CHAPTER 51

Surfaces and Curves in Space

Planes

We already know (formula (50.22)) that the equation of a plane has the form $Ax + By + Cz + D = 0$, where $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a nonzero vector perpendicular to the plane. The plane passes through the origin (0, 0, 0) when and only when $D = 0$.

Spheres

From the distance formula (50.3), we see that an equation of the sphere with radius r and center (a, b, c) is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

So a sphere with center at the origin (0, 0, 0) and radius r has the equation

$$x^2 + y^2 + z^2 = r^2$$

Cylindrical Surfaces

An equation $F(x, y) = 0$ ordinarily defines a curve $\mathscr{C}$ in the xy plane. Now, if a point (x, y) satisfies this equation, then, for any z, the point (x, y, z) in space also satisfies the equation. So, the equation $F(x, y) = 0$ determines the cylindrical *surface* obtained by moving the curve $\mathscr{C}$ parallel to the z axis. For example, the equation $x^2 + y^2 = 4$ determines a circle in the xy plane with radius 2 and center at the origin. If we move this circle parallel to the z axis, we obtain a right circular cylinder. Thus, what we ordinarily call a cylinder is a special case of a cylindrical surface.

Similarly, an equation $F(y, z) = 0$ determines the cylindrical surface obtained by moving the curve in the yz plane defined by $F(y, z) = 0$ parallel to the x axis. An equation $F(x, z) = 0$ determines the cylindrical surface obtained by moving the curve in the xz plane defined by $F(x, z) = 0$ parallel to the y axis.

More precisely, the cylindrical surfaces defined above are called right cylindrical surfaces. Other cylindrical surfaces can be obtained by moving the given curve parallel to a line that is not perpendicular to the plane of the curve.

EXAMPLE 51.1: The equation $z = x^2$ determines a cylindrical surface generated by moving the parabola $z = x^2$ lying in the xz plane parallel to the y axis.

Now we shall look at examples of surfaces determined by equations of the second degree in x, y, and z. Such surfaces are called *quadric* surfaces. Imagining what they look like is often helped by describing their intersections with planes parallel to the coordinate planes. Such intersections are called *traces.*

Ellipsoid

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

The nontrivial traces are ellipses. See Fig. 51-1. In general, the equation of an ellipsoid has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (a > 0, b > 0, c > 0)$$

When $a = b = c$, we obtain a sphere.

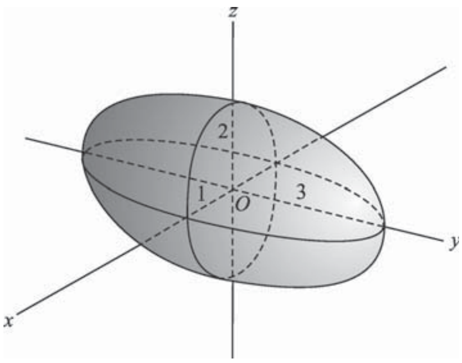

Fig. 51-1

Elliptic Paraboloid

$$z = x^2 + y^2$$

The surface lies on or above the *xy* plane. The traces parallel to the *xy* plane (for a fixed positive *z*) are circles. The traces parallel to the *xz* or *yz* plane are parabolas. See Fig. 51-2. In general, the equation of an elliptic paraboloid has the form

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad (a > 0, b > 0, c > 0)$$

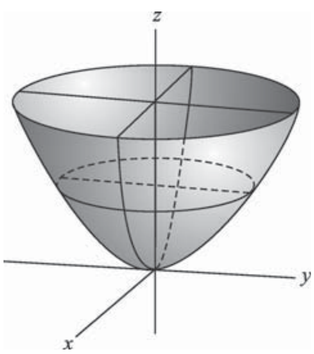

Fig. 51-2

and the traces parallel to the *xy* plane are ellipses. When $a = b$, we obtain a circular paraboloid, as in the given example.

Elliptic Cone

$$z^2 = x^2 + y^2$$

See Fig. 51-3. This is a pair of ordinary cones, meeting at the origin. The traces parallel to the *xy* plane are circles. The traces parallel to the *xz* or *yz* plane are hyperbolas. In general, the equation of an elliptic cone has the form

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad (a > 0, b > 0, c > 0)$$

and the traces parallel to the *xy* plane are ellipses. When $a = b$, we obtain a right circular cone, as in the given example.

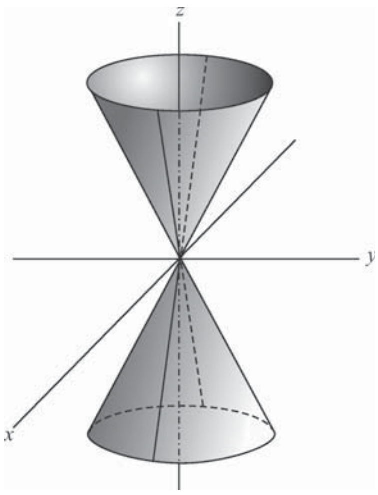

Fig. 51-3

Hyperbolic Paraboloid

$$z = 2y^2 - x^2$$

See Fig. 51-4. The surface resembles a saddle. The traces parallel to the *xy* plane are hyperbolas. The other traces are parabolas. In general, the equation of a hyperbolic paraboloid has the form

$$\frac{z}{c} = \frac{y^2}{b^2} + \frac{x^2}{a^2} \qquad (a > 0, b > 0, c \neq 0)$$

In the given example, $c = 1$, $a = 1$, and $b = 1/\sqrt{2}$.

Hyperboloid of One Sheet

$$x^2 + y^2 - \frac{z^2}{9} = 1$$

See Fig. 51-5. The traces parallel to the *xy* plane are circles and the other traces are hyperbolas. In general, a hyperboloid of one sheet has an equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

and the traces parallel to the *xy* plane are ellipses.

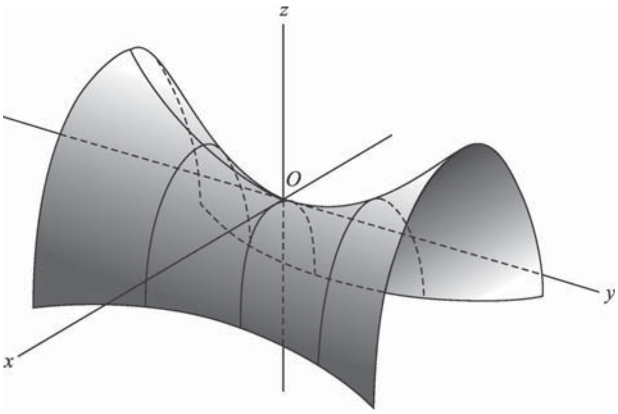

Fig. 51-4

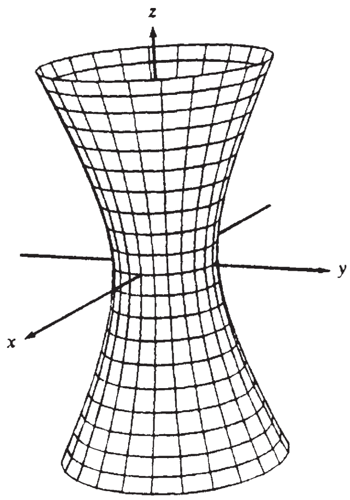

Fig. 51-5

Hyperboloid of Two Sheets

$$\frac{z^2}{4}-\frac{x^2}{9}-\frac{y^2}{9}=1$$

See Fig. 51-6. The traces parallel to the *xy* plane are circles, and the other traces are hyperbolas. In general, a hyperboloid of two sheets has an equation of the form

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (a>0, b>0, c>0)$$

and the traces parallel to the *xy* plane are ellipses.

In general equations given above for various quadric surfaces, permutation of the variables *x, y, z* is understood to produce quadric surfaces of the same type. For example, $\frac{y^2}{c^2}-\frac{z^2}{a^2}-\frac{x^2}{b^2}=1$ also determines a hyperboloid of two sheets.

Tangent Line and Normal Plane to a Space Curve

A space curve may be defined parametrically by the equations

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{51.1}$$

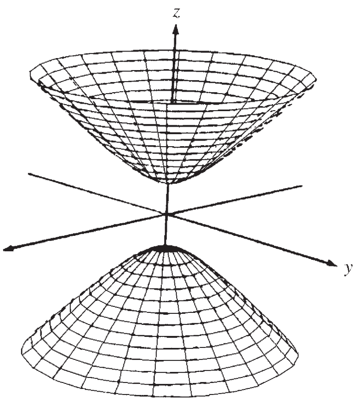

Fig. 51-6

At the point $P_0(x_0, y_0, z_0)$ of the curve (determined by $t = t_0$), the equations of the tangent line are

$$\frac{x - x_0}{dx/dt} = \frac{y - y_0}{dy/dt} = \frac{z - z_0}{dz/dt} \tag{51.2}$$

and the equations of the normal plane (the plane through P_0 perpendicular to the tangent line there) are

$$\frac{dx}{dt}(x - x_0) + \frac{dy}{dt}(y - y_0) + \frac{dz}{dt}(z - z_0) = 0 \tag{51.3}$$

See Fig. 51-7. In both (51.2) and (51.3), it is understood that the derivative has been evaluated at the point P_0. (See Problems 1 and 2.)

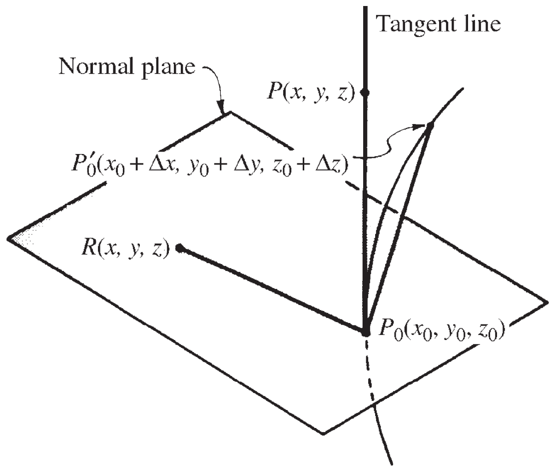

Fig. 51-7

Tangent Plane and Normal Line to a Surface

The equation of the tangent plane to the surface $F(x, y, z) = 0$ at one of its points $P_0(x_0, y_0, z_0)$ is

$$\frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) + \frac{\partial F}{\partial z}(z - z_0) = 0 \tag{51.4}$$

and the equations of the normal line at P_0 are

$$\frac{x-x_0}{\partial F/\partial x}=\frac{y-y_0}{\partial F/\partial y}=\frac{z-z_0}{\partial F/\partial z} \tag{51.5}$$

with the understanding that the partial derivatives have been evaluated at the point P_0. See Fig. 51-8. (See Problems 3–9.)

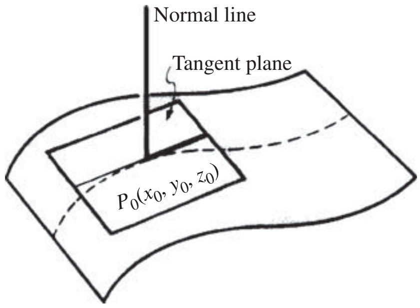

Fig. 51-8

A space curve may also be defined by a pair of equations

$$F(x, y, z) = 0, \quad G(x, y, z) = 0 \tag{51.6}$$

At the point $P_0(x_0, y_0, z_0)$ of the curve, the equations of the tangent line are

$$\frac{x-x_0}{\begin{vmatrix}\frac{\partial F}{\partial y} & \frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z}\end{vmatrix}}=\frac{y-y_0}{\begin{vmatrix}\frac{\partial F}{\partial z} & \frac{\partial F}{\partial x}\\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial x}\end{vmatrix}}=\frac{z-z_0}{\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{vmatrix}} \tag{51.7}$$

and the equation of the normal plane is

$$\begin{vmatrix}\frac{\partial F}{\partial y} & \frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z}\end{vmatrix}(x-x_0)+\begin{vmatrix}\frac{\partial F}{\partial z} & \frac{\partial F}{\partial x}\\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial x}\end{vmatrix}(y-y_0)+\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{vmatrix}(z-z_0)=0 \tag{51.8}$$

In (51.7) and (51.8), it is understood that all partial derivatives have been evaluated at the point P_0. (See Problems 10 and 11.)

Surface of Revolution

Let the graph of $y = f(x)$ in the xy plane be revolved about the x axis. As a point (x_0, y_0) on the graph revolves, a resulting point (x_0, y, z) has the distance y_0 from the point $(x_0, 0, 0)$. So, squaring that distance, we get

$$(x_0 - x_0)^2 + y^2 + z^2 = (y_0)^2 = (f(x_0))^2 \qquad \text{and, therefore,} \qquad y^2 + z^2 = (f(x_0))^2$$

Then, the equation of the surface of revolution is

$$y^2 + z^2 = (f(x))^2 \tag{51.9}$$

SOLVED PROBLEMS

1. Derive (51.2) and (51.3) for the tangent line and normal plane to the space curve $x = f(t)$, $y = g(t)$, $z = h(t)$ at the point $P_0(x_0, y_0, z_0)$ determined by the value $t = t_0$. Refer to Fig. 51-7.

Let $P'_0(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$, determined by $t = t_0 + \Delta t$, be another point on the curve. As $P_0 \to P_0$ along the curve, the chord $P_0P'_0$ approaches the tangent line to the curve at P_0 as the limiting position.

A simple set of direction numbers for the chord $P_0P'_0$ is $[\Delta x, \Delta y, \Delta z]$, but we shall use $\left[\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right]$. Then as $P_0 \to P_0$, $\Delta t \to 0$ and $\left[\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right] \to \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$, a set of direction numbers of the tangent line at P_0. Now if $P(x, y, z)$ is an arbitrary point on this tangent line, then $[x - x_0, y - y_0, z - z_0]$ is a set of direction numbers of P_0P. Thus, since the sets of direction numbers are proportional, the equations of the tangent line at P_0 are

$$\frac{x - x_0}{dx/dt} = \frac{y - y_0}{dy/dt} = \frac{z - z_0}{dz/dt}$$

If $R(x, y, z)$ is an arbitrary point in the normal plane at P_0, then, since P_0R and P_0P are perpendicular, the equation of the normal plane at P_0 is

$$(x - x_0)\frac{dx}{dt} + (y - y_0)\frac{dy}{dt} + (z - z_0)\frac{dz}{dt} = 0$$

2. Find the equations of the tangent line and normal plane to:

(a) The curve $x = t$, $y = t^2$, $z = t^3$ at the point $t = 1$.
(b) The curve $x = t - 2$, $y = 3t^2 + 1$, $z = 2t^3$ at the point where it pierces the yz plane.

(a) At the point $t = 1$ or (1, 1, 1), $dx/dt = 1$, $dy/dt = 2t = 2$, and $dz/dt = 3t^2 = 3$. Using (51.2) yields, for the equations of the tangent line, $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$; using (51.3) gives the equation of the normal plane as $(x - 1) + 2(y - 1) + 3(z - 1) = x + 2y + 3z - 6 = 0$.

(b) The given curve pierces the yz plane at the point where $x = t - 2 = 0$, that is, at the point $t = 2$ or (0, 13, 16). At this point, $dx/dt = 1$, $dy/dt = 6t = 12$, and $dz/dt = 6t^2 = 24$. The equations of the tangent line are $\frac{x}{1} = \frac{y-13}{12} = \frac{z-16}{24}$, and the equation of the normal plane is $x + 12(y - 13) + 24(z - 16) = x + 12y + 24z - 540 = 0$.

3. Derive (51.4) and (51.5) for the tangent plane to the surface $F(x, y, z) = 0$ at the point $P_0(x_0, y_0, z_0)$. Refer to Fig. 51-8.

Let $x = f(t)$, $y = g(t)$, $z = h(t)$ be the parametric equations of any curve on the surface $F(x, y, z) = 0$ and passing through the point P_0. Then, at P_0,

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

with the understanding that all derivatives have been evaluated at P_0.

This relation expresses the fact that the line through P_0 with direction numbers $\left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$ is perpendicular to the line through P_0 having direction numbers $\left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right]$. The first set of direction numbers belongs to the tangent to the curve which lies in the tangent plane of the surface. The second set defines the normal line to the surface at P_0. The equations of this normal are

$$\frac{x - x_0}{\partial F/\partial x} = \frac{y - y_0}{\partial F/\partial y} = \frac{z - z_0}{\partial F/\partial z}$$

and the equation of the tangent plane at P_0 is

$$\frac{\partial F}{\partial x}(x-x_0)+\frac{\partial F}{\partial y}(y-y_0)+\frac{\partial F}{\partial z}(z-z_0)=0$$

In Problems 4 and 5, find the equations of the tangent plane and normal line to the given surface at the given point.

4. $z = 3x^2 + 2y^2 - 11$; (2, 1, 3).

Put $F(x, y, z) = 3x^2 + 2y^2 - z - 11 = 0$. At (2, 1, 3), $\frac{\partial F}{\partial x}=6x=12, \frac{\partial F}{\partial y}=4y=4$, and $\frac{\partial F}{\partial z}=-1$. The equation of the tangent plane is $12(x-2)+4(y-1)-(z-3)=0$ or $12x+4y-z=25$.

The equations of the normal line are $\frac{x-2}{12}=\frac{y-1}{4}=\frac{z-3}{-1}$.

5. $F(x, y, z) = x^2 + 3y^2 - 4z^2 + 3xy - 10yz + 4x - 5z - 22 = 0$; (1, −2, 1).

At (1, −2, 1), $\frac{\partial F}{\partial x}=2x+3y+4=0$, $\frac{\partial F}{\partial y}=6y+3x-10z=-19$, and $\frac{\partial F}{\partial z}=-8z-10y-5=7$. The equation of the tangent plane is $0(x-1)-19(y+2)+7(z-1)=0$ or $19y-7z+45=0$.

The equations of the normal line are $x-1=0$ and $\frac{y+2}{-19}=\frac{z-1}{7}$ or $x=1$, $7y+19z-5=0$.

6. Show that the equation of the tangent plane to the surface $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ at the point $P_0(x_0, y_0, z_0)$ is $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}-\frac{zz_0}{c^2}=1$.

At P_0, $\frac{\partial F}{\partial x}=\frac{2x_0}{a^2}, \frac{\partial F}{\partial y}=-\frac{2y_0}{b^2}$, and $\frac{\partial F}{\partial z}=-\frac{2z_0}{c^2}$. The equation of the tangent plane is

$$\frac{2x_0}{a^2}(x-x_0)-\frac{2y_0}{b^2}(y-y_0)-\frac{2z_0}{c^2}(z-z_0)=0.$$

This becomes $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}-\frac{zz_0}{c^2}=\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1$, since P_0 is on the surface.

7. Show that the surfaces $F(x, y, z) = x^2 + 4y^2 - 4z^2 - 4 = 0$ and $G(x, y, z) = x^2 + y^2 + z^2 - 6x - 6y + 2z + 10 = 0$ are tangent at the point (2, 1, 1).

It is to be shown that the two surfaces have the same tangent plane at the given point. At (2, 1, 1),

$$\frac{\partial F}{\partial x}=2x-4, \qquad \frac{\partial F}{\partial y}=8y=8, \qquad \frac{\partial F}{\partial z}=-8z=-8$$

and $$\frac{\partial G}{\partial x}=2x-6=-2, \qquad \frac{\partial G}{\partial y}=2y-6=-4, \qquad \frac{\partial G}{\partial z}=2z+2=4$$

Since the sets of direction numbers [4, 8, −8] and [−2, −4, 4] of the normal lines of the two surfaces are proportional, the surfaces have the common tangent plane

$$1(x-2)+2(y-1)-2(z-1)=0 \quad \text{or} \quad x+2y-2z=2$$

8. Show that the surfaces $F(x, y, z) = xy + yz - 4zx = 0$ and $G(x, y, z) = 3z^2 - 5x + y = 0$ intersect at right angles at the point (1, 2, 1).

It is to be shown that the tangent planes to the surfaces at the point are perpendicular or, what is the same, that the normal lines at the point are perpendicular. At (1, 2, 1),

$$\frac{\partial F}{\partial x}=y-4z=-2, \qquad \frac{\partial F}{\partial y}=x+z=2, \qquad \frac{\partial F}{\partial z}=y-4x=-2$$

A set of direction numbers for the normal line to $F(x, y, z) = 0$ is $[l_1, m_1, n_1] = [1, -1, 1]$. At the same point,

$$\frac{\partial G}{\partial x} = -5, \qquad \frac{\partial G}{\partial y} = 1, \qquad \frac{\partial G}{\partial z} = 6z = 6$$

A set of direction numbers for the normal line to $G(x, y, z) = 0$ is $[l_2, m_2, n_2] = [-5, 1, 6]$.
Since $l_1l_2 + m_1m_2 + n_1n_2 = 1(-5) + (-1)1 + 1(6) = 0$, these directions are perpendicular.

9. Show that the surfaces $F(x, y, z) = 3x^2 + 4y^2 + 8z^2 - 36 = 0$ and $G(x, y, z) = x^2 + 2y^2 - 4z^2 - 6 = 0$ intersect at right angles.

At any point $P_0(x_0, y_0, z_0)$ on the two surfaces, $\frac{\partial F}{\partial x} = 6x_0, \frac{\partial F}{\partial y} = 8y_0$, and $\frac{\partial F}{\partial z} = 16z_0$; hence $[3x_0, 4y_0, 8z_0]$ is a set of direction numbers for the normal to the surface $F(x, y, z) = 0$ at P_0. Similarly, $[x_0, 2y_0, -4z_0]$ is a set of direction numbers for the normal line to $G(x, y, z) = 0$ at P_0. Now, since

$$6(x_0^2 + 2y_0^2 - 4z_0^2) - (3x_0^2 + 4y_0^2 + 8z_0^2) = 6(6) - 36 = 0,$$

these directions are perpendicular.

10. Derive (51.7) and (51.8) for the tangent line and normal plane to the space curve C: $F(x, y, z) = 0$, $G(x, y, z) = 0$ at one of its points $P_0(x_0, y_0, z_0)$.

At P_0, the directions $\left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right]$ and $\left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right]$ are normal, respectively, to the tangent planes of the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$. Now the direction

$$\left[\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix}, \begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix}, \begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix}\right]$$

being perpendicular to each of these directions, is that of the tangent line to C at P_0. Hence, the equations of the tangent line are

$$\frac{x - x_0}{\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix}} = \frac{z - z_0}{\begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix}}$$

and the equation of the normal plane is

$$\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix}(x - x_0) + \begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix}(y - y_0) + \begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix}(z - z_0) = 0$$

11. Find the equations of the tangent line and the normal plane to the curve $x^2 + y^2 + z^2 = 14$, $x + y + z = 6$ at the point (1, 2, 3).

Set $F(x,y,z) = x^2 + y^2 + z^2 - 14 = 0$ and $G(x,t,z) = x + y + z - 6 = 0$. At $(1,2,3)$,

$$\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 1 & 1 \end{vmatrix} = -2$$

$$\begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 1 & 1 \end{vmatrix} = 4, \quad \begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = -2$$

With $[1, -2, 1]$ as a set of direction numbers of the tangent, its equations are $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{1}$. The equation of the normal plane is $(x - 1) - 2(y - 2) + (z - 3) = x - 2y + z = 0$.

12. Find equations of the surfaces of revolution generated by revolving the given curve about the given axis: (a) $y = x^2$ about the x axis; (b) $y = \frac{1}{x}$ about the y axis; (c) $z = 4y$ about the y axis.

In each case, we use an appropriate form of (51.9): (a) $y^2 + z^2 = x^4$; (b) $x^2 + z^2 = \frac{1}{y^2}$; (c) $x^2 + z^2 = 16y^2$.

13. Identify the locus of all points (x, y, z) that are equidistant from the point $(0, -1, 0)$ and the plane $y = 1$.

Squaring the distances, we get $x^2 + (y + 1^2) + z^2 = (y - 1)^2$, whence $x^2 + z^2 = -4y$, a circular paraboloid.

14. Identify the surface $4x^2 - y^2 + z^2 - 8x + 2y + 2z + 3 = 0$ by completing the squares.

We have

$$4(x^2 - 2x) - (y^2 - 2y) + (z^2 + 2z) + 3 = 0$$

$$4(x - 1)^2 - (y - 1)^2 + (z + 1)^2 + 3 = 4$$

$$4(x - 1)^2 - (y - 1)^2 + (z + 1)^2 = 1$$

This is a hyperboloid of one sheet, centered at $(1, 1, -1)$.

SUPPLEMENTARY PROBLEMS

15. Find the equations of the tangent line and the normal plane to the given curve at the given point:

(a) $x = 2t,\ y = t^2,\ z = t^3;\ t = 1$ *Ans.* $\frac{x-2}{2} = \frac{y-1}{2} = \frac{z-1}{3}$; $2x + 2y + 3z - 9 = 0$

(b) $x = te^t,\ y = e^t,\ z = t;\ t = 0$ *Ans.* $\frac{x}{1} = \frac{y-1}{1} = \frac{z}{1}$; $x + y + z - 1 = 0$

(c) $x = t\cos t,\ y = t\sin t,\ z = t;\ t = 0$ *Ans.* $x = z,\ y = 0$; $x + z = 0$

16. Show that the curves (a) $x = 2 - t,\ y = -1/t,\ z = 2t^2$ and (b) $x = 1 + \theta,\ y = \sin\theta - 1,\ z = 2\cos\theta$ intersect at right angles at $P(1, -1, 2)$. Obtain the equations of the tangent line and normal plane of each curve at P.

Ans. (a) $\frac{x-1}{-1} = \frac{y+1}{1} = \frac{z-2}{4}$; $x - y - 4z + 6 = 0$; (b) $x - y = 2,\ z = 2$; $x + y = 0$

17. Show that the tangent lines to the helix $x = a\cos t,\ y = a\sin t,\ z = bt$ meet the xy plane at the same angle.

18. Show that the length of the curve (51.1) from the point $t = t_0$ to the point $t = t_1$ is given by

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Find the length of the helix of Problem 17 from $t = 0$ to $t = t_1$.

Ans. $\sqrt{a^2 + b^2}\, t_1$

19. Find the equations of the tangent line and the normal plane to the given curve at the given point:

(a) $x^2 + 2y^2 + 2z^2 = 5,\ 3x - 2y - z = 0$; $(1, 1, 1)$.

(b) $9x^2 + 4y^2 - 36z = 0,\ 3x + y + z - z^2 - 1 = 0$; $(2, -3, 2)$.

(c) $4z^2 = xy,\ x^2 + y^2 = 8z$; $(2, 2, 1)$.

Ans. (a) $\frac{x-1}{2}=\frac{y-1}{7}=\frac{z-1}{-8}$; $2x+7y-8z-1=0$; (b) $\frac{x-2}{1}=\frac{y-2}{1}$, $y+3=0$; $x+z-4=0$;
(c) $\frac{x-2}{1}=\frac{y-2}{-1}$, $z-1=0$; $x-y=0$

20. Find the equations of the tangent plane and normal line to the given surface at the given point:

(a) $x^2+y^2+z^2=14$; $(1,-2,3)$ *Ans.* $x-2y+3z=14$; $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-3}{3}$

(b) $x^2+y^2+z^2=r^2$; (x_1,y_1,z_1) *Ans.* $x_1x+y_1y+z_1z=r^2$; $\frac{x-x_1}{x_1}=\frac{y+y_1}{y_1}=\frac{z-z_1}{z_1}$

(c) $x^2+2z^3+3y^2$; $(2,-2,-2)$ *Ans.* $x+3y-2z=0$; $\frac{x-2}{1}=\frac{y+2}{3}=\frac{z+2}{-2}$

(d) $2x^2+2xy+y^2+z+1=0$; $(1,-2,-3)$ *Ans.* $z-2y=1$; $x-1=0$, $\frac{y+2}{2}=\frac{z+3}{-1}$

(e) $z=xy$; $(3,-4,-12)$ *Ans.* $4x-3y+z=12$; $\frac{x-3}{4}=\frac{y+4}{-3}=\frac{z+12}{1}$

21. (a) Show that the sum of the intercepts of the plane tangent to the surface $x^{1/2}+y^{1/2}+z^{1/2}=a^{1/2}$ at any of its points is a.

(b) Show that the square root of the sum of the squares of the intercepts of the plane tangent to the surface $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$ at any of its points is a.

22. Show that each pair of surfaces are tangent at the given point:

(a) $x^2+y^2+z^2=18$, $xy=9$; $(3,3,0)$.
(b) $x^2+y^2+z^2-8x-8y-6z+24=0$, $x^2+3y^2+2z^2=9$; $(2,1,1)$.

23. Show that each pair of surfaces are perpendicular at the given point:

(a) $x^2+2y^2-4z^2=8$, $4x^2-y^2+2z^2=14$; $(2,2,1)$.
(b) $x^2+y^2+z^2=50$, $x^2+y^2-10z+25=0$; $(3,4,5)$.

24. Show that each of the surfaces (a) $14x^2+11y^2+8z^2=66$, (b) $3z^2-5x+y=0$, and (c) $xy+yz-4zx=0$ is perpendicular to the other two at the point $(1,2,1)$.

25. Identify the following surfaces.

(a) $36y^2-x^2+36z^2=9$.
(b) $5y=-z^2+x^2$.
(c) $x^2+4y^2-4z^2-6x-16y-16z+5=0$.

Ans. (a) hyperboloid of one sheet (around the x axis); (b) hyperbolic paraboloid; (c) hyperboloid of one sheet, centered at $(3,2,-2)$

26. Find an equation of a curve that, when revolved about a suitable axis, yields the paraboloid $y^2+z^2-2x=0$.

Ans. $y=\sqrt{2x}$ or $z=\sqrt{2x}$, about the x axis

27. Find an equation of the surface obtained by revolving the given curve about the given axis. Identify the type of surface: (a) $x=y^2$ about the x axis; (b) $x=2y$ about the x axis.

Ans. (a) $x=y^2+z^2$ (circular paraboloid); (b) $y^2+z^2=\frac{x^2}{4}$ (right circular cone)