CHAPTER 52

Directional Derivatives. Maximum and Minimum Values

Directional Derivatives

Let $P(x, y, z)$ be a point on a surface $z = f(x, y)$. Through P, pass planes parallel to the xz and yz planes, cutting the surface in the arcs PR and PS, and cutting the xy plane in the lines P^*M and P^*N, as shown in Fig. 52-1. Note that P^* is the foot of the perpendicular from P to the xy plane. The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, evaluated at $P^*(x, y)$, give, respectively, the rates of change of $z = P^*P$ when y is held fixed and when x is held fixed. In other words, they give the rates of change of z in directions parallel to the x and y axes. These rates of change are the slopes of the tangent lines of the curves PR and PS at P.

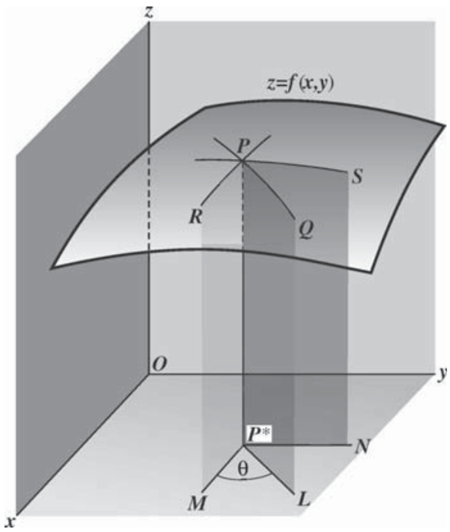

Fig. 52-1

Consider next a plane through P perpendicular to the xy plane and making an angle θ with the x axis. Let it cut the surface in the curve PQ and the xy plane in the line P^*L. The *directional derivative* of $f(x, y)$ at P^* in the direction θ is given by

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \tag{52.1}$$

The direction θ is the direction of the vector $(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$.

The directional derivative gives the rate of change of $z = P^*P$ in the direction of P^*L; it is equal to the slope of the tangent line of the curve PQ at P. (See Problem 1.)

The directional derivative at a point P^* is a function of θ. We shall see that there is a direction, determined by a vector called the *gradient* of f at P^* (see Chapter 53), for which the directional derivative at P^* has a maximum value. That maximum value is the slope of the steepest tangent line that can be drawn to the surface at P.

For a function $w = F(x, y, z)$, the directional derivative at $P(x, y, z)$ in the direction determined by the angles α, β, γ is given by

$$\frac{dF}{ds} = \frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\cos\beta + \frac{\partial F}{\partial z}\cos\gamma$$

By the direction determined by α, β, and γ, we mean the direction of the vector $(\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}$.

Relative Maximum and Minimum Values

Assume that $z = f(x, y)$ has a relative maximum (or minimum) value at $P_0(x_0, y_0, z_0)$. Any plane through P_0 perpendicular to the xy plane will cut the surface in a curve having a relative maximum (or minimum) point at P_0. Thus, the directional derivative $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$ of $z = f(x, y)$ must equal zero at P_0. In particular, when $\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$, so that $\frac{\partial f}{\partial x} = 0$. When $\theta = \frac{\pi}{2}$, $\sin\theta = 1$ and $\cos\theta = 0$, so that $\frac{\partial f}{\partial y} = 0$. Hence, we obtain the following theorem.

Theorem 52.1: If $z = f(x, y)$ has a relative extremum at $P_0(x_0, y_0, z_0)$ and $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (x_0, y_0), then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ at (x_0, y_0).

We shall cite without proof the following sufficient conditions for the existence of a relative maximum or minimum.

Theorem 52.2: Let $z = f(x, y)$ have first and second partial derivatives in an open set including a point (x_0, y_0) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Define $\Delta = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)$. Assume $\Delta < 0$ at (x_0, y_0). Then:

$$z = f(x, y) \text{ has } \begin{cases} \text{a relative } \textbf{min}\text{imum at } (x_0, y_0) & \text{if } \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} > 0 \\ \text{a relative } \textbf{max}\text{imum at } (x_0, y_0) & \text{if } \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} < 0 \end{cases}$$

If $\Delta > 0$, there is neither a relative maximum nor a relative minimum at (x_0, y_0).
If $\Delta = 0$, we have no information.

Absolute Maximum and Minimum Values

Let A be a set of points in the xy plane. We say that A is *bounded* if A is included in some disk. By the *complement* of A in the xy plane, we mean the set of all points in the xy plane that are not in A. A is said to be *closed* if the complement of A is an open set.

Example 1: The following are instances of closed and bounded sets.

(a) Any closed disk D, that is, the set of all points whose distance from a fixed point is less than or equal to some fixed positive number r. (Note that the complement of D is open because any point not in D can be surrounded by an open disk having no points in D.)

(b) The inside and boundary of any rectangle. More generally, the inside and boundary of any "simple closed curve," that is, a curve that does not interset itself except at its initial and terminal point.

Theorem 52.3: Let $f(x, y)$ be a function that is continuous on a closed, bounded set A. Then f has an absolute maximum and an absolute minimum value in A.

The reader is referred to more advanced texts for a proof of Theorem 52.3. For three or more variables, an analogous result can be derived.

SOLVED PROBLEMS

1. Derive formula (52.1).

In Fig. 52-1, let $P^{**}(x + \Delta x, y + \Delta y)$ be a second point on P^*L and denote by Δs the distance P^*P^{**}. Assuming that $z = f(x, y)$ possesses continuous first partial derivatives, we have, by Theorem 49.1,

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. The average rate of change between points P^* and P^{**} is

$$\frac{\Delta z}{\Delta s} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta s} + \epsilon_1 \frac{\Delta x}{\Delta s} + \epsilon_2 \frac{\Delta y}{\Delta s}$$

$$= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta + \epsilon_1 \cos\theta + \epsilon_2 \sin\theta$$

where θ is the angle that the line P^*P^{**} makes with the x axis. Now let $P^{**} \to P^*$ along P^*L. The directional derivative at P^*, that is, the instantaneous rate of change of z, is then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

2. Find the directional derivative of $z = x^2 - 6y^2$ at $P^*(7, 2)$ in the direction: (a) $\theta = 45°$; (b) $\theta = 135°$.

The directional derivative at any point $P^*(x, y)$ in the direction θ is

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta = 2x\cos\theta - 12y\sin\theta$$

(a) At $P^*(7, 2)$ in the direction $\theta = 45°$,

$$\frac{dz}{ds} = 2(7)(\tfrac{1}{2}\sqrt{2}) - 12(2)(\tfrac{1}{2}\sqrt{2}) = -5\sqrt{2}$$

(b) At $P^*(7, 2)$ in the direction $\theta = 135°$,

$$\frac{dz}{ds} = 2(7)(-\tfrac{1}{2}\sqrt{2}) - 12(2)(\tfrac{1}{2}\sqrt{2}) = -19\sqrt{2}$$

3. Find the directional derivative of $z = ye^x$ at $P^*(0, 3)$ in the direction (a) $\theta = 30°$; (b) $\theta = 120°$.

Here, $dz/ds = ye^x \cos\theta + e^x \sin\theta$.

(a) At (0, 3) in the direction $\theta = 30°$, $dz/ds = 3(1)(\tfrac{1}{2}\sqrt{3}) + \tfrac{1}{2} = \tfrac{1}{2}(3\sqrt{3} + 1)$.

(b) At (0, 3) in the direction $\theta = 120°$, $dz/ds = 3(1)(-\tfrac{1}{2}) + \tfrac{1}{2}\sqrt{3} = \tfrac{1}{2}(-3 + \sqrt{3})$.

4. The temperature T of a heated circular plate at any of its points (x, y) is given by $T = \dfrac{64}{x^2 + y^2 + 2}$, the origin being at the center of the plate. At the point (1, 2), find the rate of change of T in the direction $\theta = \pi/3$.

We have

$$\frac{dT}{ds} = -\frac{64(2x)}{(x^2+y^2+2)^2}\cos\theta - \frac{64(2y)}{(x^2+y^2+2)^2}\sin\theta$$

At (1, 2) in the direction $\theta = \frac{\pi}{3}$, $\frac{dT}{ds} = -\frac{128}{49}\frac{1}{2} - \frac{256}{49}\frac{\sqrt{3}}{2} = -\frac{64}{49}(1+2\sqrt{3})$.

5. The electrical potential V at any point (x, y) is given by $V = \ln\sqrt{x^2+y^2}$. Find the rate of change of V at the point (3, 4) in the direction toward the point (2, 6).

Here,

$$\frac{dV}{ds} = \frac{x}{x^2+y^2}\cos\theta + \frac{y}{x^2+y^2}\sin\theta$$

Since θ is a second-quadrant angle and $\tan\theta = (6-4)/(2-3) = -2$, $\cos\theta = -1/\sqrt{5}$ and $\sin\theta = 2/\sqrt{5}$.

Hence, at (3, 4) in the indicated direction, $\frac{dV}{ds} = \frac{3}{25}\left(-\frac{1}{\sqrt{5}}\right) + \frac{4}{25}\frac{2}{\sqrt{5}} = \frac{\sqrt{5}}{25}$.

6. Find the maximum directional derivative for the surface and point of Problem 2.

At $P^*(7, 2)$ in the direction θ, $dz/ds = 14\cos\theta - 24\sin\theta$.

To find the value of θ for which $\frac{dz}{ds}$ is a maximum, set $\frac{d}{d\theta}\left(\frac{dz}{ds}\right) = -14\sin\theta - 24\cos\theta = 0$. Then $\tan\theta = -\frac{24}{14} = -\frac{12}{7}$ and θ is either a second- or fourth-quadrant angle. For the second-quadrant angle, $\sin\theta = 12/\sqrt{193}$ and $\cos = -7/\sqrt{193}$. For the fourth-quadrant angle, $\sin\theta = -12/\sqrt{193}$ and $\cos\theta = 7/\sqrt{193}$.

Since $\frac{d^2}{d\theta^2}\left(\frac{dz}{ds}\right) = \frac{d}{d\theta}(-14\sin\theta - 24\cos\theta) = -14\cos\theta + 24\sin\theta$ is negative for the fourth-quadrant angle, the maximum directional derivative is $\frac{dz}{dz} = 14\left(\frac{7}{\sqrt{193}}\right) - 24\left(-\frac{12}{\sqrt{193}}\right) = 2\sqrt{193}$, and the direction is $\theta = 300°15'$.

7. Find the maximum directional derivative for the function and point of Problem 3.

At $P^*(0, 3)$ in the direction θ, $dz/ds = 3\cos\theta + \sin\theta$.

To find the value of θ for which $\frac{dz}{ds}$ is a maximum, set $\frac{d}{d\theta}\left(\frac{dz}{ds}\right) = -3\sin\theta + \cos\theta = 0$. Then $\tan\theta = \frac{1}{3}$ and θ is either a first- or third-quadrant angle.

Since $\frac{d^2}{d\theta^2}\left(\frac{dz}{ds}\right) = \frac{d}{d\theta}(-3\sin\theta + \cos\theta) = -3\cos\theta - \sin\theta$ is negative for the first-quadrant angle, the maximum directional derivative is $\frac{dz}{ds} = 3\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} = \sqrt{10}$, and the direction is $\theta = 18°26'$.

8. In Problem 5, show that V changes most rapidly along the set of radial lines through the origin.

At any point (x_1, y_1) in the direction θ, $\frac{dV}{ds} = \frac{x_1}{x_1^2+y_1^2}\cos\theta + \frac{y_1}{x_1^2+y_1^2}\sin\theta$. Now V changes most rapidly when $\frac{d}{d\theta}\left(\frac{dV}{ds}\right) = -\frac{x_1}{x_1^2+y_1^2}\sin\theta + \frac{y_1}{x_1^2+y_1^2}\cos\theta = 0$, and then $\tan\theta = \frac{y_1/(x_1^2+y_1^2)}{x_1/(x_1^2+y_1^2)} = \frac{y_1}{x_1}$. Thus, θ is the angle of inclination of the line joining the origin and the point (x_1, y_1).

9. Find the directional derivative of $F(x, y, z) = xy + 2xz - y^2 + z^2$ at the point (1, –2, 1) along the curve $x = t$, $y = t - 3$, $z = t^2$ in the direction of increasing z.

A set of direction numbers of the tangent to the curve at (1, –2, 1) is [1, 1, 2]; the direction cosines are $[1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}]$. The directional derivative is

$$\frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\cos\beta + \frac{\partial F}{\partial z}\cos\gamma = 0\frac{1}{\sqrt{6}} + 5\frac{1}{\sqrt{6}} + 4\frac{2}{\sqrt{6}} = \frac{13\sqrt{6}}{6}$$

10. Examine $f(x, y) = x^2 + y^2 - 4x + 6y + 25$ for maximum and minimum values.

The conditions $\frac{\partial f}{\partial x} = 2x - 4 = 0$ and $\frac{\partial f}{\partial y} = 2y + 6 = 0$ are satisfied when $x = 2$, $y = -3$. Since

$$f(x, y) = (x^2 - 4x + 4) + (y^2 + 6y + 9) + 25 - 4 - 9 = (x - 2)^2 + (y + 3)^2 + 12$$

it is evident that $f(2, -3) = 12$ is the absolute minimum value of the function. Geometrically, $(2, -3, 12)$ is the lowest point on the surface $z = x^2 + y^2 - 4x + 6y + 25$. Clearly, $f(x, y)$ has no absolute maximum value.

11. Examine $f(x,y) = x^3 + y^3 + 3xy$ for maximum and minimum values.

We shall use Theorem 52.2. The conditions $\frac{\partial f}{\partial x} = 3(x^2 + y) = 0$ and $\frac{\partial f}{\partial y} = 3(y^2 + x) = 0$ are satisfied when $x = 0$, $y = 0$ and when $x = -1$, $y = -1$.

At (0, 0), $\frac{\partial^2 f}{\partial x^2} = 6x = 0$, $\frac{\partial^2 f}{\partial x\,\partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = 6y = 0$. Then

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = 9 > 0$$

and (0, 0) yields neither a relative maximum nor minimum.

At (−1, −1), $\frac{\partial^2 f}{\partial x^2} = -6$, $\frac{\partial^2 f}{\partial x\,\partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = -6$. Then

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = -27 < 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0$$

Hence, $f(-1, -1) = 1$ is a relative maximum value of the function.

Clearly, there are no absolute maximum or minimum values. (When $y = 0$, $f(x, y) = x^3$ can be made arbitrarily large or small.)

12. Divide 120 into three nonnegative parts such that the sum of their products taken two at a time is a maximum.

Let x, y, and $120 - (x + y)$ be the three parts. The function to be maximized is $S = xy + (x + y)(120 - x - y)$. Since $0 \le x + y \le 120$, the domain of the function consists of the solid triangle shown in Fig. 52-2. Theorem 52.3 guarantees an absolute maximum.

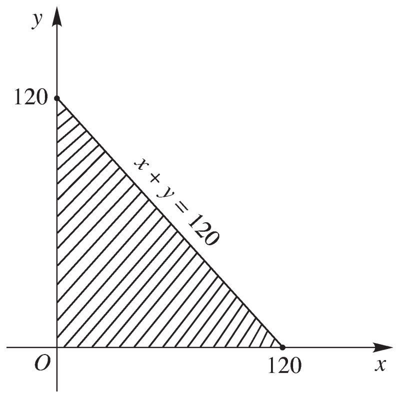

Fig. 52-2

Now,

$$\frac{\partial S}{\partial x} = y + (120 - x - y) - (x + y) = 120 - 2x - y$$

and

$$\frac{\partial S}{\partial y} = x + (120 - x - y) - (x + y) = 120 - x - 2y$$

Setting $\partial S/\partial x = \partial S/\partial y = 0$ yields $2x + y = 120$ and $x + 2y = 120$.

Simultaneous solution gives $x = 40$, $y = 40$, and $120 - (x + 4) = 40$ as the three parts, and $S = 3(40^2) = 4800$. So, if the absolute maximum occurs in the interior of the triangle, Theorem 52.1 tells us we have found it. It is still necessary to check the boundary of the triangle. When $y = 0$, $S = x(120 - x)$. Then $dS/dx = 120 - 2x$, and the critical number is $x = 60$. The corresponding maximum value of S is $60(60) = 3600$, which is < 4800. A similar result holds when $x = 0$. Finally, on the hypotenuse, where $y = 120 - x$, $S = x(120 - x)$ and we again obtain a maximum of 3600. Thus, the absolute maximum is 4800, and $x = y = z = 40$.

13. Find the point in the plane $2x - y + 2z = 16$ nearest the origin.

Let (x, y, z) be the required point; then the square of its distance from the origin is $D = x^2 + y^2 + z^2$. Since also $2x - y + 2z = 16$, we have $y = 2x + 2z - 16$ and $D = x^2 + (2x + 2z - 16)^2 + z^2$.

Then the conditions $\partial D/\partial x = 2x + 4(2x + 2z - 16) = 0$ and $\partial D/\partial z = 4(2x + 2z - 16) + 2z = 0$ are equivalent to $5x + 4z = 32$ and $4x + 5z = 32$, and $x = z = \frac{32}{9}$. Since it is known that a point for which D is a minimum exists, $(\frac{32}{9}, -\frac{16}{9}, \frac{32}{9})$ is that point.

14. Show that a rectangular parallelepiped of maximum volume V with constant surface area S is a cube.

Let the dimensions be x, y, and z. Then $V = xyz$ and $S = 2(xy + yz + zx)$.

The second relation may be solved for z and substituted in the first, to express V as a function of x and y. We prefer to avoid this step by simply treating z as a function of x and y. Then

$$\frac{\partial V}{\partial x} = yz + xy\frac{\partial z}{\partial x}, \qquad \frac{\partial V}{\partial y} = xz + xy\frac{\partial z}{\partial y}$$

$$\frac{\partial S}{\partial x} = 0 = 2\left(y + z + x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial x}\right), \qquad \frac{\partial S}{\partial y} = 0 = 2\left(x + z + x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y}\right)$$

From the latter two equations, $\frac{\partial z}{\partial x} = -\frac{y+z}{x+y}$ and $\frac{\partial z}{\partial y} = -\frac{x+z}{x+y}$. Substituting in the first two yields the conditions $\frac{\partial V}{\partial x} = yz - \frac{xy(y+z)}{x+y} = 0$ and $\frac{\partial V}{\partial y} = xz - \frac{xy(x+z)}{x+y} = 0$, which reduce to $y^2(z - x) = 0$ and $x^2(z - y) = 0$. Thus $x = y = z$, as required.

15. Find the volume V of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let $P(x, y, z)$ be the vertex in the first octant. Then $V = 8xyz$. Consider z to be defined as a function of the independent variables x and y by the equation of the ellipsoid. The necessary conditions for a maximum are

$$\frac{\partial V}{\partial x} = 8\left(yz + xy\frac{\partial z}{\partial x}\right) = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 8\left(xz + xy\frac{\partial z}{\partial y}\right) = 0 \tag{1}$$

From the equation of the ellipsoid, obtain $\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$ and $\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$. Eliminate $\partial z/\partial x$ and $\partial z/\partial y$ between these relations and (1) to obtain

$$\frac{\partial V}{\partial x} = 8\left(yz - \frac{c^2x^2y}{a^2z}\right) = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 8\left(xz - \frac{c^2xy^2}{b^2z}\right) = 0$$

and, finally,

$$\frac{x^2}{a^2} = \frac{z^2}{c^2} = \frac{y^2}{b^2} \tag{2}$$

Combine (2) with the equation of the ellipsoid to get $x = a\sqrt{3}/3$, $y = b\sqrt{3}/3$, and $z = c\sqrt{3}/3$. Then $V = 8xyz = (8\sqrt{3}/9)abc$ cubic units.

SUPPLEMENTARY PROBLEMS

16. Find the directional derivatives of the given function at the given point in the indicated direction.

(a) $z = x^2 + xy + y^2$, (3, 1), $\theta = \frac{\pi}{3}$.
(b) $z = x^3 - 3xy + y^3$, (2, 1), $\theta = \tan^{-1}(\frac{2}{3})$.
(c) $z = y + x\cos xy$, (0, 0), $\theta = \frac{\pi}{3}$.
(d) $z = 2x^2 + 3xy - y^2$, (1, −1), toward (2, 1).

Ans. (a) $\frac{1}{2}(7+5\sqrt{3})$; (b) $21\sqrt{13}/13$; (c) $\frac{1}{2}(1+\sqrt{3})$; (d) $11\sqrt{5}/5$

17. Find the maximum directional derivative for each of the functions of Problem 16 at the given point.

Ans. (a) $\sqrt{74}$; (b) $3\sqrt{10}$; (c) $\sqrt{2}$; (d) $\sqrt{26}$

18. Show that the maximal directional derivative of $V = \ln\sqrt{x^2 + y^2}$ of Problem 8 is constant along any circle $x^2 + y^2 = r^2$.

19. On a hill represented by $z = 8 - 4x^2 - 2y^2$, find (a) the direction of the steepest grade at (1, 1, 2) and (b) the direction of the contour line (the direction for which z = constant). Note that the directions are mutually perpendicular.

Ans. (a) $\tan^{-1}(\frac{1}{2})$, third quadrant; (b) $\tan^{-1}(-2)$

20. Show that the sum of the squares of the directional derivatives of $z = f(x, y)$ at any of its points is constant for any two mutually perpendicular directions and is equal to the square of the maximum directional derivative.

21. Given $z = f(x, y)$ and $w = g(x, y)$ such that $\partial z/\partial x = \partial w/\partial y$ and $\partial z/\partial y = -\partial w/\partial x$. If θ_1 and θ_2 are two mutually perpendicular directions, show that at any point $P(x, y)$, $\partial z/\partial s_1 = \partial w/\partial s_2$ and $\partial z/\partial s_2 = -\partial w/\partial s_1$.

22. Find the directional derivative of the given function at the given point in the indicated direction:

(a) xy^2z, (2, 1, 3), [1, −2, 2].
(b) $x^2 + y^2 + z^2$, (1, 1, 1), toward (2, 3, 4).
(c) $x^2 + y^2 - 2xz$, (1, 3, 2), along $x^2 + y^2 - 2xz = 6$, $3x^2 - y^2 + 3z = 0$ in the direction of increasing z.

Ans. (a) $-\frac{17}{3}$; (b) $6\sqrt{14}/7$; (c) 0

23. Examine each of the following functions for relative maximum and minimum values.

(a) $z = 2x + 4y - x^2 - y^2 - 3$ — *Ans.* maximum = 2 when $x = 1, y = 2$
(b) $z = x^3 + y^3 - 3xy$ — *Ans.* minimum = −1 when $x = 1, y = 1$
(c) $z = x^2 + 2xy + 2y^2$ — *Ans.* minimum = 0 when $x = 0, y = 0$
(d) $z = (x - y)(1 - xy)$ — *Ans.* neither maximum nor minimum
(e) $z = 2x^2 + y^2 + 6xy + 10x - 6y + 5$ — *Ans.* neither maximum nor minimum
(f) $z = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$ — *Ans.* minimum $= -\sqrt{6}$ when $x = -\sqrt{6}/6$, $y = \sqrt{6}/3$; maximum $\sqrt{6}$ when $x = \sqrt{6}/6$, $y = -\sqrt{6}/3$
(g) $z = xy(2x + 4y + 1)$ — *Ans.* maximum $\frac{1}{216}$ when $x = -\frac{1}{6}$, $y = -\frac{1}{12}$

24. Find positive numbers x, y, z such that

(a) $x + y + z = 18$ and xyz is a maximum
(b) $xyz = 27$ and $x + y + z$ is a minimum
(c) $x + y + z = 20$ and xyz^2 is a maximum
(d) $x + y + z = 12$ and xy^2z^3 is a maximum

Ans. (a) $x = y = z = 6$; (b) $x = y = z = 3$; (c) $x = y = 5$, $z = 10$; (d) $x = 2$, $y = 4$, $z = 6$

25. Find the minimum value of the square of the distance from the origin to the plane $Ax + By + Cz + D = 0$.

Ans $D^2/(A^2 + B^2 + C^2)$

26. (a) The surface area of a rectangular box without a top is to be 108 ft^2. Find the greatest possible volume.
(b) The volume of a rectangular box without a top is to be 500 ft^3. Find the minimum surface area.

Ans. (a) 108 ft^3; (b) 300 ft^2

27. Find the point on $z = xy - 1$ nearest the origin.

Ans. $(0, 0, -1)$

28. Find the equation of the plane through (1, 1, 2) that cuts off the least volume in the first octant.

Ans. $2x + 2y + z = 6$

29. Determine the values of p and q so that the sum S of the squares of the vertical distances of the points (0, 2), (1, 3), and (2, 5) from the line $y = px + q$ is a minimum. (*Hint:* $S = (q - 2)^2 + (p + q - 3)^2 + (2p + q - 5)^2$.)

Ans. $p = \frac{3}{2}$; $q = \frac{11}{6}$