CHAPTER 7

# Limits

## Limit of a Function

If $f$ is a function, then we say:

*A is the limit of f(x) as x approaches a*

if the value of $f(x)$ gets arbitrarily close to $A$ as $x$ approaches $a$. This is written in mathematical notation as:

$$\lim_{x \to a} f(x) = A$$

For example, $\lim_{x \to 3} x^2 = 9$, since $x^2$ gets arbitrarily close to 9 as $x$ approaches as close as one wishes to 3. The definition of $\lim_{x \to a} f(x) = A$ was stated above in ordinary language. The definition can be stated in more precise mathematical language as follows: $\lim_{x \to a} f(x) = A$ if and only if, for any given positive number $\epsilon$, however small, there exists a positive number $\delta$ such that, whenever $0 < |x - a| < \delta$, then $|f(x) - A| < \epsilon$.

The gist of the definition is illustrated in Fig. 7-1. After $\epsilon$ has been chosen [that is, after interval (ii) has been chosen], then $\delta$ can be found [that is, interval (i) can be determined] so that, whenever $x \neq a$ is on interval (i), say at $x_0$, then $f(x)$ is on interval (ii), at $f(x_0)$. Notice the important fact that whether or not $\lim_{x \to a} f(x) = A$ is true does not depend upon the value of $f(x)$ when $x = a$. In fact, $f(x)$ need not even be defined when $x = a$.

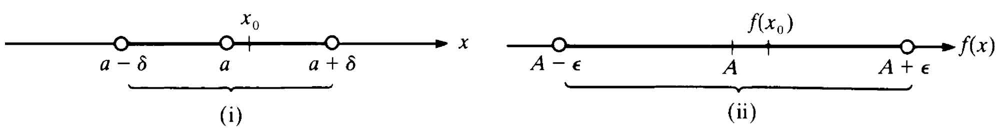


Fig. 7-1

**EXAMPLE 7.1:** $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$, although $\frac{x^2 - 4}{x - 2}$ is not defined when $x = 2$. Since

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

we see that $\frac{x^2 - 4}{x - 2}$ approaches 4 as $x$ approaches 2.

**EXAMPLE 7.2:** Let us use the precise definition to show that $\lim_{x \to 2} (4x - 5) = 3$. Let $\epsilon > 0$ be chosen. We must produce some $\delta > 0$ such that, whenever $0 < |x - 2| < \delta$, then $|(4x - 5) - 3| < \epsilon$.

First we note that $|(4x - 5) - 3| = |4x - 8| = 4|x - 2|$.

If we take $\delta$ to be $\epsilon/4$, then, whenever $0 < |x - 2| < \delta$, $|(4x - 5) - 3| = 4|x - 2| < 4\delta = \epsilon$.

## Right and Left Limits

Next we want to talk about one-sided limits of $f(x)$ as $x$ approaches $a$ from the right-hand side or from the left-hand side. By $\lim_{x\to a^-} f(x) = A$ we mean that $f$ is defined in some open interval $(c, a)$ and $f(x)$ approaches $A$ as $x$ approaches $a$ through values less than $a$, that is, as $x$ approaches $a$ *from the left*. Similarly, $\lim_{x\to a^+} f(x) = A$ means that $f$ is defined in some open interval $(a, d)$ and $f(x)$ approaches $A$ as $x$ approaches *a from the right*. If $f$ is defined in an interval to the left of $a$ and in an interval to the right of $a$, then the statement $\lim_{x\to a} f(x) = A$ is equivalent to the conjunction of the two statements $\lim_{x\to a^-} f(x) = A$ and $\lim_{x\to a^+} f(x) = A$. We shall see by examples below that the existence of the limit from the left does not imply the existence of the limit from the right, and conversely.

When a function is defined only on one side of a point $a$, then we shall identify $\lim_{x\to a} f(x)$ with the one-sided limit, if it exists. For example, if $f(x) = \sqrt{x}$, then $f$ is defined only at and to the right of 0. Hence, since $\lim_{x\to 0^+} \sqrt{x} = 0$, we will also write $\lim_{x\to 0} \sqrt{x} = 0$. Of course, $\lim_{x\to 0^-} \sqrt{x}$ does not exist, since $\sqrt{x}$ is not defined when $x < 0$. This is an example where the existence of the limit from one side does not entail the existence of the limit from the other side. As another interesting example, consider the function $g(x) = \sqrt{1/x}$, which is defined only for $x > 0$. In this case, $\lim_{x\to 0^+} \sqrt{1/x}$ does not exist, since $1/x$ gets larger and larger without bound as $x$ approaches 0 from the right. Therefore, $\lim_{x\to 0} \sqrt{1/x}$ does not exist.

**EXAMPLE 7.3:** The function $f(x) = \sqrt{9-x^2}$ has the interval $-3 \le x \le 3$ as its domain. If $a$ is any number on the interval $(-3, 3)$, then $\lim_{x\to a} \sqrt{9-x^2}$ exists and is equal to $\sqrt{9-a^2}$. Now consider $a = 3$. Let $x$ approach 3 from the left; then $\lim_{x\to 3^-} \sqrt{9-x^2} = 0$. For $x > 3$, $\sqrt{9-x^2}$ is not defined, since $9 - x^2$ is negative. Hence, $\lim_{x\to 3} \sqrt{9-x^2} = \lim_{x\to 3^-} \sqrt{9-x^2} = 0$. Similarly, $\lim_{x\to -3} \sqrt{9-x^2} = \lim_{x\to -3^+} \sqrt{9-x^2} = 0$.

## Theorems on Limits

The following theorems are intuitively clear. Proofs of some of them are given in Problem 11.

**Theorem 7.1:** If $f(x) = c$, a constant, then $\lim_{x\to a} f(x) = c$.

For the next five theorems, assume $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$.

**Theorem 7.2:** $\lim_{x\to a} c \cdot f(x) = c \lim_{x\to a} f(x) = cA$.

**Theorem 7.3:** $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$.

**Theorem 7.4:** $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$.

**Theorem 7.5:** $\lim_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{A}{B}$, if $B \ne 0$.

**Theorem 7.6:** $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A}$, if $\sqrt[n]{A}$ is defined.

## Infinity

Let

$$\lim_{x\to a} f(x) = +\infty$$

mean that, as $x$ approaches $a$, $f(x)$ eventually becomes greater than any preassigned positive number, however large. In such a case, we say that $f(x)$ approaches $+\infty$ as $x$ approaches $a$. More precisely, $\lim_{x\to a} f(x) = +\infty$ if and only if, for any positive number $M$, there exists a positive number $\delta$ such that, whenever $0 < |x - a| < \delta$, then $f(x) > M$.

Similarly, let

$$\lim_{x \to a} f(x) = -\infty$$

mean that, as $x$ approaches $a$, $f(x)$ eventually becomes less than any preassigned number. In that case, we say that $f(x)$ approaches $-\infty$ as $x$ approaches $a$.

Let

$$\lim_{x \to a} f(x) = \infty$$

mean that, as $x$ approaches $a$, $|f(x)|$ eventually becomes greater than any preassigned positive number. Hence, $\lim_{x \to a} f(x) = \infty$ if and only if $\lim_{x \to a} |f(x)| = +\infty$.

These definitions can be extended to one-sided limits in the obvious way.

**EXAMPLE 7.4:**

(a) $\lim_{x \to 0} \frac{1}{x^2} = +\infty$ (b) $\lim_{x \to 1} \frac{-1}{(x-1)^2} = -\infty$ (c) $\lim_{x \to 0} \frac{1}{x} = \infty$

**EXAMPLE 7.5:**

(a) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$. As $x$ approaches 0 from the right (that is, through positive numbers), $1/x$ is positive and eventually becomes larger than any preassigned number.

(b) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ since, as $x$ approaches 0 from the left (that is, through negative numbers), $1/x$ is negative and eventually becomes smaller than any preassigned number.

The limit concepts already introduced can be extended in an obvious way to the case in which the variable approaches $+\infty$ or $-\infty$. For example,

$$\lim_{x \to +\infty} f(x) = A$$

means that $f(x)$ approaches $A$ as $x \to +\infty$, or, in more precise terms, given any positive $\epsilon$, there exists a number $N$ such that, whenever $x > N$, then $|f(x) - A| < \epsilon$. Similar definitions can be given for the statements $\lim_{x \to -\infty} f(x) = A$, $\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to a} f(x) = -\infty$, and $\lim_{x \to -\infty} f(x) = +\infty$.

**EXAMPLE 7.6:** $\lim_{x \to +\infty} \frac{1}{x} = 0$ and $\lim_{x \to +\infty} \left(2 + \frac{1}{x^2}\right) = 2$.

*Caution*: When $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, Theorems 7.3–7.5 do not make sense and cannot be used.

For example, $\lim_{x \to 0} \frac{1}{x^2} = +\infty$ and $\lim_{x \to 0} \frac{1}{x^4} = +\infty$, but

$$\lim_{x \to 0} \frac{1/x^2}{1/x^4} = \lim_{x \to 0} x^2 = 0$$

*Note*: We say that a limit, such as $\lim_{x \to a} f(x)$ or $\lim_{x \to +\infty} f(x)$ exists when the limit is a real number, but not when the limit is $+\infty$ or $-\infty$ or $\infty$. For example, since $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$, we say that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ exists. However, although $\lim_{x \to 0} \frac{1}{x^2} = +\infty$, we do not say that $\lim_{x \to 0} \frac{1}{x^2}$ exists.

## SOLVED PROBLEMS

**1.** Verify the following limit computations:

(a) $\lim_{x \to 2} 5x = 5 \lim_{x \to 2} x = 5 \cdot 2 = 10$

(b) $\lim_{x\to 2}(2x+3) = 2\lim_{x\to 2} x + \lim_{x\to 2} 3 = 2\cdot 2 + 3 = 7$

(c) $\lim_{x\to 2}(x^2-4x+1) = 4-8+1 = -3$

(d) $\lim_{x\to 3}\dfrac{x-2}{x+2} = \dfrac{\lim_{x\to 3}(x-2)}{\lim_{x\to 3}(x+2)} = \dfrac{1}{5}$

(e) $\lim_{x\to -2}\dfrac{x^2-4}{x^2+4} = \dfrac{4-4}{4+4} = 0$

(f) $\lim_{x\to 4}\sqrt{25-x^2} = \sqrt{\lim_{x\to 4}(25-x^2)} = \sqrt{9} = 3$

[*Note*: Do not assume from these problems that $\lim_{x\to a} f(x)$ is invariably $f(a)$.]

(g) $\lim_{x\to -5}\dfrac{x^2-25}{x+5} = \lim_{x\to -5}(x-5) = -10$

**2.** Verify the following limit computations:

(a) $\lim_{x\to 4}\dfrac{x-4}{x^2-x-12} = \lim_{x\to 4}\dfrac{x-4}{(x+3)(x-4)} = \lim_{x\to 4}\dfrac{1}{x+3} = \dfrac{1}{7}$

The division by $x-4$ before passing to the limit is valid since $x \neq 4$ as $x \to 4$; hence, $x-4$ is never zero.

(b) $\lim_{x\to 3}\dfrac{x^3-27}{x^2-9} = \lim_{x\to 3}\dfrac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \lim_{x\to 3}\dfrac{x^2+3x+9}{x+3} = \dfrac{9}{2}$

(c) $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\dfrac{x^2+2hx+h^2-x^2}{h} = \lim_{h\to 0}\dfrac{2hx+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x$

Here, and again in Problems 4 and 5, $h$ is a variable, so that it might be thought that we are dealing with functions of two variables. However, the fact that $x$ is a variable plays no role in these problems; for the moment, $x$ can be considered a constant.

(d) $\lim_{x\to 2}\dfrac{4-x^2}{3-\sqrt{x^2+5}} = \lim_{x\to 2}\dfrac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})} = \lim_{x\to 2}\dfrac{(4-x^2)(3+\sqrt{x^2+5})}{4-x^2} = \lim_{x\to 2}(3+\sqrt{x^2+5}) = 6$

(e) $\lim_{x\to 1}\dfrac{x^2+x-2}{(x-1)^2} = \lim_{x\to 1}\dfrac{(x-1)(x+2)}{(x-1)^2} = \lim_{x\to 1}\dfrac{x+2}{x-1} = \infty$ ; no limit exists.

**3.** In the following problems (a)–(c), you can interpret $\lim_{x\to\pm\infty}$ as either $\lim_{x\to+\infty}$ or $\lim_{x\to-\infty}$; it will not matter which. Verify the limit computations.

(a) $\lim_{x\to\pm\infty}\dfrac{3x-2}{9x+7} = \lim_{x\to\pm\infty}\dfrac{3-2/x}{9+7/x} = \dfrac{3-0}{9+0} = \dfrac{1}{3}$

(b) $\lim_{x\to\pm\infty}\dfrac{6x^2+2x+1}{5x^2-3x+4} = \lim_{x\to\pm\infty}\dfrac{6+2/x+1/x^2}{5-3/x+4/x^2} = \dfrac{6+0+0}{5-0+0} = \dfrac{6}{5}$

(c) $\lim_{x\to\pm\infty}\dfrac{x^2+x-2}{4x^3-1} = \lim_{x\to\pm\infty}\dfrac{1/x+1/x^2-2/x^3}{4-1/x^3} = \dfrac{0}{4} = 0$

(d) $\lim_{x\to-\infty}\dfrac{2x^3}{x^2+1} = \lim_{x\to-\infty}\dfrac{2x}{1+1/x^2} = -\infty$

(e) $\lim_{x\to+\infty}\dfrac{2x^3}{x^2+1} = \lim_{x\to+\infty}\dfrac{2x}{1+1/x^2} = +\infty$

(f) $\lim_{x\to+\infty}(x^5-7x^4-2x+5) = \lim_{x\to+\infty} x^5\left(1-\dfrac{7}{x}-\dfrac{2}{x^4}+\dfrac{5}{x^5}\right) = +\infty$ since

$\lim_{x\to+\infty}\left(1-\dfrac{7}{x}-\dfrac{2}{x^4}+\dfrac{5}{x^5}\right) = (1-0-0+0) = 1$ and $\lim_{x\to+\infty} x^5 = +\infty$

(g) $\lim_{x\to-\infty}(x^5-7x^4-2x+5) = \lim_{x\to-\infty} x^5\left(1-\dfrac{7}{x}-\dfrac{2}{x^4}+\dfrac{5}{x^5}\right) = -\infty$ since

$$\lim_{x\to-\infty}\left(1-\frac{7}{x}-\frac{2}{x^4}+\frac{5}{x^5}\right)=(1-0-0+0)=1 \text{ and } \lim_{x\to-\infty}x^5=-\infty$$

**4.** Given $f(x)=x^2-3x$, find $\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$.

Since $f(x)=x^2-3x$, we have $f(x+h)=(x+h)^2-3(x+h)$ and

$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{(x^2+2hx+h^2-3x-3h)-(x^2-3x)}{h}=\lim_{h\to0}\frac{2hx+h^2-3h}{h}$$
$$=\lim_{h\to0}(2x+h-3)=2x-3.$$

**5.** Given $f(x)=\sqrt{5x+1}$, find $\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$ when $x>-\frac{1}{5}$.

$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\sqrt{5x+5h+1}-\sqrt{5x+1}}{h}$$
$$=\lim_{h\to0}\frac{\sqrt{5x+5h+1}-\sqrt{5x+1}}{h}\,\frac{\sqrt{5x+5h+1}+\sqrt{5x+1}}{\sqrt{5x+5h+1}+\sqrt{5x+1}}$$
$$=\lim_{h\to0}\frac{(5x+5h+1)-(5x+1)}{h(\sqrt{5x+5h+1}+\sqrt{5x+1})}$$
$$=\lim_{h\to0}\frac{5}{\sqrt{5x+5h+1}+\sqrt{5x+1}}=\frac{5}{2\sqrt{5x+1}}$$

**6.** (a) In each of the following, (a) to (e), determine the points $x=a$ for which each denominator is zero. Then see what happens to $y$ as $x\to a^-$ and as $x\to a^+$, and verify the given solutions.
(b) (GC) Check the answers in (*a*) with a graphing calculator.

(a) $y=f(x)=2/x$: The denominator is zero when $x=0$. As $x\to0^-$, $y\to-\infty$; as $x\to0^+$, $y\to+\infty$.

(b) $y=f(x)=\frac{x-1}{(x+3)(x-2)}$: The denominator is zero for $x=-3$ and $x=2$. As $x\to-3^-$, $y\to-\infty$; as $x\to-3^+$, $y\to+\infty$. As $x\to2^-$, $y\to-\infty$; as $x\to2^+$, $y\to+\infty$.

(c) $y=f(x)=\frac{x-3}{(x+2)(x-1)}$: The denominator is zero for $x=-2$ and $x=1$. As $x\to-2^-$, $y\to-\infty$; as $x\to-2^+$, $y\to+\infty$. As $x\to1^-$, $y\to+\infty$; as $x\to1^+$, $y\to-\infty$.

(d) $y=f(x)=\frac{(x+2)(x-1)}{(x-3)^2}$: The denominator is zero for $x=3$. As $x\to3^-$, $y\to+\infty$; as $x\to3^+$, $y\to+\infty$.

(e) $y=f(x)=\frac{(x+2)(1-x)}{x-3}$: The denominator is zero for $x=3$. As $x\to3^-$, $y\to+\infty$; as $x\to3^+$, $y\to-\infty$.

**7.** For each of the functions of Problem 6, determine what happens to $y$ as $x\to-\infty$ and $x\to+\infty$.

(a) As $x\to\pm\infty$, $y=2/x\to0$. When $x<0$, $y<0$. Hence, as $x\to-\infty$, $y\to0^-$. Similarly, as $x\to+\infty$, $y\to0^+$.

(b) Divide numerator and denominator of $\frac{x-1}{(x+3)(x-2)}$ by $x^2$ (the highest power of $x$ in the denominator), obtaining

$$\frac{1/x-1/x^2}{(1+3/x)(1-2/x)}$$

Hence, as $x\to\pm\infty$,

$$y\to\frac{0-0}{(1+0)(1-0)}=\frac{0}{1}=0$$

As $x\to-\infty$, the factors $x-1$, $x+3$, and $x-2$ are negative, and, therefore, $y\to0^-$. As $x\to+\infty$, those factors are positive, and, therefore, $y\to0^+$.

(c) Similar to (*b*).

(d) $\dfrac{(x+2)(x-1)}{(x-3)^2} = \dfrac{x^2+x-2}{x^2-6x+9} = \dfrac{1+1/x-2/x^2}{1-6/x+9/x^2}$, after dividing numerator and denominator by $x^2$ (the highest power of $x$ in the denominator). Hence, as $x \to \pm\infty$, $y \to \dfrac{1+0-0}{1-0+0} = \dfrac{1}{1} = 1$. The denominator $(x-3)^2$ is always nonnegative. As $x \to -\infty$, both $x+2$ and $x-1$ are negative and their product is positive; hence, $y \to 1^+$. As $x \to +\infty$, both $x+2$ and $x-1$ are positive, as is their product; hence, $y \to 1^+$.

(e) $\dfrac{(x+2)(1-x)}{x-3} = \dfrac{-x^2-x+2}{x-3} = \dfrac{-x-1+2/x}{1-3/x}$, after dividing numerator and denominator by $x$ (the highest power of $x$ in the denominator). As $x \to \pm\infty$, $2/x$ and $3/x$ approach 0, and $-x-1$ approaches $\pm\infty$. Thus, the denominator approaches 1 and the numerator approaches $\pm\infty$. As $x \to -\infty$, $x+2$ and $x-3$ are negative and $1-x$ is positive; so, $y \to +\infty$. As $x \to +\infty$, $x+2$ and $x-3$ are positive and $1-x$ is negative; so, $y \to -\infty$.

**8.** Examine the function of Problem 4 in Chapter 6 as $x \to a^-$ and as $x \to a^+$ when $a$ is any positive integer.

Consider, as a typical case, $a = 2$. As $x \to 2^-$, $f(x) \to 10$. As $x \to 2^+$, $f(x) \to 15$. Thus, $\lim_{x\to 2} f(x)$ does not exist. In general, the limit fails to exist for all positive integers. (Note, however, that $\lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x) = 5$, since $f(x)$ is not defined for $x \le 0$.)

**9.** Use the precise definition to show that $\lim_{x\to 2}(x^2+3x) = 10$.

Let $\epsilon > 0$ be chosen. Note that $(x-2)^2 = x^2 - 4x + 4$, and so, $x^2 + 3x - 10 = (x-2)^2 + 7x - 14 = (x-2)^2 + 7(x-2)$. Hence $|(x^2+3x) - 10| = |(x-2)^2 + 7(x-2)| \le |x-2|^2 + 7|x-2|$. If we choose $\delta$ to be the minimum of 1 and $\epsilon/8$, then $\delta^2 \le \delta$, and, therefore, $0 < |x-2| < \delta$ implies $|(x^2+3x)-10| < \delta^2 + 7\delta \le \delta + 7\delta = 8\delta \le \epsilon$.

**10.** If $\lim_{x\to a} g(x) = B \ne 0$, prove that there exists a positive number $\delta$ such that $0 < |x-a| < \delta$ implies $|g(x)| > \dfrac{|B|}{2}$.

Letting $\epsilon = |B|/2$ we obtain a positive $\delta$ such that $0 < |x-a| < \delta$ implies $|g(x) - B| < |B|/2$. Now, if $0 < |x-a| < \delta$, then $|B| = |g(x) + (B - g(x))| \le |g(x)| + |B - g(x)| < |g(x)| + |B|/2$ and, therefore, $|B|/2 < |g(x)|$.

**11.** Assume (I) $\lim_{x\to a} f(x) = A$ and (II) $\lim_{x\to a} g(x) = B$. Prove:

(a) $\lim_{x\to a}[f(x)+g(x)] = A+B$ (b) $\lim_{x\to a} f(x)g(x) = AB$ (c) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{A}{B}$ if $B \ne 0$

(a) Let $\epsilon > 0$ be chosen. Then $\epsilon/2 > 0$. By (I) and (II), there exist positive $\delta_1$ and $\delta_2$ such that $0 < |x-a| < \delta_1$ implies $|f(x) - A| < \epsilon/2$ and $0 < |x-a| < \delta_2$ implies $|g(x) - B| < \epsilon/2$. Let $\delta$ be the minimum of $\delta_1$ and $\delta_2$. Thus, for $0 < |x-a| < \delta$, $|f(x)-A| < \epsilon/2$ and $|g(x) - B| < \epsilon/2$. Therefore, for $0 < |x-a| < \delta$,

$$|(f(x)+g(x)-(A+B)| = |(f(x)-A)+(g(x)-B)|$$
$$\le |f(x)-A| + |g(x)-B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(b) Let $\epsilon > 0$ be chosen. Choose $\epsilon^*$ to be the minimum of $\epsilon/3$ and 1 and $\epsilon/(3|B|)$ (if $B \ne 0$), and $\epsilon/(3|A|)$ (if $A \ne 0$). Note that $(\epsilon^*)^2 \le \epsilon^*$ since $\epsilon^* \le 1$ Moreover, $|B|\epsilon^* \le \epsilon/3$ and $|A|\epsilon^* \le \epsilon/3$. By (I) and (II), there exist positive $\delta_1$ and $\delta_2$ such that $0 < |x-a| < \delta_1$ implies $|f(x)-A| < \epsilon^*$ and $0 < |x-a| < \delta_2$ implies $|g(x)-B| < \epsilon^*$. Let $\delta$ be the minimum of $\delta_1$ and $\delta_2$. Now, for $0 < |x-a| < \delta$,

$$|f(x)g(x) - AB| = |(f(x)-A)(g(x)-B) + B(f(x)-A) + A(g(x)-B)|$$
$$\le |f(x)-A)(g(x)-B)| + |B(f(x)-A)| + |A(g(x)-B)|$$
$$= |f(x)-A||g(x)-B| + |B||f(x)-A| + |A||g(x)-B|$$
$$\le (\epsilon^*)^2 + |B|\epsilon^* + |A|\epsilon^* \le \epsilon^* + \frac{\epsilon}{3} + \frac{\epsilon}{3} \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

(c) By part (b), it suffices to show that $\lim_{x\to a}\dfrac{1}{g(x)} = \dfrac{1}{B}$. Let $\epsilon > 0$ be chosen. Then $B^2\epsilon/2 > 0$. Hence, there exists a positive $\delta_1$ such that $0 < |x-a| < \delta_1$ implies $|g(x) - B| < \dfrac{|B|^2\epsilon}{2}$.

By Problem 10, there exists a positive $\delta_2$ such that $0 < |x - a| < \delta_2$ implies $|g(x)| > |B|/2$. Let $\delta$ be the minimum of $\delta_1$ and $\delta_2$. Then $0 < |x - a| < \delta$ implies that

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|B - g(x)|}{|B\|g(x)|} < \frac{|B|^2 \in}{2} \cdot \frac{2}{|B|^2} = \in$$

**12.** Prove that, for any polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad \lim_{x\to a} f(x) = f(a)$$

This follows from Theorems 7.1–7.4 and the obvious fact that $\lim_{x\to a} x = a$.

**13.** Prove the following generalizations of the results of Problem 3. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ be two polynomials.

(a) $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{a_n}{b_k}$ if $n = k$

(b) $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$ if $n < k$

(c) $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \pm\infty$ if $n > k$. (It is $+\infty$ if and only if $a_n$ and $b_k$ have the same sign.)

(d) $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \pm\infty$ if $n > k$. (The correct sign is the sign of $a_n b_k (-1)^{n-k}$.)

**14.** Prove (a) $\lim_{x\to 2^-} \frac{1}{(x-2)^3} = -\infty$; (b) $\lim_{x\to+\infty} \frac{x}{x+1} = 1$; (c) $\lim_{x\to+\infty} \frac{x^2}{x-1} = +\infty$.

(a) Let $M$ be any negative number. Choose $\delta$ positive and equal to the minimum of 1 and $\frac{1}{|M|}$. Assume $x < 2$ and $0 < |x - 2| < \delta$. Then $|x-2|^3 < \delta^3 \le \delta \le \frac{1}{|M|}$. Hence, $\frac{1}{|x-2|^3} > |M| = -M$. But $(x - 2)^3 < 0$. Therefore, $\frac{1}{(x-2)^3} = -\frac{1}{|x-2|^3} < M$.

(b) Let $\in$ be any positive number, and let $M = 1/\in$. Assume $x > M$. Then

$$\left|\frac{x}{x+1} - 1\right| = \left|\frac{1}{x+1}\right| = \frac{1}{x+1} < \frac{1}{x} < \frac{1}{M} = \in$$

(c) Let $M$ be any positive number. Assume $x > M + 1$. Then $\frac{x^2}{x-1} \ge \frac{x^2}{x} = x > M$.

**15.** Evaluate: (a) $\lim_{x\to 0^+} \frac{|x|}{x}$; (b) $\lim_{x\to 0^-} \frac{|x|}{x}$; (c) $\lim_{x\to 0} \frac{|x|}{x}$

(a) When $x > 0$, $|x| = x$. Hence, $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} 1 = 1$.

(b) When $x < 0$, $|x| = -x$. Hence, $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} -1 = -1$.

(c) $\lim_{x\to 0} \frac{|x|}{x}$ does not exist, since $\lim_{x\to 0^-} \frac{|x|}{x} \ne \lim_{x\to 0^+} \frac{|x|}{x}$.

## SUPPLEMENTARY PROBLEMS

**16.** Evaluate the following limits:

(a) $\lim_{x\to 2} (x^2 - 4x)$

(b) $\lim_{x\to -1} (x^3 + 2x^2 - 3x - 4)$

(c) $\lim_{x\to 1} \frac{(3x-1)^2}{(x+1)^3}$

(d) $\lim_{x\to 0}\frac{3^x-3^{-x}}{3^x+3^{-x}}$

(e) $\lim_{x\to 2}\frac{x-1}{x^2-1}$

(f) $\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}$

(g) $\lim_{x\to -1}\frac{x^2+3x+2}{x^2+4x+3}$

(h) $\lim_{x\to 2}\frac{x-2}{x^2-4}$

(i) $\lim_{x\to 2}\frac{x-2}{\sqrt{x^2-4}}$

(j) $\lim_{x\to 2}\frac{\sqrt{x-2}}{x^2-4}$

(k) $\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$

(l) $\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}$

*Ans.* (a) –4; (b) 0; (c) $\frac{1}{2}$; (d) 0; (e) $\frac{1}{3}$; (f) –4; (g) $\frac{1}{2}$; (h) $\frac{1}{4}$; (i) 0; (j) ∞, no limit; (k) $3x^2$; (l) 2

**17.** Evalute the following limits:

(a) $\lim_{x\to +\infty}\frac{7x^9-4x^5+2x-13}{-3x^9+x^8-5x^2+2x}$

(b) $\lim_{x\to +\infty}\frac{14x^3-5x+27}{x^4+10}$

(c) $\lim_{x\to -\infty}\frac{2x^5+12x+5}{7x^3+6}$

(d) $\lim_{x\to +\infty}\frac{-2x^3+7}{5x^2-3x-4}$

(e) $\lim_{x\to +\infty}(3x^3-25x^2-12x-17)$

(f) $\lim_{x\to -\infty}(3x^3-25x^2-12x-17)$

(g) $\lim_{x\to -\infty}(3x^4-25x^3-8)$

*Ans.* (a) $-\frac{7}{3}$; (b) 0; (c) $+\infty$; (d) $-\infty$; (e) $+\infty$; (f) $-\infty$; (g) $+\infty$

**18.** Evaluate the following limits:

(a) $\lim_{x\to +\infty}\frac{2x+3}{4x-5}$

(b) $\lim_{x\to +\infty}\frac{2x^2+1}{6+x-3x^2}$

(c) $\lim_{x\to +\infty}\frac{x}{x^2+5}$

(d) $\lim_{x\to +\infty}\frac{x^2+5x+6}{x+1}$

(e) $\lim_{x\to +\infty}\frac{x+3}{x^2+5x+6}$

(f) $\lim_{x\to +\infty}\frac{3^x-3^{-x}}{3^x+3^{-x}}$

(g) $\lim_{x\to-\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$

*Ans.* (a) $\frac{1}{2}$; (b) $-\frac{2}{3}$; (c) 0; (d) $+\infty$; (e) 0; (f) 1; (g) $-1$

**19.** Find $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ for the functions $f$ in Problems 11, 12, 13, 15, and 16 ($a$, $b$, $d$, $g$) of Chapter 6.

*Ans.* (11) $2a - 4$; (12) $\frac{2}{(a+1)^2}$; (13) $2a - 1$; (15) $-\frac{27}{(4a-5)^2}$; (16) (a) $2a$, (b) $\frac{a}{\sqrt{a^2+4}}$, (d) $\frac{3}{(a+3)^2}$, (g) $\frac{4a}{(a^2+1)^2}$

**20.** (GC) Investigate the behavior of

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ x+1 & \text{if } x \le 0 \end{cases}$$

as $x \to 0$. Draw a graph and verify it with a graphing calculator.

*Ans.* $\lim_{x\to 0^+} f(x) = 0$; $\lim_{x\to 0^-} f(x) = 1$; $\lim_{x\to 0} f(x)$ does not exist.

**21.** Use Theorem 7.4 and mathematical induction to prove $\lim_{x\to a} x^n = a^n$ for all positive integers $n$.

**22.** For $f(x) = 5x - 6$, find $\delta > 0$ such that, whenever $0 < |x - 4| < \delta$, then $|f(x) - 14| < \epsilon$, when (a) $\epsilon = \frac{1}{2}$ and (b) $\epsilon = 0.001$.

*Ans.* (a) $\frac{1}{10}$; (b) 0.0002

**23.** Use the precise definition to prove: (a) $\lim_{x\to 3} 5x = 15$; (b) $\lim_{x\to 2} x^2 = 4$; (c) $\lim_{x\to 2}(x^2 - 3x + 5) = 3$.

**24.** Use the precise definition to prove:

(a) $\lim_{x\to 0} \frac{1}{x} = \infty$ (b) $\lim_{x\to 1} \frac{x}{x-1} = \infty$ (c) $\lim_{x\to+\infty} \frac{x}{x-1} = 1$ (d) $\lim_{x\to-\infty} \frac{x^2}{x+1} = -\infty$

**25.** Let $f(x)$, $g(x)$, and $h(x)$ be such that (1) $f(x) \le g(x) \le h(x)$ for all values in certain intervals to the left and right of $a$, and (2) $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = A$. Prove $\lim_{x\to a} g(x) = A$.

(*Hint*: For $\epsilon > 0$, there exists $\delta > 0$ such that, whenever $0 < |x - a| < \delta$, then $|f(x) - A| < \epsilon$ and $|h(x) - A| < \epsilon$ and, therefore, $A - \epsilon < f(x) \le g(x) \le h(x) < A + \epsilon$.)

**26.** Prove: If $f(x) \le M$ for all $x$ in an open interval containing $a$ and if $\lim_{x\to a} f(x) = A$, then $A \le M$.

(*Hint*: Assume $A > M$. Choose $\epsilon = \frac{1}{2}(A - M)$ and derive a contradiction.)

**27.** (GC) Use a graphing calculator to confirm the limits found in Problems 1($d$, $e$, $f$), 2($a$, $b$, $d$), 16, and 18.

**28.** (a) Show that $\lim_{x\to+\infty}(x - \sqrt{x^2 - 1}) = 0$.

(*Hint*: Multiply and divide by $x + \sqrt{x^2 - 1}$.)

(b) Show that the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ gets arbitrarily close to the asymptote $y = \frac{b}{a}x$ as $x$ approaches $\infty$.

**29.** (a) Find $\lim_{x\to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$.

(*Hint*: Multiply the numerator and denominator by $\sqrt{x+3} + \sqrt{3}$.)

(b) (GC) Use a graphing calculator to confirm the result of part (a).

**30.** Let $f(x) = \sqrt{x} - 1$ if $x > 4$ and $f(x) = x^2 - 4x + 1$ if $x < 4$. Find:

(a) $\lim_{x\to 4^+} f(x)$ (*b*) $\lim_{x\to 4^-} f(x)$ (c) $\lim_{x\to 4} f(x)$

*Ans.* (a) 1; (b) 1; (c) 1

**31.** Let $g(x) = 10x - 7$ if $x > 1$ and $g(x) = 3x + 2$ if $x < 1$. Find:

(a) $\lim_{x\to 1^+} g(x)$ (b) $\lim_{x\to 1^-} g(x)$ (c) $\lim_{x\to 1} g(x)$

*Ans.* (a) 3; (b) 5; (c) It does not exist.