CHAPTER 9

The Derivative

Delta Notation

Let f be a function. As usual, we let x stand for any argument of f, and we let y be the corresponding value of f. Thus, $y = f(x)$. Consider any number x_0 in the domain of f. Let Δx (read "delta x") represent a small change in the value of x, from x_0 to $x_0 + \Delta x$, and then let Δy (read "delta y") denote the corresponding change in the value of y. So, $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Then the ratio

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called the *average rate of change* of the function f on the interval between x_0 and $x_0 + \Delta x$.

EXAMPLE 9.1: Let $y = f(x) = x^2 + 2x$. Starting at $x_0 = 1$, change x to 1.5. Then $\Delta x = 0.5$. The corresponding change in y is $\Delta y = f(1.5) - f(1) = 5.25 - 3 = 2.25$. Hence, the average rate of change of y on the interval between $x = 1$ and $x = 1.5$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{2.25}{0.5} = 4.5$.

The Derivative

If $y = f(x)$ and x_0 is in the domain of f, then by the *instantaneous rate of change* of f at x_0 we mean the limit of the average rate of change between x_0 and $x_0 + \Delta x$ as Δx approaches 0:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

provided that this limit exists. This limit is also called the *derivative* of f at x_0.

Notation for Derivatives

Let us consider the derivative of f at an arbitrary point x in its domain:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The value of the derivative is a function of x, and will be denoted by any of the following expressions:

$$D_x y = \frac{dy}{dx} = y' = f'(x) = \frac{d}{dx} y = \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The value $f'(a)$ of the derivative of f at a particular point a is sometimes denoted by $\left.\dfrac{dy}{dx}\right|_{x=a}$

Differentiability

A function is said to be *differentiable* at a point x_0 if the derivative of the function exists at that point. Problem 2 of Chapter 8 shows that differentiability implies continuity. That the converse is false is shown in Problem 11.

SOLVED PROBLEMS

1. Given $y = f(x) = x^2 + 5x - 8$, find Δy and $\Delta y/\Delta x$ as x changes (a) from $x_0 = 1$ to $x_1 = x_0 + \Delta x = 1.2$ and (b) from $x_0 = 1$ to $x_1 = 0.8$.

(a) $\Delta x = x_1 - x_0 = 1.2 - 1 = 0.2$ and $\Delta y = f(x_0 + \Delta x) - f(x_0) = f(1.2) - f(1) = -0.56 - (-2) = 1.44$. So $\frac{\Delta y}{\Delta x} = \frac{1.44}{0.2} = 7.2$.

(b) $\Delta x = 0.8 - 1 = -0.2$ and $\Delta y = f(0.8) - f(1) = -3.36 - (-2) = -1.36$. So $\frac{\Delta y}{\Delta x} = \frac{-1.36}{-0.2} = 6.8$.

Geometrically, $\Delta y/\Delta x$ in (a) is the slope of the secant line joining the points (1, –2) and (1.2, –0.56) of the parabola $y = x^2 + 5x - 8$, and in (b) is the slope of the secant line joining the points (0.8, –3.36) and (1, –2) of the same parabola.

2. If a body (that is, a material object) starts out at rest and then falls a distance of s feet in t seconds, then physical laws imply that $s = 16t^2$. Find $\Delta s/\Delta t$ as t changes from t_0 to $t_0 + \Delta t$. Use the result to find $\Delta s/\Delta t$ as t changes: (a) from 3 to 3.5, (b) from 3 to 3.2, and (c) from 3 to 3.1.

$$\frac{\Delta s}{\Delta t} = \frac{16(t_0 + \Delta t)^2 - 16t_0^2}{\Delta t} = \frac{32t_0\Delta t + 16(\Delta t)^2}{\Delta t} = 32t_0 + 16\,\Delta t$$

(a) Here $t_0 = 3$, $\Delta t = 0.5$, and $\Delta s/\Delta t = 32(3) + 16(0.5) = 104$ ft/sec.
(b) Here $t_0 = 3$, $\Delta t = 0.2$, and $\Delta s/\Delta t = 32(3) + 16(0.2) = 99.2$ ft/sec.
(c) Here $t_0 = 3$, $\Delta t = 0.1$, and $\Delta s/\Delta t = 97.6$ ft/sec.

Since Δs is the displacement of the body from time $t = t_0$ to $t = t_0 + \Delta t$,

$$\frac{\Delta s}{\Delta t} = \frac{\text{displacement}}{\text{time}} = \text{average velocity of the body over the time interval}$$

3. Find dy/dx, given $y = x^3 - x^2 - 4$. Find also the value of dy/dx when (a) $x = 4$, (b) $x = 0$, (c) $x = -1$.

$$y + \Delta y = (x + \Delta x)^3 - (x + \Delta x)^2 - 4$$
$$= x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^2 - 2x(\Delta x) - (\Delta x)^2 - 4$$
$$\Delta y = (3x^2 - 2x)\Delta x + (3x - 1)(\Delta x)^2 + (\Delta x)^3$$
$$\frac{\Delta y}{\Delta x} = 3x^2 - 2x + (3x - 1)\Delta x + (\Delta x)^2$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0}[3x^2 - 2x + (3x - 1)\Delta x + (\Delta x)^2] = 3x^2 - 2x$$

(a) $\left.\frac{dy}{dx}\right|_{x=4} = 3(4)^2 - 2(4) = 40$; (b) $\left.\frac{dy}{dx}\right|_{x=0} = 3(0)^2 - 2(0) = 0$; (c) $\left.\frac{dy}{dx}\right|_{x=-1} = 3(-1)^2 - 2(-1) = 5$

4. Find the derivative of $y = f(x) = x^2 + 3x + 5$.

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x) = [(x+\Delta x)^2 + 3(x+\Delta x) + 5)] - [x^2 + 3x + 5]\\ &= [x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x + 5] - [x^2 + 3x + 5] = 2x\Delta x + (\Delta x)^2 + 3\Delta x\\ &= (2x + \Delta x + 3)\Delta x\end{aligned}$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 3$$

So, $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} (2x + \Delta x + 3) = 2x + 3$.

5. Find the derivative of $y = f(x) = \dfrac{1}{x-2}$ at $x = 1$ and $x = 3$.

$$\Delta y = f(x+\Delta x) - f(x) = \frac{1}{(x+\Delta x)-2} - \frac{1}{x-2} = \frac{(x-2)-(x+\Delta x-2)}{(x-2)(x+\Delta x-2)}$$

$$= \frac{-\Delta x}{(x-2)(x+\Delta x-2)}$$

$$\frac{\Delta y}{\Delta x} = \frac{-1}{(x-2)(x+\Delta x-2)}$$

So, $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{-1}{(x-2)(x+\Delta x-2)} = \dfrac{-1}{(x-2)^2}$.

At $x = 1$, $\dfrac{dy}{dx} = \dfrac{-1}{(1-2)^2} = -1$. At $x = 3$, $\dfrac{dy}{dx} = \dfrac{-1}{(3-2)^2} = -1$.

6. Find the derivative of $f(x) = \dfrac{2x-3}{3x+4}$.

$$f(x+\Delta x) = \frac{2(x+\Delta x)-3}{3(x+\Delta x)+4}$$

$$f(x+\Delta x) - f(x) = \frac{2x+2\Delta x-3}{3x+3\Delta x+4} - \frac{2x-3}{3x+4}$$

$$= \frac{(3x+4)[(2x-3)+2\Delta x] - (2x-3)[(3x+4)+3\Delta x]}{(3x+4)(3x+3\Delta x+4)}$$

$$= \frac{(6x+8-6x+9)\Delta x}{(3x+4)(3x+3\Delta x+4)} = \frac{17\Delta x}{(3x+4)(3x+3\Delta x+4)}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{17}{(3x+4)(3x+3\Delta x+4)}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{17}{(3x+4)(3x+3\Delta x+4)} = \frac{17}{(3x+4)^2}$$

7. Find the derivative of $y = f(x) = \sqrt{2x+1}$.

$$y + \Delta y = (2x + 2\Delta x + 1)^{1/2}$$

$$\Delta y = (2x + 2\Delta x + 1)^{1/2} - (2x+1)^{1/2}$$

$$= [(2x+2\Delta x+1)^{1/2} - (2x+1)^{1/2}]\frac{(2x+2\Delta x+1)^{1/2} + (2x+1)^{1/2}}{(2x+2\Delta x+1)^{1/2} + (2x+1)^{1/2}}$$

$$= \frac{(2x+2\Delta x+1)-(2x+1)}{(2x+2\Delta x+1)^{1/2} + (2x+1)^{1/2}} = \frac{2\Delta x}{(2x+2\Delta x+1)^{1/2} + (2x+1)^{1/2}}$$

$$\frac{\Delta y}{\Delta x} = \frac{2}{(2x+2\Delta x+1)^{1/2} + (2x+1)^{1/2}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2}{(2x+2\Delta x+1)^{1/2} + (2x+1)^{1/2}} = \frac{1}{(2x+1)^{1/2}}$$

8. Find the derivative of $f(x) = x^{1/3}$. Examine $f'(0)$.

$$f(x+\Delta x) = (x+\Delta x)^{1/3}$$

$$f(x+\Delta x) - f(x) = (x+\Delta x)^{1/3} - x^{1/3}$$

$$= \frac{[(x+\Delta x)^{1/3} - x^{1/3}][(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}]}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}}$$

$$= \frac{x+\Delta x - x}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}}$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{1}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}}$$

The derivative does not exist at $x = 0$ because the denominator is zero there. Note that the function f is continuous at $x = 0$.

9. Interpret dy/dx geometrically.

From Fig. 9-1 we see that $\Delta y/\Delta x$ is the slope of the secant line joining an arbitrary but fixed point $P(x, y)$ and a nearby point $Q(x + \Delta x, y + \Delta y)$ of the curve. As $\Delta x \to 0$, P remains fixed while Q moves along the curve toward P, and the line PQ revolves about P toward its limiting position, the tangent line PT moves to the curve at P. Thus, dy/dx gives the slope of the tangent line at P to the curve $y = f(x)$.

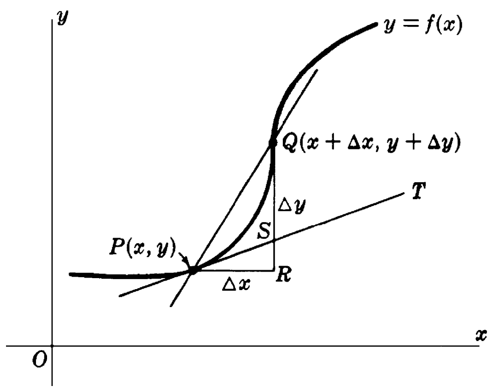

Fig. 9-1

For example, from Problem 3, the slope of the cubic $y = x^3 - x^2 - 4$ is $m = 40$ at the point $x = 4$; it is $m = 0$ at the point $x = 0$; and it is $m = 5$ at the point $x = -1$.

10. Find ds/dt for the function of Problem 2 and interpret the result.

$$\frac{\Delta s}{\Delta t} = 32t_0 + 16\Delta t. \quad \text{Hence,} \quad \frac{ds}{dt} = \lim_{\Delta t \to 0}(32t_0 + 16\Delta t) = 32t_0$$

As $\Delta t \to 0$, $\Delta s/\Delta t$ gives the average velocity of the body for shorter and shorter time intervals Δt. Then we can consider ds/dt to be the *instantaneous velocity* v of the body at time t_0.

For example, at $t = 3$, $v = 32(3) = 96$ ft/sec. In general, if an object is moving on a straight line, and its position on the line has coordinate s at time t, then its instantaneous velocity at time t is ds/dt. (See Chapter 19.)

11. Find $f'(x)$ when $f(x) = |x|$.

The function is continuous for all values of x. For $x < 0$, $f(x) = -x$ and

$$f'(x) = \lim_{\Delta x \to 0} \frac{-(x+\Delta x) - (-x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x \to 0} -1 = -1$$

Similarly, for $x > 0$, $f(x) = x$ and

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

At $x = 0$, $f(x) = 0$ and $\lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$.

As $\Delta x \to 0^-$, $\frac{|\Delta x|}{\Delta x} = \frac{-\Delta x}{\Delta x} = -1 \to -1$. But, as $\Delta x \to 0^+$, $\frac{|\Delta x|}{\Delta x} = \frac{\Delta x}{\Delta x} = 1 \to 1$. Hence, the derivative does not exist at $x = 0$.

Since the function is continuous at 0, this shows that continuity does not imply differentiability.

12. Compute $\epsilon = \frac{\Delta y}{\Delta x} - \frac{dy}{dx}$ for the function of (a) Problem 3 and (b) Problem 5. Verify that $\epsilon \to 0$ as $\Delta x \to 0$.

(a) $\epsilon = [3x^2 - 2x + (3x - 1)\Delta x + (\Delta x)^2] - (3x^2 - 2x) = (3x - 1 + \Delta x)\,\Delta x$

(b) $\epsilon = \frac{-1}{(x-2)(x + \Delta x - 2)} - \frac{-1}{(x-2)^2} = \frac{-(x-2) + (x + \Delta x - 2)}{(x-2)^2(x + \Delta x - 2)} = \frac{1}{(x-2)^2(x + \Delta x - 2)}\Delta x$

Both obviously go to zero as $\Delta x \to 0$.

13. Interpret $\Delta y = \frac{dy}{dx}\Delta x + \epsilon\,\Delta x$ of Problem 12 geometrically.

In Fig. 9-1, $\Delta y = RQ$ and $\frac{dy}{dx}\Delta x = PR \tan \angle TPR = RS$; thus, $\epsilon\,\Delta x = SQ$. For a change Δx in x from $P(x, y)$, Δy is the corresponding change in *y along the curve* while $\frac{dy}{dx}\Delta x$ is the corresponding change in *y along the tangent line PT*. Since their difference $\epsilon\,\Delta x$ is a multiple of $(\Delta x)^2$, it goes to zero faster than Δx, and $\frac{dy}{dx}\Delta x$ can be used as an approximation of Δy when $|\Delta x|$ is small.

SUPPLEMENTARY PROBLEMS

14. Find Δy and $\Delta y/\Delta x$, given

(a) $y = 2x - 3$ and x changes from 3.3 to 3.5.
(b) $y = x^2 + 4x$ and x changes from 0.7 to 0.85.
(c) $y = 2/x$ and x changes from 0.75 to 0.5.

Ans. (a) 0.4 and 2; (b) 0.8325 and 5.55; (c) $\frac{4}{3}$ and $-\frac{16}{3}$

15. Find Δy, given $y = x^2 - 3x + 5$, $x = 5$, and $\Delta x = -0.01$. What then is the value of y when $x = 4.99$?

Ans. $\Delta y = -0.0699$; $y = 14.9301$

16. Find the average velocity (see Problem 2), given: (a) $s = (3t^2 + 5)$ feet and t changes from 2 to 3 seconds. (b) $s = (2t^2 + 5t - 3)$ feet and t changes from 2 to 5 seconds.

Ans. (a) 15 ft/sec; (b) 19 ft/sec

17. Find the increase in the volume of a spherical balloon when its radius is increased (a) from r to $r + \Delta r$ inches; (b) from 2 to 3 inches. (Recall that volume $V = \frac{4}{3}\pi r^3$.)

Ans. (a) $\frac{4}{3}\pi[3r^2 + 3r\,\Delta r + (\Delta r)^2]\Delta r$ in^3; (b) $\frac{76}{3}\pi$ in^3

18. Find the derivative of each of the following:

(a) $y = 4x - 3$ (b) $y = 4 - 3x$ (c) $y = x^2 + 2x - 3$
(d) $y = 1/x^2$ (e) $y = (2x - 1)/(2x + 1)$ (f) $y = (1 + 2x)/(1 - 2x)$
(g) $y = \sqrt{x}$ (h) $y = 1/\sqrt{x}$ (i) $y = \sqrt{1+2x}$
(j) $y = 1/\sqrt{2+x}$

Ans. (a) 4; (b) -3; (c) $2(x + 1)$; (d) $-2/x^3$; (e) $\frac{4}{(2x+1)^2}$; (f) $\frac{4}{(1-2x)^2}$; (g) $\frac{1}{2\sqrt{x}}$; (h) $-\frac{1}{2x\sqrt{x}}$; (i) $\frac{1}{\sqrt{1+2x}}$; (j) $-\frac{1}{2(2+x)^{3/2}}$

19. Find the slope of the tangent line to the following curves at the point $x = 1$ (see Problem 9): (a) $y = 8 - 5x^2$; (b) $y = \frac{4}{x+1}$; (c) $\frac{2}{x+3}$.

Ans. (a) -10; (b) -1; (c) $-\frac{1}{8}$

20. **(GC)** Use a graphing calculator to verify your answers in Problem 19. (Graph the curve and the tangent line that you found.)

21. Find the coordinates of the vertex (that is, the turning point) of the parabola $y = x^2 - 4x + 1$ by making use of the fact that, at the vertex, the slope of the tangent line is zero. (See Problem 9.) (GC) Check your answer with a graphing calculator.

Ans. $(2, -3)$

22. Find the slope m of the tangent lines to the parabola $y = -x^2 + 5x - 6$ at its points of intersection with the x axis.

Ans. At $x = 2$, $m = 1$. At $x = 3$, $m = -1$.

23. When an object is moving on a straight line and its coordinate on that line is s at time t (where s is measured in feet and t in seconds), find the velocity at time $t = 2$ in the following cases:

(a) $s = t^2 + 3t$ (b) $s = t^3 - 3t^2$ (c) $s = \sqrt{t+2}$
(See Problem 10.)

Ans. (a) 7 ft/sec; (b) 0 ft/sec; (c) $\frac{1}{4}$ ft/sec

24. Show that the instantaneous rate of change of the volume V of a cube with respect to its edge x (measured in inches) is 12 in^3/in when $x = 2$ in.