CHAPTER 8

Continuity

Continuous Function

A function f is defined to be continuous at x_0 if the following three conditions hold:

(i) $f(x_0)$ is defined;

(ii) $\lim_{x \to x_0} f(x)$ exists;

(iii) $\lim_{x \to x_0} f(x) = f(x_0)$.

For example, $f(x) = x^2 + 1$ is continuous at 2, since $\lim_{x \to 2} f(x) = 5 = f(2)$. Condition (i) implies that a function can be continuous only at points of its domain. Thus, $f(x) = \sqrt{4 - x^2}$ is not continuous at 3 because $f(3)$ is not defined.

Let f be a function that is defined on an interval (a, x_0) to the left of x_0 and/or on an interval (x_0, b) to the right of x_0. We say that f is discontinuous at x_0 if f is not continuous at x_0, that is, if one or more of the conditions (i)–(iii) fails.

EXAMPLE 8.1:

(a) $f(x) = \dfrac{1}{x-2}$ is discontinuous at 2 because $f(2)$ is not defined and also because $\lim_{x \to 2} f(x)$ does not exist (since $\lim_{x \to 2} f(x) = \infty$). See Fig. 8-1.

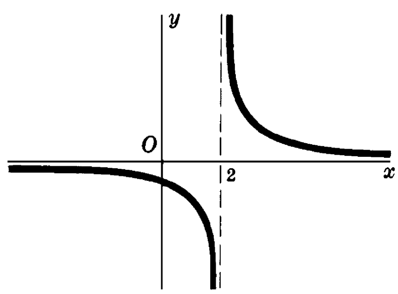


Fig. 8-1

(b) $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is not defined. However, $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{(x+2)(x-2)}{x-2} = \lim_{x \to 2}(x+2) = 4$ so that condition (ii) holds.

The discontinuity at 2 in Example 8.1(b) is said to be *removable* because, if we extended the function f by defining its value at $x = 2$ to be 4, then the extended function g would be continuous at 2. Note that $g(x) = x + 2$ for all x. The graphs of $f(x) = \dfrac{x^2 - 4}{x - 2}$ and $g(x) = x + 2$ are identical except at $x = 2$, where the former has a "hole." (See Fig. 8-2.) Removing the discontinuity consists simply of filling the "hole."

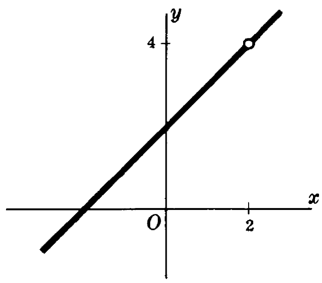


Fig. 8-2

The discontinuity at 2 in Example 8.1(a) is not removable. Redefining the value of f at 2 cannot change the fact that $\lim_{x\to 2}\frac{1}{x-2}$ does not exist.

We also call a discontinuity of a function f at x_0 *removable* when $f(x_0)$ is defined and changing the value of the function at x_0 produces a function that is continuous at x_0.

EXAMPLE 8.2: Define a function f as follows:

$$f(x)=\begin{cases} x^2 & \text{if } x\neq 2 \\ 0 & \text{if } x=2 \end{cases}$$

Here $\lim_{x\to 2} f(x)=4$, but $f(2)=0$. Hence, condition (iii) fails, so that f has a discontinuity at 2. But if we change the value of f at 2 to be 4, then we obtain a function h such that $h(x)=x^2$ for all x, and h is continuous at 2. Thus, the discontinuity of f at 2 was removable.

EXAMPLE 8.3: Let f be the function such that $f(x)=\frac{|x|}{x}$ for all $x\neq 0$. The graph of f is shown in Fig. 8-3. f is discontinuous at 0 because $f(0)$ is not defined. Moreover,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x}{x}=1 \quad\text{and}\quad \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{-x}{x}=-1$$

Thus, $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$. Hence, the discontinuity of f at 0 is not removable.

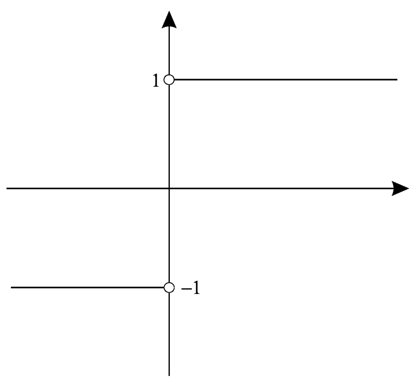


Fig. 8-3

The kind of discontinuity shown in Example 8.3 is called a *jump discontinuity*. In general, a function f has a jump discontinuity at x_0 if $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ both exist and $\lim_{x\to x_0^-} f(x)\neq \lim_{x\to x_0^+} f(x)$. Such a discontinuity is not removable.

EXAMPLE 8.4: The function of Problem 4 in Chapter 6 has a jump discontinuity at every positive integer.

Properties of limits lead to corresponding properties of continuity.

Theorem 8.1: Assume that f and g are continuous at x_0. Then:

(a) The constant function $h(x) = c$ for all x is continuous at every x_0.
(b) cf is continuous at x_0, for any constant c. (Recall that cf has the value $c \cdot f(x)$ for each argument x.)
(c) $f + g$ is continuous at x_0.
(d) $f - g$ is continuous at x_0.
(e) fg is continuous at x_0.
(f) f/g is continuous at x_0 if $g(x_0) \neq 0$.
(g) $\sqrt[n]{f}$ is continuous at x_0 if $\sqrt[n]{f(x_0)}$ is defined.

These results follow immediately from Theorems 7.1–7.6. For example, (*c*) holds because

$$\lim_{x\to x_0} (f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = f(x_0) + g(x_0)$$

Theorem 8.2: The identify function $I(x) = x$ is continuous at every x_0.

This follows from the fact that $\lim_{x\to x_0} x = x_0$.

We say that a function f is *continuous on a set A* if f is continuous at every point of A. Moreover, if we just say that *f is continuous*, we mean that f is continuous at every real number.

The original intuitive idea behind the notion of continuity was that the graph of a continuous function was supposed to be "continuous" in the intuitive sense that one could draw the graph without taking the pencil off the paper. Thus, the graph would not contain any "holes" or "jumps." However, it turns out that our precise definition of continuity goes well beyond that original intuitive notion; there are very complicated continuous functions that could certainly not be drawn on a piece of paper.

Theorem 8.3: Every polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is continuous.

This is a consequence of Theorems 8.1 (*a–e*) and 8.2.

EXAMPLE 8.5: As an instance of Theorem 8.3, consider the function $x^2 - 2x + 3$. Note that, by Theorem 8.2, the identity function x is continuous and, therefore, by Theorem 8.1(*e*), x^2 is continuous, and, by Theorem 8.1(*b*), $-2x$ is continuous. By Theorem 8.1(*a*), the constant function 3 is continuous. Finally, by Theorem 8.1(*c*), $x^2 - 2x + 3$ is continuous.

Theorem 8.4: Every *rational function* $H(x) = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions, is continuous on the set of all points at which $g(x) \neq 0$.

This follows from Theorems 8.1(*f*) and 8.3. As examples, the function $H(x) = \dfrac{x}{x^2 - 1}$ is continuous at all points except 1 and −1, and the function $G(x) = \dfrac{x-7}{x^2+1}$ is continuous at all points (since $x^2 + 1$ is never 0).

We shall use a special notion of continuity with respect to a closed interval $[a, b]$. First of all, we say that a function f is *continuous on the right at a* if $f(a)$ is defined and $\lim_{x\to a^+} f(x)$ exists, and $\lim_{x\to a^+} f(x) = f(a)$. We say that f is *continuous on the left at b* if $f(b)$ is defined and $\lim_{x\to b^-} f(x)$ exists, and $\lim_{x\to b^-} f(x) = f(b)$.

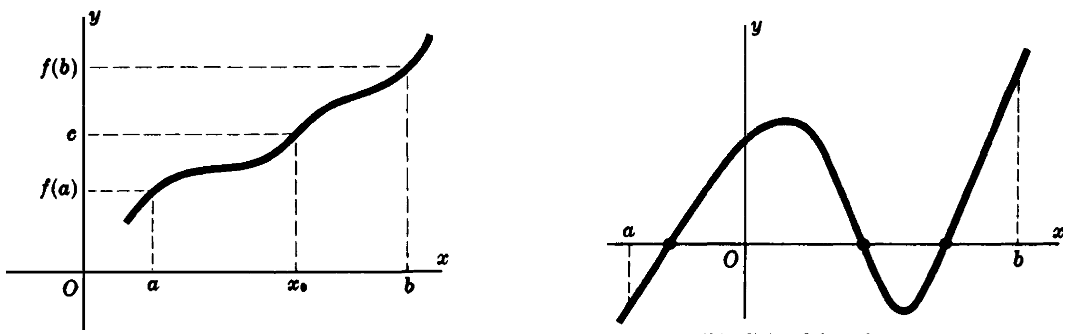
Definition: f is *continuous* on $[a, b]$ if f is continuous at each point on the open interval (a, b), f is continuous on the right at a, and f is continuous on the left at b.

Note that whether f is continuous on $[a, b]$ does not depend on the values of f, if any, outside of $[a, b]$. Note also that every continuous function (that is, a function continuous at all real numbers) must be continuous on any closed interval. In particular, every polynomial function is continuous on any closed interval.

We want to discuss certain deep properties of continuous functions that we shall use but whose proofs are beyond the scope of this book.

Theorem 8.5 (Intermediate Value Theorem): If f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then, for any number c between $f(a)$ and $f(b)$, there is at least one number x_0 in the open interval (a, b) for which $f(x_0) = c$.

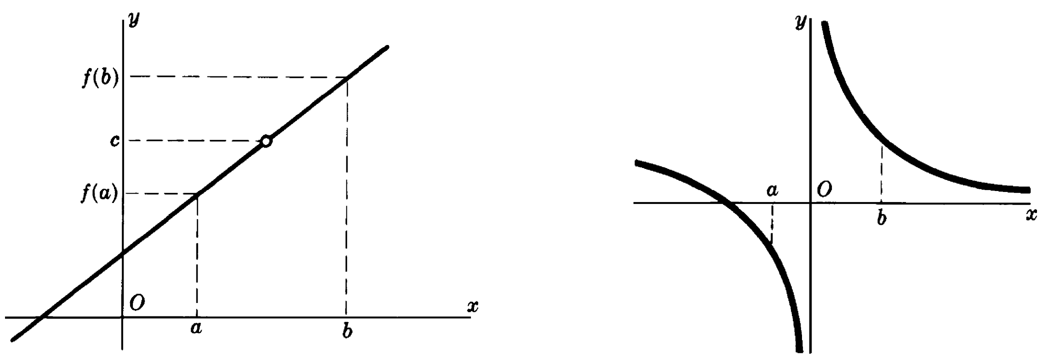
Figure 8-4(a) is an illustration of Theorem 8.5. Fig. 8-5 shows that continuity throughout the interval is essential for the validity of the theorem. The following result is a special case of the Intermediate Value Theorem.



(a)

(b) $f(x) = 0$ has three roots between $x = a$ and $x = b$.

Fig. 8-4



(a)

(b) $f(x) = 0$ has no root between $x = a$ and $x = b$.

Fig. 8-5

Corollary 8.6: If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one root in the open interval (a, b), and, therefore, the graph of f crosses the x-axis at least once between a and b. (See Fig. 8-4(b).)

Theorem 8.7 (Extreme Value Theorem): If f is continuous on $[a, b]$, then f takes on a least value m and a greatest value M on the interval.

As an illustration of the Extreme Value Theorem, look at Fig. 8-6(a), where the minimum value m occurs at $x = c$ and the maximum value M occurs at $x = d$. In this case, both c and d lie inside the interval. On the other hand, in Fig. 8-6(b), the minimum value m occurs at the endpoint $x = a$ and the maximum value M occurs inside the interval. To see that continuity is necessary for the Extreme Value Theorem to be true, consider the function whose graph is indicated in Fig. 8-6(c). There is a discontinuity at c inside the interval; the function has a minimum value at the left endpoint $x = a$ but the function has no maximum value.

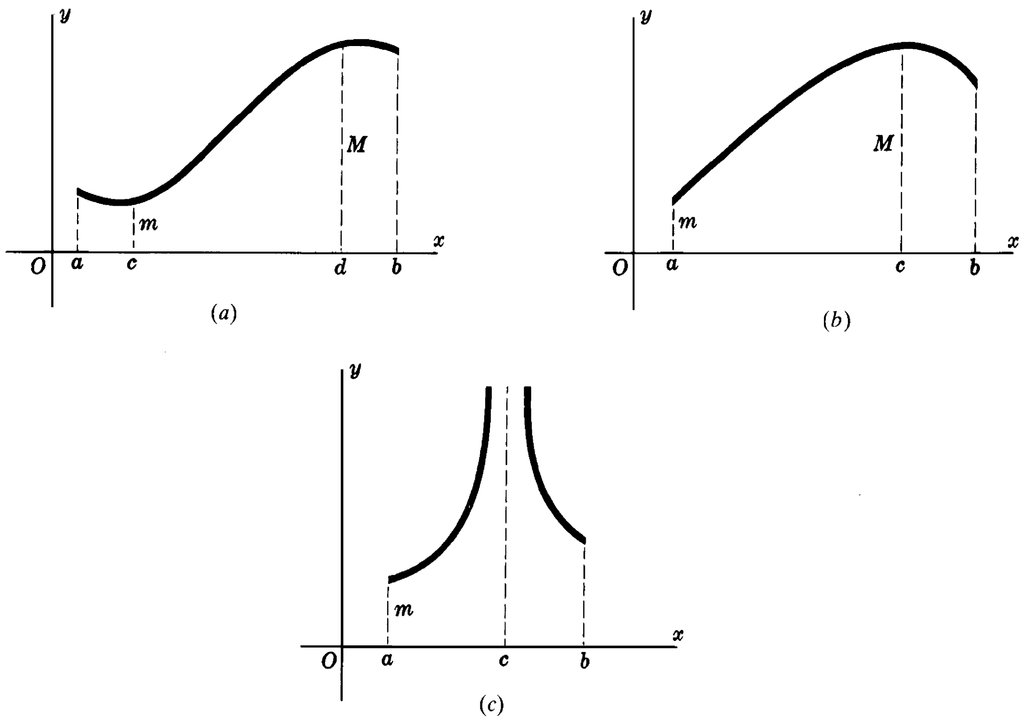


Fig. 8-6

Another useful property of continuous functions is given by the following result.

Theorem 8.8: If f is continuous at c and $f(c) > 0$, then there is a positive number δ such that, whenever $c - \delta < x < c + \delta$, then $f(x) > 0$.

This theorem is illustrated in Fig. 8-7. For a proof, see Problem 3.

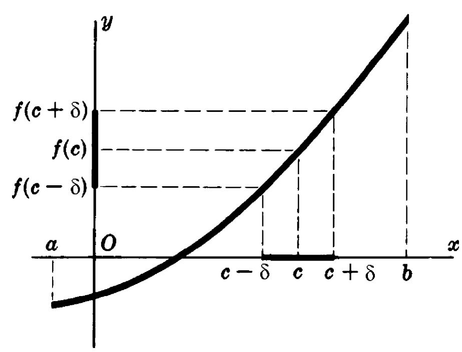


Fig. 8-7

SOLVED PROBLEMS

1. Find the discontinuities of the following functions. Determine whether they are removable. If not removable, determine whether they are jump discontinuities. (GC) Check your answers by showing the graph of the function on a graphing calculator.

   (a) $f(x) = \frac{2}{x}$. Nonremovable discontinuity at $x = 0$.

(b) $f(x) = \frac{x-1}{(x+3)(x-2)}$. Nonremovable discontinuities at $x = -3$ and $x = 2$.

(c) $f(x) = \frac{(x+2)(x-1)}{(x-3)^2}$. Nonremovable discontinuity at $x = 3$.

(d) $f(x) = \frac{x^3-27}{x^2-9}$. Has a removable discontinuity at $x = 3$. (Note that $x^3 - 27 = (x-3)(x^2 + 3x + 9)$.) Also has a nonremovable discontinuity at $x = -3$.

(e) $f(x) = \frac{4-x^2}{3-\sqrt{x^2+5}}$. Has a removable discontinuity at $x = \pm 2$. Note that

$$\frac{4-x^2}{3-\sqrt{x^2+5}}\,\frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}} = 3+\sqrt{x^2+5}.$$

(f) $f(x) = \frac{x^2+x-2}{(x-1)^2}$. Has a nonremovable discontinuity at $x = 1$.

(g) $f(x) = [x]$ = the greatest integer $\leq x$. Has a jump discontinuity at every integer.

(h) $f(x) = x - [x]$. Has a nonremovable discontinuity at every integer.

(i) $f(x) = 3x^3 - 7x^2 + 4x - 2$. A polynomial has no discontinuities.

(j) $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2 & \text{if } x \neq 0 \end{cases}$ Removable discontinuity at $x = 0$.

(k) $f(x) = \begin{cases} x & \text{if } x \leq 0. \\ x^2 & \text{if } 0 < x < 1 \\ 2-x & \text{if } x \geq 1. \end{cases}$ No discontinuities.

2. Show that the existence of $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ implies that f is continuous at $x = a$.

$$\lim_{h\to 0}(f(a+h)-f(a)) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\cdot h\right) =$$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\lim_{h\to 0} h = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot 0 = 0$$

But

$$\lim_{h\to 0}(f(a+h)-f(a)) = \lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) = \lim_{h\to 0} f(a+h) - f(a)$$

Hence, $\lim_{h\to 0} f(a+h) = f(a)$. Note that $\lim_{h\to 0} f(a+h) = \lim_{x\to a} f(x)$. So, $\lim_{x\to a} f(x) = f(a)$.

3. Prove Theorem 8.8.

By the continuity of f at c, $\lim_{x\to c} f(x) = f(c)$. If we let $\epsilon = f(c)/2 > 0$, then there exists a positive δ such that $0 < |x - c| < \delta$ implies that $|f(x) - f(c)| < f(c)/2$. The latter inequality also holds when $x = c$. Thus, $|x - c| < \delta$ implies $|f(x) - f(c)| < f(c)/2$. The latter implies $-f(c)/2 < f(x) - f(c) < f(c)/2$. Adding $f(c)$ to the left-hand inequality, we obtain $f(c)/2 < f(x)$.

SUPPLEMENTARY PROBLEMS

4. Determine the discontinuities of the following functions and state why the function fails to be continuous at those points. (GC) Check your answers by graphing the function on a graphing calculator.

(a) $f(x) = \frac{x^2-3x-10}{x+2}$

(b) $f(x) = \begin{cases} x+3 & \text{if } x \geq 2 \\ x^2+1 & \text{if } x < 2 \end{cases}$

(c) $f(x) = |x| - x$

(d) $f(x) = \begin{cases} 4-x & \text{if } x < 3 \\ x-2 & \text{if } 0 < x < 3 \\ x-1 & \text{if } x \leq 0 \end{cases}$

(e) $f(x) = \frac{x^4-1}{x^2-1}$

(f) $f(x) = \frac{x^3+x^2-17x+15}{x^2+2x-15}$

(g) $f(x) = x^3 - 7x$ (h) $f(x) = \dfrac{x^2 - 4}{x^2 - 5x + 6}$

(i) $f(x) = \dfrac{x^2 + 3x + 2}{x^2 + 4x + 3}$ (j) $f(x) = \dfrac{x - 2}{x^2 - 4}$

(k) $f(x) = \dfrac{x - 1}{\sqrt{x^2 + 3} - 2}$

Ans. (a) Removable discontinuity at $x = -2$. (Note that $x^2 - 3x - 10 = (x + 2)(x - 5)$.)
(b, c, g) None.
(d) Jump discontinuity at $x = 0$.
(e) Removable discontinuities at $x = \pm 1$.
(f) Removable discontinuities at $x = 3$, $x = -5$. (Note that $x^2 + 2x - 5 = (x + 5)(x - 3)$ and $x^3 + x^2 - 17x + 15 = (x + 5)(x - 3)(x - 1)$.)
(h) Removable discontinuity at $x = 2$ and nonremovable discontinuity at $x = 3$.
(i) Removable discontinuity at $x = -1$ and nonremovable discontinuity at $x = -3$.
(j) Removable discontinuity at $x = 2$ and nonremovable discontinuity at $x = -2$.
(k) Removable discontinuity at $x = 1$ and nonremovable discontinuity at $x = -1$.

5. Show that $f(x) = |x|$ is continuous.

6. If Fig. 8-5(*a*) is the graph of $f(x) = \dfrac{x^2 - 4x - 21}{x - 7}$, show that there is a removable discontinuity at $x = 7$ and that $c = 10$ there.

7. Prove: If f is continuous on the interval $[a, b]$ and c is a number in (a, b) such that $f(c) < 0$, then there exists a positive number δ such that, whenever $c - \delta < x < c + \delta$, then $f(x) < 0$.

(*Hint*: Apply Theorem 8.8 to $-f$.)

8. Sketch the graphs of the following functions and determine whether they are continuous on the closed interval [0, 1]:

(a) $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$ (b) $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ 1 & \text{if } x \le 0 \end{cases}$

(c) $f(x) = \begin{cases} -x^2 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$ (d) $f(x) = 1$ if $0 < x \le 1$

(e) $f(x) = \begin{cases} x & \text{if } x \le 0 \\ 0 & \text{if } 0 < x < 1 \\ x & \text{if } x \ge 1 \end{cases}$

Ans. (a) Yes. (b) No. Not continuous on the right at 0. (c) Yes. (d) No. Not defined at 0. (*e*) No. Not continuous on the left at 1.