2b, ab^2
2. $3x^2, 6xy$
3. $x + y, x - y$
4. $a + 1, a^2 - 1$
5. $8x^2y^2, 12y^3$
6. $12xy^2, 16x^3y$
7. $x^2 + 16, x - 2$
8. $x^2 - 2x + 1, 1 - x^2$

Computer Activity

Use computer software to investigate the measure of an angle formed by two chords intersecting inside a circle.

Draw $\overline{BC}$ and $\overline{DE}$. Name the intersection of $\overline{BC}$ and $\overline{DE}$ point F.
Measure $\angle CFD$.
Measure $\angle CAD$ and $\angle BAE$.

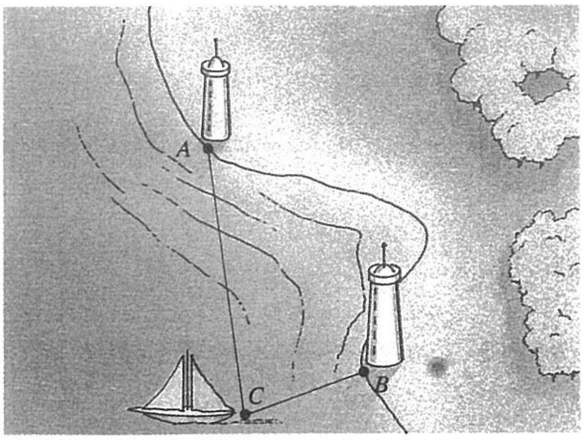


What generalization can you make about the measure of angle formed by two chords intersecting inside a circle?

OBJECTIVE: *State and apply theorems concerning measures of angles formed by two chords or two secants.*

8-7 Angles of Chords, Secants, and Tangents

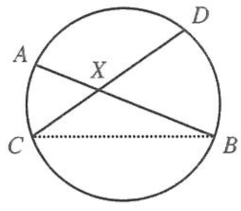
In the previous section you studied inscribed angles and tangent-chord angles. In both types of angles, the vertex of the angle is on the circle. In the angles formed by chords, secants, and tangents, vertices are either in the interior of the circle or on the exterior of the circle. Theorem 8.13 deals with the case in which the vertex is on the interior of the circle. This theorem also helps answer the question to the right.



If a ship's captain measures $\angle ACB$ and finds it to be greater than the published "danger angle" how would he know that he was located inside the "danger circle"?

◆ THEOREM 8.13

The measure of an angle formed by two chords intersecting inside a circle is one half the sum of the intercepted arcs.



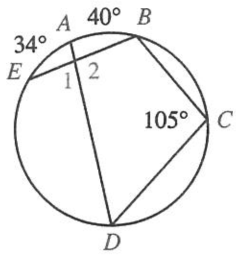
Given: chords $\overline{AB}$ and $\overline{CD}$ intersecting at point X

Prove: $m\angle AXC = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{BD})$

Proof Draw $\overline{BC}$. Then $\angle AXC$ is an exterior angle of $\triangle XCB$ and hence $m\angle AXC = m\angle XCB + m\angle XBC$. Furthermore, $m\angle XCB = \frac{1}{2}m\overset{\frown}{BD}$ and $m\angle XBC = \frac{1}{2}m\overset{\frown}{AC}$. Therefore, $m\angle AXC = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{BD})$.

Example 1

Find $m\angle 1$.



Solution

$m\angle 1 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{ED})$
$= \frac{1}{2}(40 + 136)$
$= 88$

Since $m\angle BCD = 105°$, you know that $m\overset{\frown}{BED} = 210$. So $m\overset{\frown}{ED} = 210 - (34 + 40) = 136$.

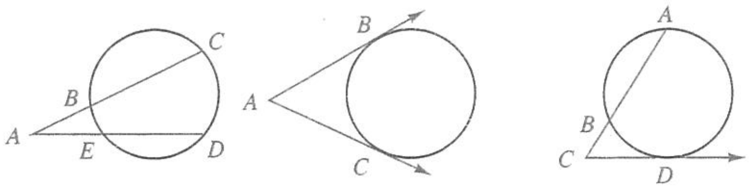
Try This

Find $m\angle 2$.

The next theorem deals with the measure of an angle whose vertex is outside the circle and whose sides intersect the circle. There are three cases. The sides are (1) both secants or (2) both tangents or (3) one secant and one tangent.

◆ THEOREM 8.14

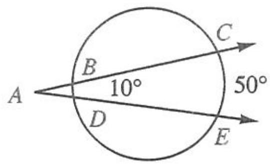
The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point in the exterior of a circle is equal to half the difference of the measures of the intercepted arcs.



Example 2

Find

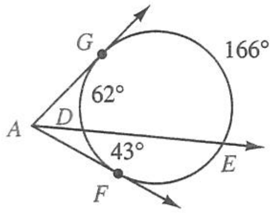
a. $m\angle CAE$ if $m\widehat{BD} = 10$ and $m\widehat{CE} = 50$.
b. $m\angle GAF$ if $m\widehat{GF} = 105$ and $m\widehat{GEF} = 255$.



Solution

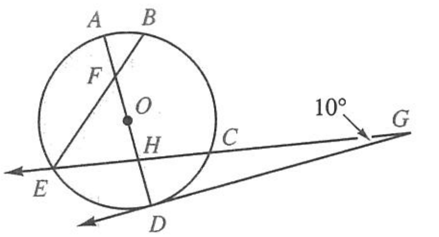
a. $m\angle CAE = \frac{1}{2}(m\widehat{CE} - m\widehat{BD})$
$= \frac{1}{2}(50 - 10)$
$= 20$

b. $m\angle GAF = \frac{1}{2}(m\widehat{GEF} - m\widehat{GF})$
$= \frac{1}{2}(255 - 105)$
$= \frac{1}{2}(150) = 75$



Try This

Find $m\angle GAE$ if $m\widehat{GE} = 166$ and $m\widehat{GD} = 62$.



Example 3

$\overline{AD}$ is a diameter and $\overline{GD}$ is tangent at D.
$m\widehat{AB} = 25$, $m\widehat{CD} = 40$ Find $m\angle AFB$.

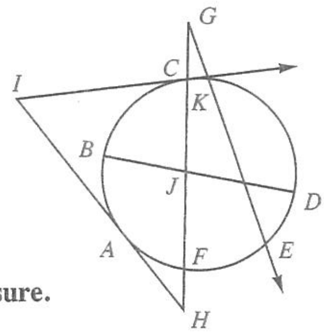
Solution

$m\angle EGD = \frac{1}{2}(m\widehat{ED} - m\widehat{CD})$ *Apply the tangent-secant form of Theorem 8.18.*
$10 = \frac{1}{2}(m\widehat{ED} - 40)$
$10 + 20 = \frac{1}{2}m\widehat{ED}$ *Use this fact in a application of Theorem 8.17*
$m\widehat{ED} = 60$ *to find $m\angle AFB$.*
$m\angle AFB = \frac{1}{2}(m\widehat{AB} + m\widehat{ED})$
$= \frac{1}{2}(25 + 60)$
$= 42.5$

Try This

Find $m\angle AHC$.

Class Exercises



Short Answer

Name an angle whose sides are as follows.

1. two secants of the circle
2. two tangents of the circle
3. a secant and a tangent of the circle
4. on intersecting chords of the circle

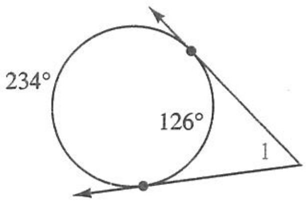
Tell what arc measures you would need to know to find each measure.

5. $m\angle BJF$ 6. $m\angle FGE$ 7. $m\angle AIC$ 8. $m\angle IHG$

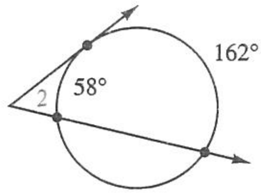
Sample Exercises

Find the measure of each numbered angle.

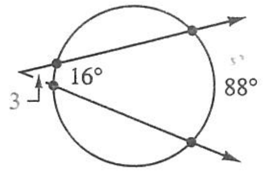
9.



10.

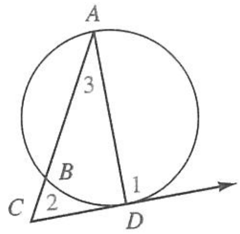


11.



Discussion

12. This figure illustrates which one of the three cases of Theorem 8.14? Draw figures that illustrate the other two cases and give a convincing argument why each case is true.

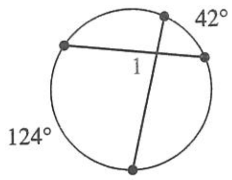


Exercises

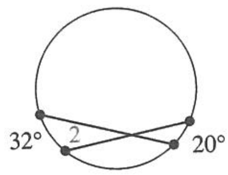
A

Find the measure of each numbered angle.

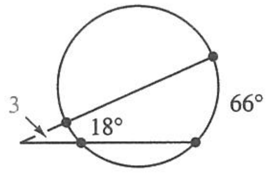
1.



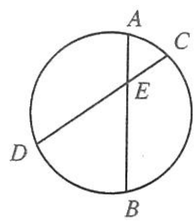
2.



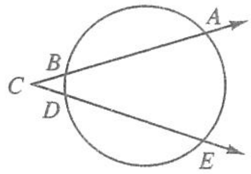
3.



4. If $m\overparen{AC} = 39$ and $m\overparen{BD} = 73$, find $m\angle AED$.
5. If $m\angle AEC = 48$ and $m\overparen{AC} = 34$, find $m\overparen{BD}$.
6. If $m\angle BEC = 130$ and $m\overparen{AD} = 120$, find $m\overparen{BC}$.

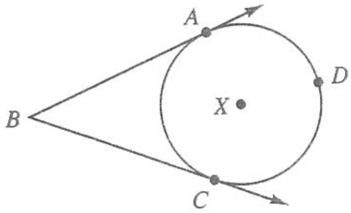


7. If $m\overset{\frown}{BD} = 15$ and $m\overset{\frown}{AE} = 85$, then find $m\angle ACE$.
8. If $m\angle ACE = 32$ and $m\overset{\frown}{BD} = 22$, find $m\overset{\frown}{AE}$.
9. If $m\angle ACE = 28$ and $m\overset{\frown}{AE} = 94$, find $m\overset{\frown}{BD}$.

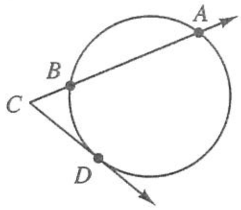


$\overrightarrow{BA}$ and $\overrightarrow{BC}$ are tangent to $\odot X$.

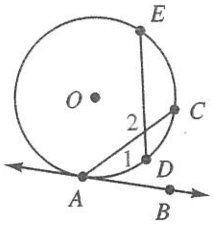
10. If $m\overset{\frown}{AC} = 160$, find $m\angle ABC$.
11. If $m\overset{\frown}{ADC} = 240$, find $m\angle ABC$.
12. If $m\overset{\frown}{AC} = b$ and $m\overset{\frown}{ADC} = 3b$, find $m\angle ABC$.



13. If $m\overset{\frown}{AD} = 165$ and $m\overset{\frown}{BD} = 63$, find $m\angle ACD$.
14. If $m\overset{\frown}{BD} = 55$ and $m\angle ACD = 43$, find $m\overset{\frown}{AD}$.
15. If $m\overset{\frown}{AD} = x$ and $m\overset{\frown}{BD} = y$, find $m\angle ACD$.



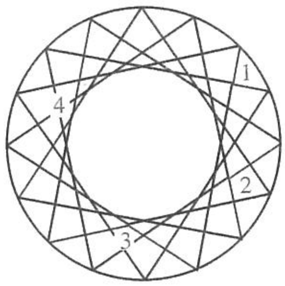
16. $\overleftrightarrow{AB}$ is tangent to $\odot O$. $m\overset{\frown}{AE} = 160$, $m\overset{\frown}{AD} = 50$, $m\overset{\frown}{DC} = 60$
Find $m\angle 1$ and $m\angle 2$.



B

This design was drawn by using 16 points equally spaced around the circle. Find the measure of each angle.

17. $m\angle 1$ 18. $m\angle 2$
19. $m\angle 3$ 20. $m\angle 4$

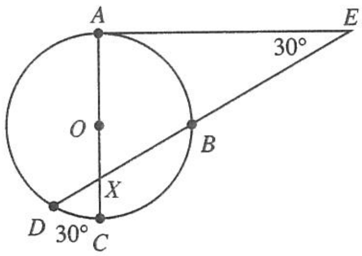


A diameter $\overline{AB}$ and a chord $\overline{CD}$ intersect inside $\odot O$ at X. If $m\overset{\frown}{AD} = 128$ and $m\angle AXD = 74$, find each measure.

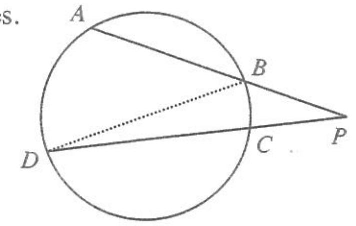
21. $m\overset{\frown}{AC}$ 22. $m\overset{\frown}{CB}$ 23. $m\overset{\frown}{BD}$

In $\odot O$, $\overline{AC}$ is a diameter and $\overline{AE}$ is a tangent. $m\overset{\frown}{DC} = 30$, $m\angle AED = 30$ Find each measure.

24. $m\overset{\frown}{AB}$ 25. $m\overset{\frown}{BC}$ 26. $m\angle AXD$

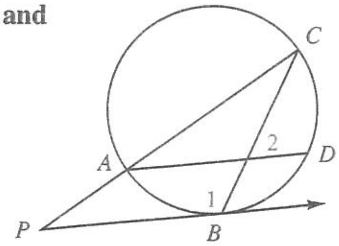


27. Prove Theorem 8.16 for the case of two secant lines.
Given: $\overrightarrow{PA}$ and $\overrightarrow{PD}$ secants
Prove: $m\angle P = \frac{1}{2}(m\widehat{AD} - m\widehat{BC})$.
(HINT: Use the auxiliary line BD.)



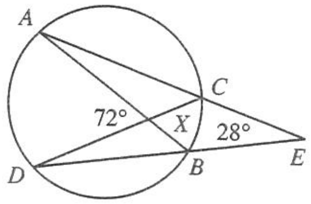
$m\widehat{AB} = 55$, $m\widehat{BD} = 40$, $\overline{AC}$ is a diameter and $\overline{PB}$ is tangent to the circle at B.

28. Find $m\angle P$.
29. Find $m\angle 2$.
30. Find $m\angle 1$.
31. Is $\overline{AD} \parallel \overline{PB}$? Explain.

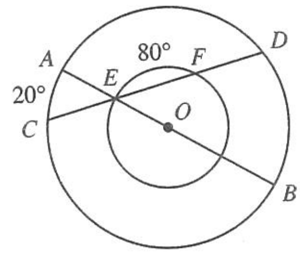


C

32. Given $m\angle AXD = 72$, $m\angle AED = 28$
Find $m\widehat{AD}$ and $m\widehat{CB}$.

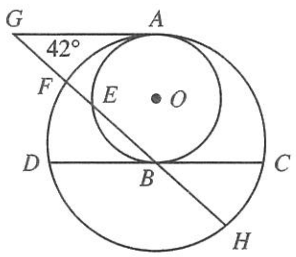


33. Given concentric circles centered at O. $\overline{AB}$ and $\overline{CD}$ are chords of the large circle that intersect at point E on the small circle. $m\widehat{EF} = 80$, $m\widehat{AC} = 20$ Find $m\widehat{BD}$.



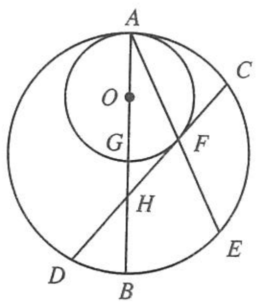
$\odot O$ is tangent to $\odot B$ at point A. $\overline{GA}$ is tangent to both circles at A and $\overline{CD}$ is tangent to $\odot O$ at B. $\overline{GA} \parallel \overline{CD}$, $m\angle AGB = 42$

34. Find $m\widehat{AH}$.
35. Find $m\widehat{AF}$.
36. Find $m\widehat{AE}$.



$\odot O$ is tangent to the larger circle at A. Chords $\overline{AE}$ and $\overline{CD}$ intersect at F on $\odot O$ and $\overline{CD}$ is tangent to $\odot O$.

37. If $m\angle EAB = 25$, find $m\angle AHF$.
38. If $m\angle DFE = 65$, find $m\angle BAE$.
39. If $m\angle DFE = 65$, find $m\widehat{BD}$.

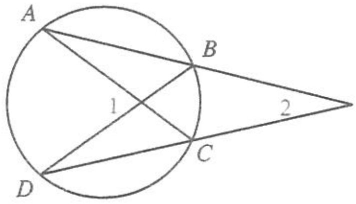


Critical Thinking

Complete the following statements by considering special cases and making generalizations.

40. If $m\widehat{AD} = 2m\widehat{BC}$, then $m\angle 2 =$ ___ $m\widehat{BC}$.

41. If $m\widehat{AD} = 3m\widehat{BC}$, then $m\angle 1 =$ ___ $m\widehat{BC}$.

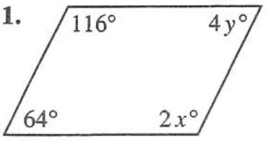


42. Reread the question asked in the caption under the map at the beginning of the lesson. Give a convincing argument that if $m\angle ACB$ is greater than the "danger angle," the captain will know that the ship was located inside the danger circle.

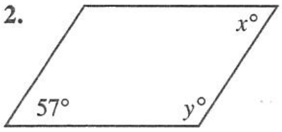
Mixed Review

Determine the values of x and y so that each figure is a parallelogram.

1.



2.



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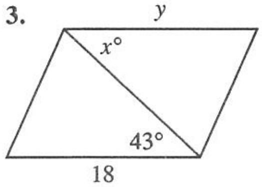

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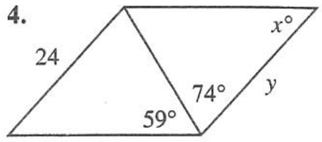


5. A tower 160 ft high casts a shadow 220 ft long. Find the angle of elevation of the sun.

6. A ramp is 100 ft long and rises vertically 12 ft. Find the angle of elevation of the ramp.

Biographical Note

Emmy Noether (1882–1935)

Emmy Noether has been recognized as the most creative abstract algebraist of modern times. At the age of 31 she began to lecture on mathematics, sometimes substituting for her father, a professor of mathematics at Erlangen University in Germany. After years of unsuccessful attempts to overcome the objections of those faculty members who wanted to exclude women from the faculty, she won formal admission as an academic lecturer in 1919.

From 1930 to 1933 she was at the center of the mathematical activity at Gottingen. When the Nazis came to power in 1933, Noether and many others were prohibited from taking part in any academic activities. Within a year she came to the United States and became professor of mathematics at Bryn Mawr College.

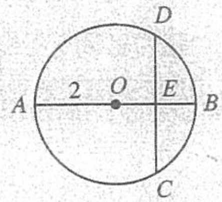
The extent and significance of Noether's work cannot be judged from only her writing. Many of her remarks and suggestions revealed her great insights and influenced the work of students and colleagues.

OBJECTIVE: *State and apply theorems concerning measures of intersecting chords, secants, and tangents.*

8-8 Segments of Chords, Secants, and Tangents

EXPLORE

Construct on paper or on a computer screen $\odot O$ with diameter $\overline{AB}$ of length 4 and construct $\overline{CD}$ perpendicular to $\overline{OA}$ at its midpoint. Find $AE \cdot BE$ and $CE \cdot DE$. What do you discover?



In the last lesson you studied how the measure of an angle formed (1) by chords intersecting in the interior of a circle, (2) by two secants from an external point, and (3) by a tangent and a secant from an external point are related to the measures of the intercepted arcs. In this lesson the same three situations are presented, but you will focus on lengths of chords and segments.

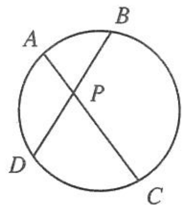
THEOREM 8.15

If two chords intersect in a circle, then the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the second chord.

Given: $\overline{AC}$ and $\overline{BD}$ are chords intersecting at P.

Prove: $AP \cdot PC = BP \cdot PD$

Plan You can prove that $\triangle ABP \sim \triangle DCP$ by the AA Similarity Theorem. Use the fact that ratios of lengths of corresponding sides in similar triangles are proportional to conclude $\frac{AP}{DP} = \frac{BP}{CP}$ and rewrite this equation.



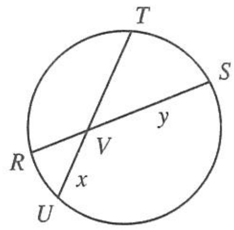
Example 1

Find x if $RV = 3$, $TV = 7$, and $SV = 8$.

Solution

$RV \cdot VS = UV \cdot VT$

$3 \cdot 8 = x \cdot 7 \quad x = \frac{24}{7}$

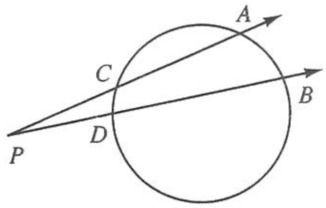


Try This

Find y if $UV = 8$, $VT = 15$, and $RV = 7$.

The next theorem is about secants from an external point. The conclusion of the theorem is similar to that for Theorem 8.15.

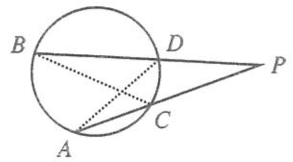
Suppose that two secants from an external point P intersect a circle at points A, B, C, and D as shown to the right. $\overline{PA}$ and $\overline{PB}$ are called secant segments and $\overline{PC}$ and $\overline{PD}$ are called external segments.



THEOREM 8.16

If two secant segments are drawn to a circle from an exterior point, then the product of the lengths of one secant segment and its external segment equals the product of the lengths of the other secant segment and its external segment.

Given: $\overline{PA}$ and $\overline{PB}$ are secant segments from the external point P and intersect the circle at points C and D to form external segments $\overline{PC}$ and $\overline{PD}$.

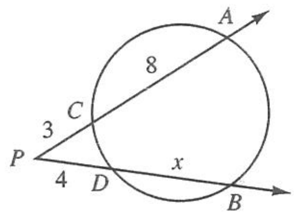


Prove: $PA \cdot PC = PB \cdot PD$

Plan You can prove that $\triangle ADP \sim \triangle BCP$ by the AA Similarity Theorem. Ratios of corresponding sides in similar triangles are proportional so $\frac{PA}{PB} = \frac{PD}{PC}$. Rewrite this proportion to obtain the desired form.

Example 2

Find x in the figure to the right.



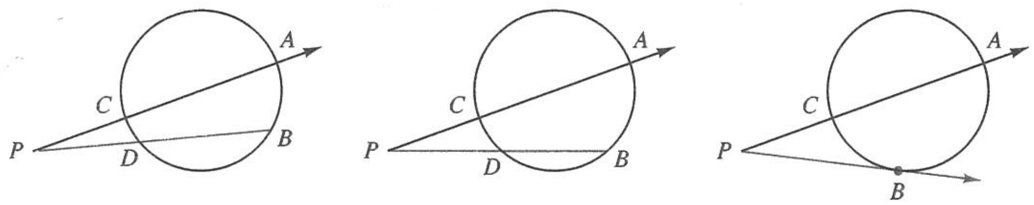
Solution

$$PA \cdot PC = PB \cdot PD$$
$$(8 + 3)3 = (x + 4)4$$
$$33 = 4x + 16$$
$$17 = 4x$$
$$x = \frac{17}{4}$$

Try This

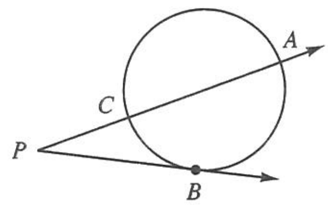
If $PC = 7$, $PD = 9$, and $DB = 15$, find AC.

As the secant $\overline{PB}$ moves closer and closer to the position of tangent from P, the product $PB \cdot PD$ becomes $PB \cdot PB$. This result is stated in Theorem 8.17.



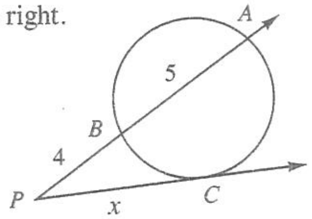
THEOREM 8.17

If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the length of the tangent segment equals the product of the lengths of the secant segment and its external secant segment.



Example 3

Find x in the figure to the right.



Solution

$$PA \cdot PB = PC^2$$
$$(5 + 4)4 = x^2$$
$$36 = x^2$$
$$x = 6$$

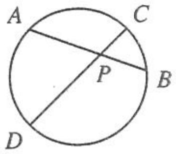
Try This

If $PC = 10$ and $PB = 6$, find AB.

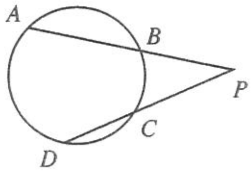
Class Exercises

Short Answer

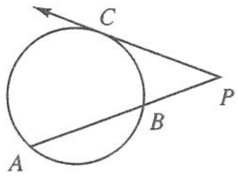
Complete each statement.

1. 

$AP \cdot PB =$ ___

2. 

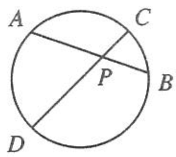
$DP \cdot PC =$ ___

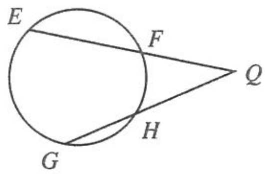
3. 

$AP \cdot PB =$ ___

Sample Exercises

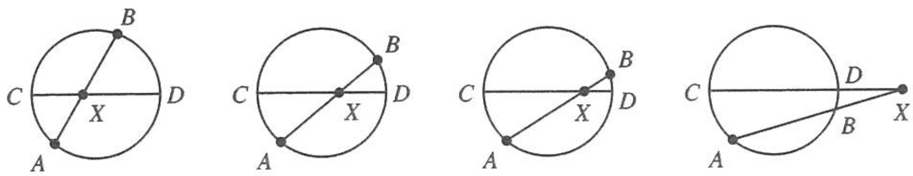
4. $AP = 8$, $BP = 6$, $CP = 5$ Find DP.
5. $AB = 11$, $AP = 6$, $DP = 7$ Find CP.
6. $EQ = 8$, $FQ = 6$, $GQ = 5$ Find HQ.
7. $EQ = 7$, $HQ = 9$, $GH = 6$ Find FQ.





Discussion

8. Imagine that points A, C, and D are fixed and that point B is moving clockwise around the circle. When B is near D is the value of $AX \cdot XB$ almost 1 or almost 0? Explain. If $\overline{CD}$ is a diameter, where would point B be located if $AX \cdot XB$ were exactly equal to one?



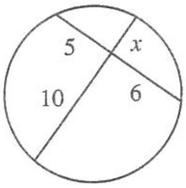
9. Give a convincing argument that for some locations of point B on the circles above $AX \cdot XB$ is greater than 10.

Exercises

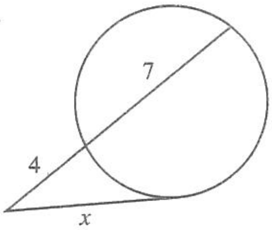
A

Find x.

1.



2.



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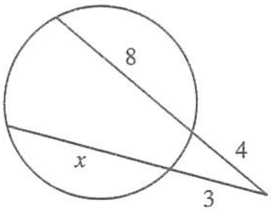
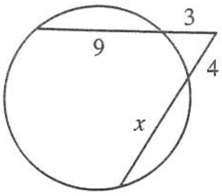
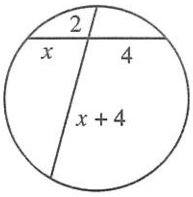


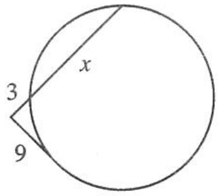
4.



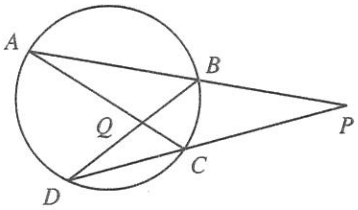
5.



6.



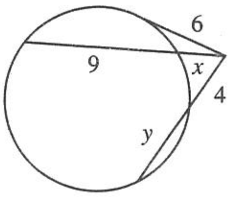
7. $AP = 12$, $PB = 7$, $PD = 11$ Find PC.
8. $AQ = 8$, $QC = 3$, $BQ = 5$ Find QD.
9. $AB = 9$, $BP = 10$, $PD = 18$ Find CD.
10. $AC = 12$, $AQ = 8$, $BQ = 5$ Find BD.
11. $AB = 8$, $AP = 17$, $CD = 7$ Find DP.
12. $BD = 18$, $AC = 14$, $CQ = 5$ Find BQ.
13. $AC = 15$, $BQ = 9$, $QD = 4$ Find AQ and CQ.



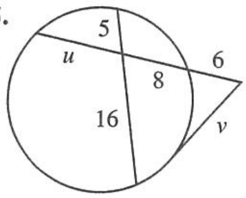
B

Find each variable.

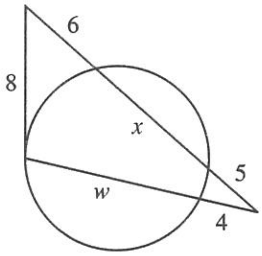
14.



15.

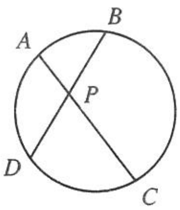


16.

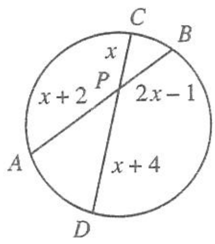


17. Is the triangle in Exercise 16 equilateral, isosceles, or scalene? Explain your reasoning.

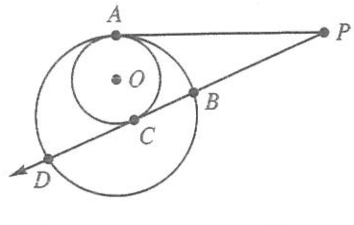
18. Prove Theorem 8.15.
Given: $\overline{AC}$ and $\overline{BD}$ are chords intersecting at P.
Prove: $AP \cdot PC = BP \cdot PD$



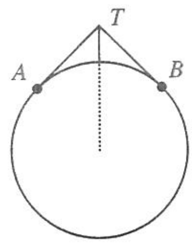
19. Find the lengths of chords $\overline{AB}$ and $\overline{CD}$.



20. Suppose that $\overline{PA}$ and $\overline{PC}$ are both tangent to $\odot O$. Prove that $PC^2 = PB \cdot PD$.

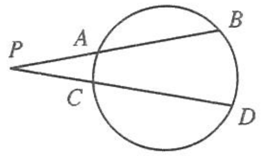


21. Suppose a radio tower T is 800 ft tall. Assuming that the diameter of the earth is 8000 mi, how far is it from the top of the tower to the horizon point A?

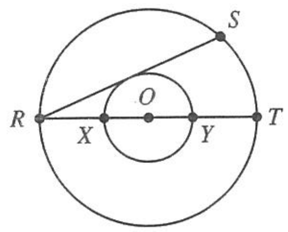


C

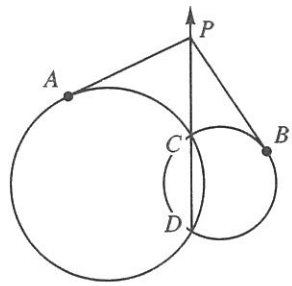
22. If $\overline{PB}$ and $\overline{PD}$ are secant segments and $PB = PD$, prove that $PA = PC$.



23. In the figure O is the center of two concentric circles. $\overline{RS}$ is tangent to the smaller circle. If $RX = 5$ and $RS = 30$, find XY.

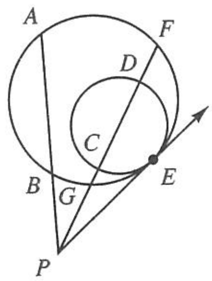


24. Suppose that two circles intersect at C and D and that point P is a point on $\overleftrightarrow{CD}$ exterior to both circles. $\overline{PA}$ and $\overline{PB}$ are tangent to the circles as shown. Prove that $\triangle APB$ is isosceles.

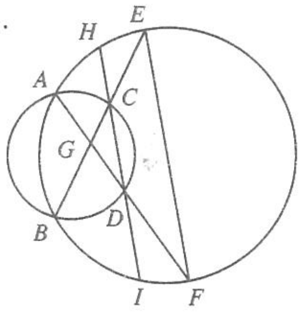


25. The two circles are tangent at E and $\overline{PE}$ is tangent to both circles at E. If $AP = 15$ and $PB = 6$, find $DP \cdot PC$. Explain your answer.

26. If $PF = 18$ and $PE = 5$, find $DP \cdot PC$. Explain your answer.

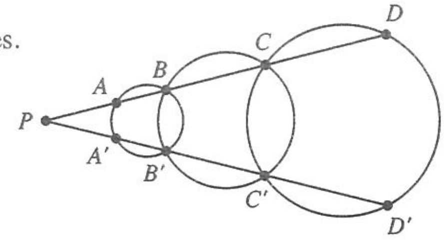


27. If $AG = 5$ and $BG = 7$, find $\frac{CG}{DG}$ and $\frac{EG}{FG}$.
28 If $m\angle BEF = 38$, find $m\angle BCI$.
29. Prove that $\overline{CD} \parallel \overline{EF}$.
30. Prove that $m\widehat{EH} = m\widehat{FI}$.



Critical Thinking

31. Suppose that $\overline{PD}$ and $\overline{PD}'$ are secants of all three circles. Decide whether or not each equation is always true, sometimes true, or never true. Give a convincing argument for each answer.
  a. $PA \cdot PB = PA' \cdot PB'$
  b. $PA \cdot PC = PA' \cdot PC'$
  c. $PA \cdot AD = PA' \cdot PD'$



Algebra Review

Multiply.

1. $(x + 1)^2$
2. $(x - 3)^2$
3. $(a + 5)^2$
4. $(y - 5)^2$
5. $(m + 3)(m - 2)$
6. $(2a + 3)(a - 1)$
7. $(s + 3)(3s - 4)$
8. $(r - 5)(r + 5)$

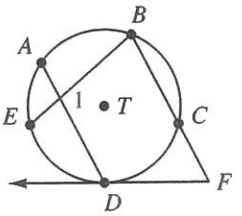
Quiz

Determine whether each statement is true or false.

1. The measure of an inscribed angle is one half the measure of the intercepted arc.
2. If two inscribed angles in a circle are congruent, then they intercept the same arc.
3. If a right angle is inscribed in a circle, then its intercepted arc is a semicircle.
4. The measurement of an angle formed by two secants intersecting in the exterior of a circle is equal to one half the difference of the two intercepted arcs.

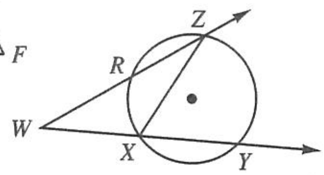
$\overline{DF}$ is tangent to $\odot T$.

5. If $m\angle B = 32$, find $m\widehat{EC}$.
6. If $m\widehat{AD} = 76$, find $m\angle ADF$.
7. If $m\widehat{AE} = 28$ and $m\widehat{BD} = 170$, find $m\angle 1$.
8. If $CF = 3$ and $BC = 9$, find DF.



$\overline{WZ}$ and $\overline{WY}$ are secants.

9. If $m\widehat{ZY} = 88$ and $m\widehat{RX} = 26$, find $m\angle W$.
10. If $m\angle ZXY = 32$ and $m\angle W = 20$, find $m\widehat{RX}$.
11. If $WZ = 15$, $WR = 4$, and $WX = 3$, find XY.



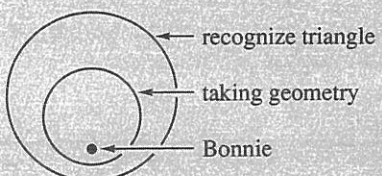
CRITICAL THINKING

Venn Diagrams and Reasoning Patterns

Venn diagrams (see page 38) can be used to test whether an argument is logically correct. An if-then statement can be represented in terms of circles, which help illustrate the validity of arguments based on the conditional. Consider the following examples.

Example 1

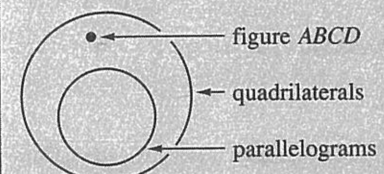
If a student is taking geometry, then she recognizes triangles.	$p \rightarrow q$
Bonnie is taking geometry.	p (premise)
Bonnie recognizes triangles.	q (conclusion)



The outer circle defines those students that recognize triangles. The inner circle indicates that all of the students taking geometry recognize triangles. Since Bonnie is in the inner circle, she is automatically among those students that recognize triangles.

Example 2 *Assuming the Converse*

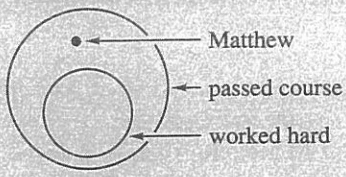
If a figure is a parallelogram, then it is a quadrilateral.	$p \rightarrow q$
Figure *ABCD* is a quadrilateral.	q
Figure *ABCD* is a parallelogram.	p



The Venn diagram shows that a point representing figure *ABCD* can be outside the parallelogram circle, so the reasoning pattern is invalid. Here, the converse of the original conditional is assumed.

Example 3 *Assuming the Inverse*

If a student worked hard, then he passed the course.	$p \rightarrow q$
Matthew did not work hard.	$\sim p$
Matthew did not pass the course.	$\sim q$



The Venn diagram shows that although Matthew did not work hard, the point representing him could still be inside the circle representing those who passed the course. The error in this case is assuming the inverse of the original conditional.

Exercises

Draw Venn diagrams to show each reasoning pattern and decide whether it is valid.

1. If you are over 21, then you can vote.
   Marika cannot vote.
   Marika is not over 21.
2. If you do not pay taxes, then you will go to jail.
   Peter paid his taxes.
   Peter went to jail.
3. If you have enough, then you must share.
   Theresa has enough.
   Theresa must share.
4. If you are a citizen, then you want power.
   If you want power, then you must vote.
   If you are a citizen, then you must vote.
5. If you are a doctor, then you have gone to college.
   Pat is not a doctor.
   Pat did not go to college.
6. If x is divisible by 4, then x is even.
   10 is even.
   10 is divisible by 4.

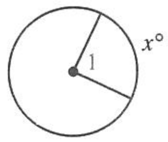
CHAPTER SUMMARY

Vocabulary

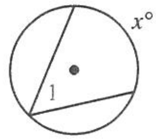
arc (8-4)
center (8-1)
central angle (8-4)
chord (8-1)
circle (8-1)
circumscribed polygon (8-1)
common tangents (8-3)
concentric circles (8-1)
congruent arcs (8-4)
congruent circles (8-1)
diameter (8-1)
inscribed angle (8-6)
inscribed polygon (8-1)
intercepted arc (8-6)
major arc (8-4)
minor arc (8-4)
radius (8-1)
secant (8-1)
semicircle (8-4)
sphere (8-1)
tangent (8-1)
tangent-chord angle (8-6)

Key Ideas

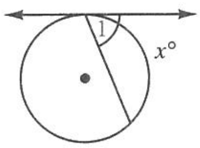
1. If a line is perpendicular to a radius of a circle at a point on the circle, then the line is tangent to the circle.
2. If a line is tangent to a circle, then it is perpendicular to the radius at the point of tangency.
3. In the same circle or congruent circles
   a. congruent chords have congruent minor chords.
   b. congruent minor arcs have congruent chords and congruent central angles.
   c. congruent chords are equidistant from the center.
   d. chords equidistant from the center are congruent.
4. If a diameter is perpendicular to a chord, then it bisects the chord and its minor and major arcs.
5. If two inscribed angles intercept the same arc, then the angles are congruent.
6. The following are angle-arc relationships.



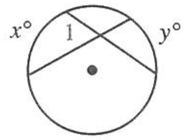
$m\angle 1 = x$



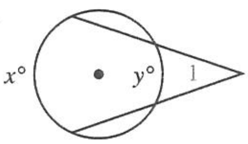
$m\angle 1 = \frac{1}{2}x$



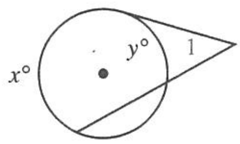
$m\angle 1 = \frac{1}{2}x$



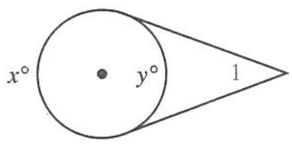
$m\angle 1 = \frac{1}{2}(x + y)$



$m\angle 1 = \frac{1}{2}(x - y)$

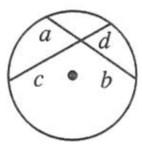


$m\angle 1 = \frac{1}{2}(x - y)$

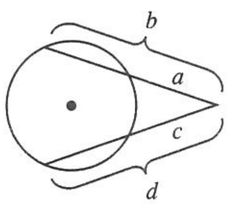


$m\angle 1 = \frac{1}{2}(x - y)$

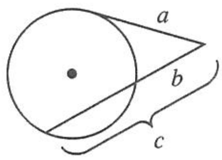
7. The following are chord, secant, and tangent relationships.



$a \cdot b = c \cdot d$



$a \cdot b = c \cdot d$

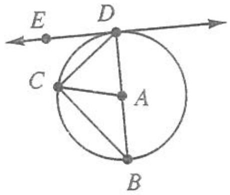


$a^2 = b \cdot c$

CHAPTER REVIEW

8-1

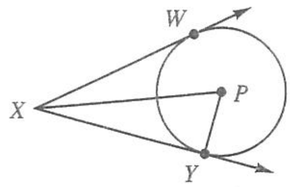
Determine whether each statement is true or false.



1. $\overline{BD}$ is a chord.
2. $\overline{BC}$ is a secant.
3. $\overline{CD}$ is a radius.
4. $\angle ACD$ is inscribed in $\odot A$.

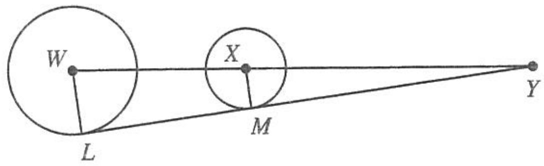
8-2

$\overline{XW}$ and $\overline{XY}$ are tangent to $\odot P$ at W and Y.



5. Find $m\angle PYX$.
6. If $XW = 10$, find XY.
7. If $m\angle WXY = 60$ and $XY = 6\sqrt{3}$, find PY.

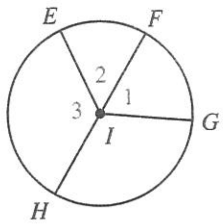
8-3



8. If $WL = 6$, $LM = 8$, and $MY = 4$, find XM.
9. If $m\angle Y = 30$, $WL = 7$, and $WX = 8$, find XM.
10. Draw a pair of circles that have only one common internal tangent.

8-4

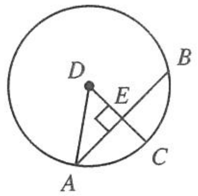
In $\odot I$, HF is a diameter.



11. If $m\angle 1 = 43$, find $m\overset{\frown}{FG}$.
12. If $m\overset{\frown}{EF} = 53$, find $m\angle 3$.
13. If $m\overset{\frown}{HG} = 140$ and $\angle 1 = \angle 2$, find $m\overset{\frown}{HE}$.

8-5

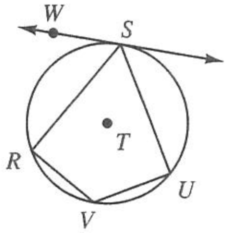
In $\odot D$, $AB \perp DC$ and $\overline{AE} \cong \overline{EB}$.



14. If $AD = 26$ and $DE = 10$, find AB.
15. If $m\overset{\frown}{BC} = 45$ and $AE = 7$, find AD.
16. If $AB = 16$ and $AD = 9$, find DE.

8-6

$\overline{SW}$ is tangent at S. Determine whether each statement is true or false.

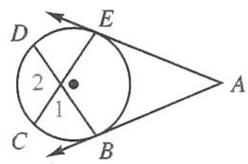


17. If $m\angle U = 103$, then $m\overset{\frown}{VRS} = 206$.
18. If $m\overset{\frown}{RU} = 150$, then $m\angle S = 150$.
19. If $m\angle WSR = 44$, then $m\overset{\frown}{RS} = 22$.

8-7

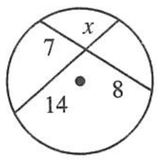
$\overline{AB}$ and $\overline{AE}$ are tangent at B and E. $m\overset{\frown}{CB} = 50$, $m\overset{\frown}{CD} = 30$, $m\overset{\frown}{DE} = 116$

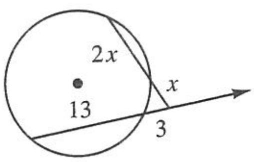
20. Find $m\angle 1$.
21. Find $m\overset{\frown}{BE}$.
22. Find $m\angle A$.
23. Find $m\angle 2$.



8-8

24. Find x in the figure on the right.
25. Find x in the figure on the left.



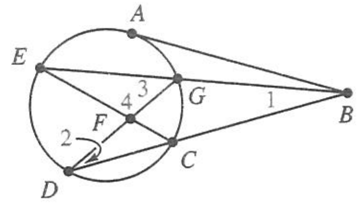


CHAPTER TEST

Determine whether each statement is true or false.

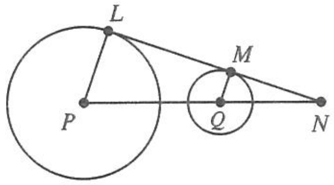
1. A radius is a segment that joins two points on a circle.
2. Concentric circles have the same center.
3. Two circles are congruent if their radii are congruent.
4. If a line is perpendicular to a radius of a circle at a point on the circle, then the line is tangent to the circle.
5. If the measure of an arc in one circle is equal to the measure of an arc in another circle, the chords of these arcs are congruent.
6. If a quadrilateral is inscribed in a circle, then its opposite angles are complementary.

$\overline{AB}$ is tangent at A.



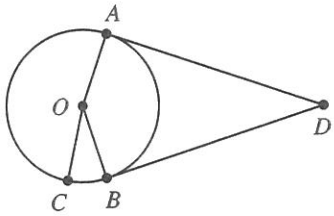
7. If $m\overset{\frown}{ED} = 48$, find $m\angle 3$.
8. If $m\overset{\frown}{DC} = 42$ and $m\angle 4 = 100$, find $m\overset{\frown}{EG}$.
9. If $m\overset{\frown}{ED} = 74$ and $m\overset{\frown}{GC} = 38$, find $m\angle 1$.
10. If $m\overset{\frown}{DEA} = 210$ and $m\overset{\frown}{AC} = 70$, find $m\angle ABD$.
11. If $m\angle 2 = 28$ and $m\angle 3 = 34$, find $m\angle 4$.

$\overline{LM}$ is a common tangent.



12. If $QM = 6$, $MN = 8$ and $LN = 20$, find PL.
13. If $PL = 5$ and $LN = 10$, find PN.
14. If $PL = 10$, $QM = 2$, and $LM = 20$, find LN.

$\overline{AD}$ and $\overline{BD}$ are tangent to $\odot O$.



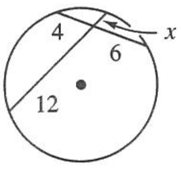
15. Find $m\angle OAD$.
16. If $BD = 8$, find AD.
17. If $m\angle BOC = 37$ and $m\angle AOB = 120$, find $m\overset{\frown}{AC}$.
18. If $\angle AOB$ is supplementary to $\angle ADB$ and $m\overset{\frown}{AB} = 100$, find $m\angle ADB$.

Complete each statement.

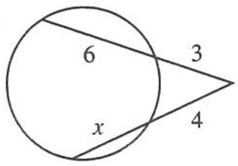
19. The perpendicular bisector of a chord contains the ___ of the circle.
20. In a circle, chords equidistant from the center are ___.
21. If a diameter bisects a chord that is not a diameter, then it is ___ to the chord.

Find x in each figure.

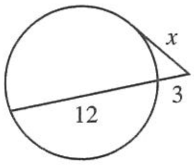
22.



23.



24.



PREPARING FOR COLLEGE ENTRANCE EXAMS

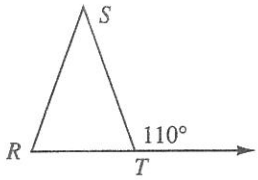
Give the one correct answer for each question.

1. If $m = x + 7$ and $n = x - 12$, then $m - n$ is equal to which of the following?

(A) 5 (B) 19 (C) -5
(D) -19 (E) $2x - 5$

2. If $z < 0$, which of the following must be negative?

(A) z^2 (B) $-2z$ (C) $z + 1$
(D) $|z| - z$ (E) $\frac{z}{2}$



3. In the figure above, if $\overline{RS} \cong \overline{ST}$, then $m\angle RST =$

(A) 40 (B) 140 (C) 70
(D) 110 (E) 60

4. A long string of colored lanterns is prepared for a party. The colors of the lanterns form a repeating pattern starting with red, yellow, green, blue, white; red, yellow, green, blue, white, and so on. What color is the 37th lantern?

(A) red (B) yellow (C) green
(D) blue (E) white

5. If the perimeter of a rectangular yard is 200 ft, which of the following could be the length of one of its sides?
I. 40 ft
II. 51 ft
III. 100 ft

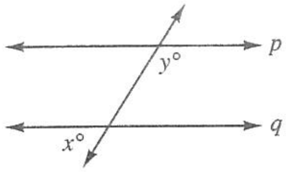
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

6. If the sum of five consecutive integers is 50, what is the largest of these integers?

(A) 16 (B) 12 (C) 10
(D) 9 (E) Cannot be determined from the information given.

7. A florist sells b bouquets of roses at d dollars per bouquet in h hours. Which of the following represents the amount of money received in two hours from the sale of roses?

(A) $\frac{2bd}{h}$ (B) $\frac{bd}{2h}$ (C) $2bdh$
(D) $\frac{2bh}{d}$ (E) $\frac{bh}{2d}$

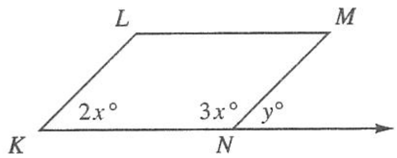


8. In the figure above, if $p \parallel q$, then $y - x =$

(A) 180 (B) 90 (C) 45
(D) 0 (E) Cannot be determined from the information given.

9. If M is odd, which of the following CANNOT be a whole number?

(A) $\sqrt{M}$ (B) $\frac{M-1}{2}$ (C) $\frac{M^2}{2}$
(D) M^2 (E) $\frac{M}{5}$



KLMN is a parallelogram.

10. In the figure above, what is the value of y?

(A) 36 (B) 60 (C) 72
(D) 108 (E) 45

11. If $x - y > 0$, which of the following MUST be true?

(A) $|x| > y$ (B) $-x > -y$
(C) $x > 0$ (D) $x^2 > y^2$ (E) $y < 0$

12. If 12 liters of pure alcohol are mixed with 8 liters of pure water, what percent of the resulting mixture is alcohol?

(A) $66\frac{2}{3}\%$ (B) 80% (C) 40%
(D) 75% (E) 60%