CHAPTER 50

Space Vectors

Vectors in Space

As in the plane (see Chapter 39), a vector in space is a quantity that has both magnitude and direction. Three vectors **a**, **b**, and **c**, not in the same plane and no two parallel, issuing from a common point are said to form a *right-handed system* or *triad* if **c** has the direction in which the right-threaded screw would move when rotated through the smaller angle in the direction from **a** to **b**, as in Fig. 50-1. Note that, as seen from a point on **c**, the rotation through the smaller angle from **a** to **b** is counterclockwise.

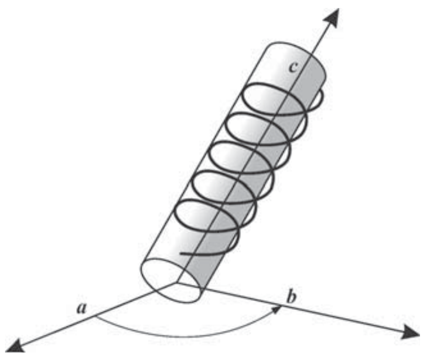

Fig. 50-1

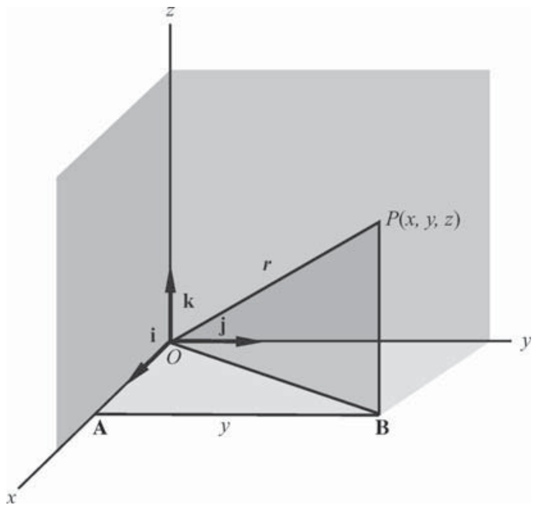

Fig. 50-2

We choose a right-handed rectangular coordinate system in space and let **i**, **j**, and **k** be unit vectors along the positive *x, y* and *z* axes, respectively, as in Fig. 50-2. The coordinate axes divide space into eight parts, called *octants.* The *first octant,* for example, consists of all points (x, y, z) for which $x > 0$, $y > 0$, $z > 0$.

As in Chapter 39, any vector **a** may be written as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

If $P\ (x, y, z)$ is a point in space (Fig. 50-2), the vector **r** from the origin O to P is called the *position vector* of P and may be written as

$$\mathbf{r} = \mathbf{OP} = \mathbf{OB} + \mathbf{BP} = \mathbf{OA} + \mathbf{AB} + \mathbf{BP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{50.1}$$

The algebra of vectors developed in Chapter 39 holds here with only such changes as the difference in dimensions requires. For example, if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. then

$$k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j} + ka_3\mathbf{k} \text{ for } k \text{ any scalar}$$

$$\mathbf{a} = \mathbf{b} \text{ if and only if } a_1 = b_1, a_2 = b_2, and\ a_3 = b_3$$

$$\mathbf{a} \pm \mathbf{b} = (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\mathbf{j} + (a_3 \pm b_3)\mathbf{k}$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta, \text{where } \theta \text{ is the smaller angle between } \mathbf{a} \text{ and } \mathbf{b}$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \text{ and } \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\mathbf{a} \cdot \mathbf{b} = 0 \text{ if and only if } \mathbf{a} = \mathbf{0}, \text{ or } \mathbf{b} = \mathbf{0}, \text{ or } \mathbf{a} \text{ and } \mathbf{b} \text{ are perpendicular}$$

From (50.1), we have

$$|\mathbf{r}| = \sqrt{\mathbf{r} \, . \, \mathbf{r}} = \sqrt{x^2 + y^2 + z^2} \tag{50.2}$$

as the distance of the point $P\ (x, y, z)$ from the origin. Also, if $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points (see Fig. 50-3), then

$$\mathbf{P_1P_2} = \mathbf{P_1B} + \mathbf{BP_2} = \mathbf{P_1A} + \mathbf{AB} + \mathbf{BP_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

and

$$|\mathbf{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{50.3}$$

is the familiar formula for the distance between two points. (See Problems 1–3.)

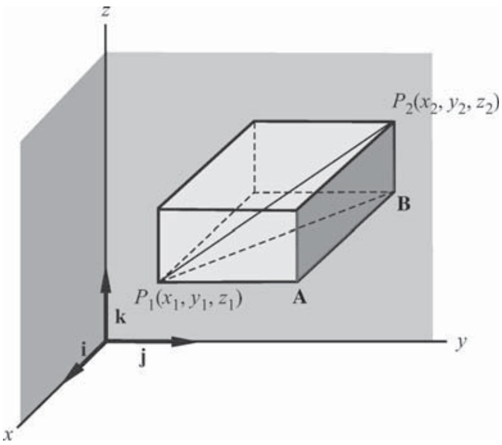

Fig. 50-3

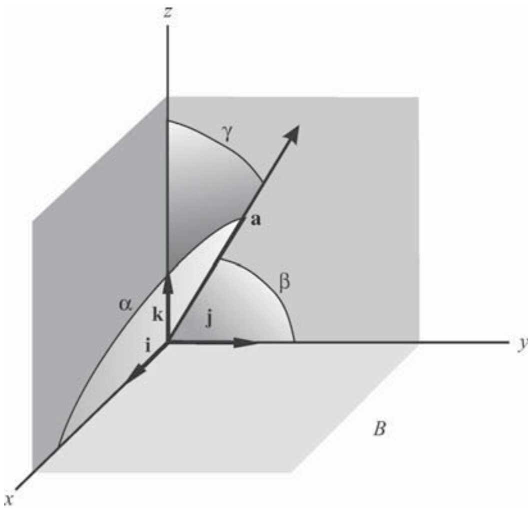

Fig. 50-4

Direction Cosines of a Vector

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ make angles α, β, and γ, respectively, with the positive x, y, and z axes, as in Fig. 50-4. From

$$\mathbf{i} \cdot \mathbf{a} = |\mathbf{i}|\,|\mathbf{a}| \cos\alpha = |\mathbf{a}| \cos\alpha, \qquad \mathbf{j} \cdot \mathbf{a} = |\mathbf{a}| \cos\beta, \quad \mathbf{k} \cdot \mathbf{a} = |\mathbf{a}| \cos\gamma$$

we have

$$\cos\alpha = \frac{\mathbf{i}\,.\,\mathbf{a}}{|\mathbf{a}|} = \frac{a_1}{|\mathbf{a}|}, \qquad \cos\beta = \frac{\mathbf{j}\,.\,\mathbf{a}}{|\mathbf{a}|} = \frac{a_2}{|\mathbf{a}|}, \qquad \cos\gamma = \frac{\mathbf{k}\,.\,\mathbf{a}}{|\mathbf{a}|} = \frac{a_3}{|\mathbf{a}|}$$

These are the *direction cosines* of **a**. Since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2 + a_2^2 + a_3^2}{|\mathbf{a}|^2} = 1$$

the vector $\mathbf{u} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma$ is a unit vector parallel to **a**.

Determinants

We shall assume familiarity with 2×2 and 3×3 determinants. In particular,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \text{and} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

That expansion of the 3×3 determinant is said to be "along the first row." It is equal to suitable expansions along the other rows and down the columns.

Vector Perpendicular to Two Vectors

Let

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

be two nonparallel vectors with common initial point P. By an easy computation, it can be shown that

$$\mathbf{c} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{50.4}$$

is perpendicular to (normal to) both **a** and **b** and, hence, to the plane of these vectors.

In Problems 5 and 6, we show that

$$|\mathbf{c}| = |\mathbf{a}||\mathbf{b}|\sin\theta = \text{area of a parallelogram with nonparallel sides } \mathbf{a} \text{ and } \mathbf{b} \tag{50.5}$$

If **a** and **b** are parallel, then $\mathbf{b} = k\mathbf{a}$, and (50.4) shows that $\mathbf{c} = \mathbf{0}$; that is, **c** is the zero vector. The zero vector, by definition, has magnitude 0 but no specified direction.

Vector Product of Two Vectors

Take

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

with initial point P and denote by **n** the unit vector normal to the plane of **a** and **b**, so directed that **a**, **b**, and **n** (in that order) form a right-handed triad at P, as in Fig. 50-5. The *vector product* or *cross product* of **a** and **b** is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\ \mathbf{n} \tag{50.6}$$

where θ is again the smaller angle between **a** and **b**. Thus, $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both **a** and **b**.

We show in Problem 6 that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ is the area of the parallelogram having **a** and **b** as nonparallel sides.

If **a** and **b** are parallel, then $\theta = 0$ or π and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. Thus,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \tag{50.7}$$

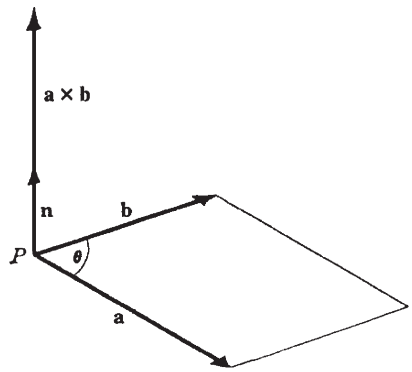

Fig. 50-5

In (50.6), if the order of **a** and **b** is reversed, then **n** must be replaced by **–n**; hence,

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \tag{50.8}$$

Since the coordinate axes were chosen as a right-handed system, it follows that

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k}, & \mathbf{j} \times \mathbf{k} = \mathbf{i}, & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}, & \mathbf{k} \times \mathbf{j} = -\mathbf{i}, & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array} \tag{50.9}$$

In Problem 8, we prove for any vectors **a**, **b**, and **c**, the distributive law

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) \tag{50.10}$$

Multiplying (50.10) by –1 and using (50.8), we have the companion distributive law

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = (\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b}) \tag{50.11}$$

Then, also,

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d} \tag{50.12}$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{50.13}$$

(See Problems 9 and 10.)

Triple Scalar Product

In Fig. 50-6, let θ be the smaller angle between **b** and **c** and let ϕ be the smaller angle between **a** and $\mathbf{b}\times\mathbf{c}$. Let h denote the height and A the area of the base of the parallelepiped. Then the triple scalar product is by definition

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot|\mathbf{b}|\,|\mathbf{c}|\sin\theta\;\mathbf{n} = |\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\phi = (|\mathbf{a}|\cos\phi)(|\mathbf{b}||\mathbf{c}|\sin\theta) = hA$$
$$= \text{volume of parallelepiped}$$

It may be shown (see Problem 11) that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \qquad (50.14)$$

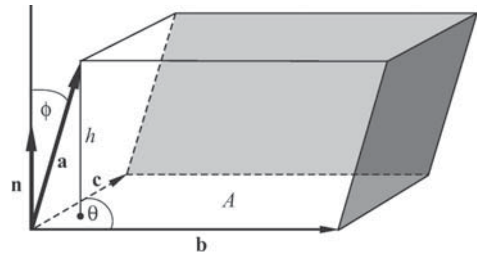

Fig. 50-6

Also

$$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

whereas

$$\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

Similarly, we have

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \qquad (50.15)$$

and

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}) = -\,\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) \qquad (50.16)$$

From the definition of $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ as a volume, it follows that if **a**, **b**, and **c** are coplanar, then $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$, and conversely.

The parentheses in $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ are not necessary. For example, $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ can be interpreted only as $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ or $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$. But $\mathbf{a}\cdot\mathbf{b}$ is a scalar, so $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$ is without meaning. (See Problem 12.)

Triple Vector Product

In Problem 13, we show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \qquad (50.17)$$

Similarly,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \qquad (50.18)$$

Thus, except when **b** is perpendicular to both **a** and **c**, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and the use of parentheses is necessary.

The Straight Line

A line in space through a given point $P_0(x_0, y_0, z_0)$ may be defined as the locus of all points $P(x, y, z)$ such that P_0P is parallel to a given direction $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (Fig. 50-7). Then

$$\mathbf{r} - \mathbf{r}_0 = k\mathbf{a} \quad \text{where } k \text{ is a scalar variable} \qquad (50.19)$$

is the vector equation of line PP_0. Writing (50.19) as

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = k(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$$

then separating components to obtain

$$x - x_0 = ka_1, \quad y - y_0 = ka_2, \quad z - z_0 = ka_3$$

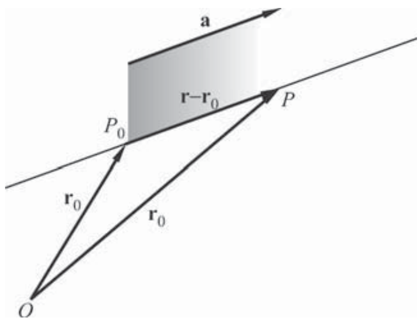

Fig. 50-7

and eliminating k, we have

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \qquad (50.20)$$

as the equations of the line in rectangular coordinates. Here, $[a_1, a_2, a_3]$ is a set of *direction numbers* for the line and $\left[\frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|}, \frac{a_3}{|\mathbf{a}|}\right]$ is a set of *direction cosines* of the line.

If any one of the numbers a_1, a_2, or a_3 is zero, the corresponding numerator in (50.20) must be zero. For example, if $a_1 = 0$ but $a_2, a_3 \neq 0$, the equations of the line are

$$x - x_0 = 0 \quad \text{and} \quad \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}$$

The Plane

A plane in space through a given point $P_0(x_0, y_0, z_0)$ can be defined as the locus of all lines through P_0 and perpendicular (normal) to a given line (direction) $\mathbf{a} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ (Fig. 50-8). Let $P(x, y, z)$ be any other point in the plane. Then $\mathbf{r} - \mathbf{r}_0 = \mathbf{P}_0\mathbf{P}$ is perpendicular to $\mathbf{a}$, and the equation of the plane is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a} = 0 \tag{50.21}$$

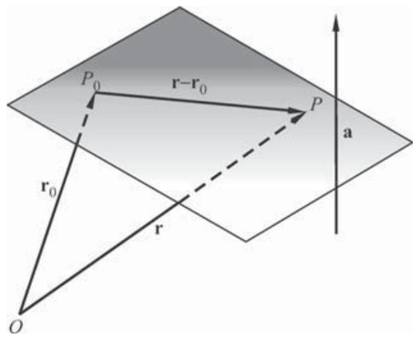

Fig. 50-8

In rectangular coordinates, this becomes

$$[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] \cdot (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$Ax + By + Cz + D = 0 \tag{50.22}$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

Conversely, let $P_0(x_0, y_0, z_0)$ be a point on the surface $Ax + By + Cz + D = 0$. Then also $Ax_0 + By_0 + Cz_0 + D = 0$. Subtracting the second of these equations from the first yields $A(x - x_0) + B(y - y_0) + C(z - z_0) = (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$ and the constant vector $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the surface at each of its points. Thus, the surface is a plane.

SOLVED PROBLEMS

1. Find the distance of the point $P_1(1, 2, 3)$ from (a) the origin, (b) the x axis, (c) the z axis, (d) the xy plane, and (e) the point $P_2(3, -1, 5)$.

In Fig. 50-9,

(a) $r = \mathbf{OP}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$; hence, $|\mathbf{r}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

(b) $\mathbf{AP}_1 = \mathbf{AB} + \mathbf{BP}_1 = 2\mathbf{j} + 3\mathbf{k}$; hence, $|\mathbf{AP}_1| = \sqrt{4 + 9} = \sqrt{13}$.

(c) $\mathbf{DP}_1 = \mathbf{DE} + \mathbf{EP}_1 = 2\mathbf{j} + \mathbf{i}$; hence, $|\mathbf{DP}_1| = \sqrt{5}$.

(d) $\mathbf{BP}_1 = 3\mathbf{k}$, so $|\mathbf{BP}_1| = 3$.

(e) $\mathbf{P}_1\mathbf{P}_2 = (3 - 1)\mathbf{i} + (-1 - 2)\mathbf{j} + (5 - 3)\mathbf{k} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; hence, $|\mathbf{P}_1\mathbf{P}_2| = \sqrt{4 + 9 + 4} = \sqrt{17}$.

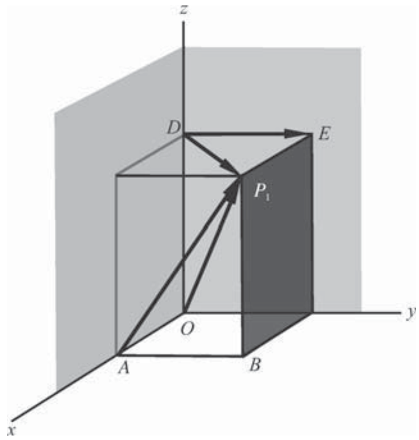

Fig. 50-9

2. Find the angle θ between the vectors joining O to $P_1(1, 2, 3)$ and $P_2(2, -3, -1)$.

Let $\mathbf{r}_1 = \mathbf{OP}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{r}_2 = \mathbf{OP}_2 = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$. Then

$$\cos\theta = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1||\mathbf{r}_2|} = \frac{1(2) + 2(-3) + 3(-1)}{\sqrt{14}\sqrt{14}} = -\frac{1}{2} \quad \text{and} \quad \theta = 120°.$$

3. Find the angle $\alpha = \angle BAC$ of the triangle ABC (Fig. 50-10) whose vertices are $A(1, 0, 1)$, $B(2, -1, 1)$, $C(-2, 1, 0)$.

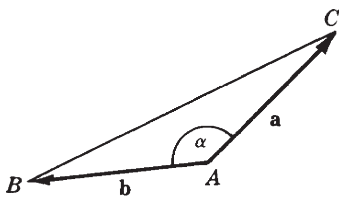

Fig. 50-10

Let $\mathbf{a} = \mathbf{AC} = -3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{AB} = \mathbf{i} - \mathbf{j}$. Then

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-3 - 1}{\sqrt{22}} \sim -0.85280 \quad \text{and} \quad \alpha \sim 148°31'.$$

4. Find the direction cosines of $\mathbf{a} = 3\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$.

The direction cosines are $\cos\alpha = \frac{\mathbf{i} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{3}{13}, \cos\beta = \frac{\mathbf{j} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{12}{13}, \cos\gamma = \frac{\mathbf{k} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{4}{13}.$

5. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors issuing from a point P and if

$$\mathbf{c} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k},$$

show that $|\mathbf{c}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, where θ is the smaller angle between $\mathbf{a}$ and $\mathbf{b}$.

We have $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ and

$$\sin\theta = \sqrt{1 - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)^2} = \frac{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2}}{|\mathbf{a}||\mathbf{b}|} = \frac{|\mathbf{c}|}{|\mathbf{a}||\mathbf{b}|}$$

Hence, $|\mathbf{c}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ as required.

6. Find the area of the parallelogram whose nonparallel sides are **a** and **b**.

From Fig. 50-11, $h = |\mathbf{b}| \sin\theta$ and the area is $h|\mathbf{a}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.

7. Let $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, be the components of **a** parallel and perpendicular to **b**, as in Fig. 50-12. Show that $\mathbf{a}_2 \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{a}_1 \times \mathbf{b} = 0$.

If θ is the angle between **a** and **b**, then $|\mathbf{a}_1| = |\mathbf{a}| \cos\theta$ and $|\mathbf{a}_2| = |\mathbf{a}| \sin\theta$. Since **a**, $\mathbf{a}_2$, and **b** are coplanar,

$$\mathbf{a}_2 \times \mathbf{b} = |\mathbf{a}_2||\mathbf{b}| \sin\phi\, \mathbf{n} = |\mathbf{a}| \sin\theta\, |\mathbf{b}|\, \mathbf{n} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \mathbf{n} = \mathbf{a} \times \mathbf{b}$$

Since $\mathbf{a}_1$ and **b** are parallel, $\mathbf{a}_1 \times \mathbf{b} = \mathbf{0}$.

8. Prove: $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$.

In Fig. 50-13, the initial point P of the vectors **a**, **b**, and **c** is in the plane of the paper, while their endpoints are above this plane, The vectors $\mathbf{a}_1$ and $\mathbf{b}_1$ are, respectively, the components of **a** and **b** perpendicular to **c**. Then $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_1 + \mathbf{b}_1$, $\mathbf{a}_1 \times \mathbf{c}$, $\mathbf{b}_1 \times \mathbf{c}$, and $(\mathbf{a}_1 + \mathbf{b}_1) \times \mathbf{c}$ all lie in the plane of the paper.

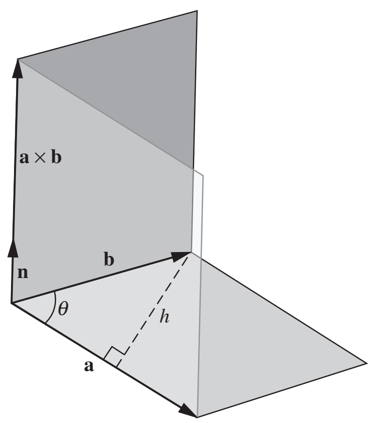

Fig. 50-11

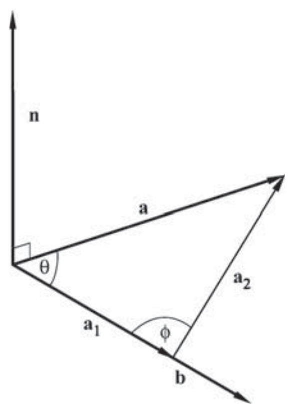

Fig. 50-12

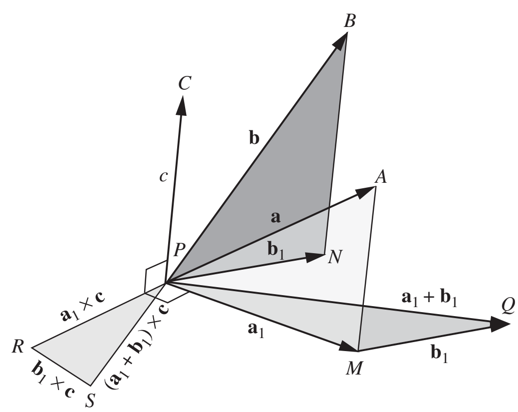

Fig. 50-13

In triangles PRS and PMQ,

$$\frac{RS}{PR} = \frac{|\mathbf{b}_1 \times \mathbf{c}|}{|\mathbf{a}_1 \times \mathbf{c}|} = \frac{|\mathbf{b}_1||\mathbf{c}|}{|\mathbf{a}_1||\mathbf{c}|} = \frac{|\mathbf{b}_1|}{|\mathbf{a}_1|} = \frac{MQ}{PM}$$

Thus, *PRS* and *PMQ* are similar. Now *PR* is perpendicular to *PM,* and *RS* is perpendicular to *MQ;* hence *PS* is perpendicular to *PQ* and **PS** = **PQ** × **c**. Then, since **PS** = **PQ** × **c** = **PR** + **RS**, we have

$$(\mathbf{a}_1 + \mathbf{b}_1) \times \mathbf{c} = (\mathbf{a}_1 \times \mathbf{c}) + (\mathbf{b}_1 \times \mathbf{c})$$

By Problem 7, $\mathbf{a}_1$ and $\mathbf{b}_1$ may be replaced by **a** and **b**, respectively, to yield the required result.

9. When $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, show that $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

We have, by the distributive law,

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= (a_1b_2\mathbf{k} - a_1b_3\mathbf{j}) + (-a_2b_1\mathbf{k} + a_2b_3\mathbf{i}) + (a_3b_1\mathbf{j} - a_3b_2\mathbf{i}) \\
&= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$

10. Derive the law of sines of plane trigonometry.

Consider the triangle *ABC,* whose sides **a**, **b**, **c** are of magnitudes *a, b, c,* respectively, and whose interior angles are α, β, γ. We have

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

Then $\quad \mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0} \quad$ or $\quad \mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$

and $\quad \mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = \mathbf{0} \quad$ or $\quad \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$

Thus, $\quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$

so that $\quad |\mathbf{a}||\mathbf{b}| \sin \gamma = |\mathbf{b}||\mathbf{c}| \sin \alpha = |\mathbf{c}||\mathbf{a}| \sin \beta$

or $\quad ab \sin \gamma = bc \sin \alpha = ca \sin \beta$

and $$\frac{\sin \gamma}{c} = \frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

11. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

By (50.13),

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot [(b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}] \\
&= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

12. Show that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) = 0$.

By (50.14), $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{c} = 0$.

13. For the vectors **a**, **b**, and **c** of Problem 11, show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Here

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$= (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times[(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c_3)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}]$$

$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1\end{vmatrix}$$

$$= \mathbf{i}(a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3)+\mathbf{j}(a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1)$$
$$\quad +\mathbf{k}(a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2)$$

$$= \mathbf{i}b_1(a_1c_1+a_2c_2+a_3c_3)+\mathbf{j}b_2(a_1c_1+a_2c_2+a_3c_3)+\mathbf{k}b_3(a_1c_1+a_2c_2+a_3c_3)$$
$$\quad -[\mathbf{i}c_1(a_1b_1+a_2b_2+a_3b_3)+\mathbf{j}c_2(a_1b_1+a_2b_2+a_3b_3)+\mathbf{k}c_3(a_1b_1+a_2b_2+a_3b_3)]$$

$$= (b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})(\mathbf{a}\cdot\mathbf{c})-(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})(\mathbf{a}\cdot\mathbf{b})$$

$$= \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

14. If l_1 and l_2 are two nonintersecting lines in space, show that the shortest distance d between them is the distance from any point on l_1 to the plane through l_2 and parallel to l_1; that is, show that if P_1 is a point on l_1 and P_2 is a point on l_2 then, apart from sign, d is the scalar projection of $\mathbf{P}_1\mathbf{P}_2$ on a common perpendicular to l_1 and l_2.

Let l_1 pass through $P_1(x_1, y_1, z_1)$ in the direction $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and let l_2 pass through $P_2(x_2, y_2, z_2)$ in the direction $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$.

Then $\mathbf{P}_1\mathbf{P}_2 = (x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$, and the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to both l_1 and l_2. Thus,

$$d = \left|\frac{\mathbf{P}_1\mathbf{P}_2\cdot(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}\right| = \left|\frac{(\mathbf{r}_2-\mathbf{r})\cdot(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}\right|$$

15. Write the equation of the line passing through $P_0(1, 2, 3)$ and parallel to $\mathbf{a} = 2\mathbf{i}-\mathbf{j}-4\mathbf{k}$. Which of the points $A(3, 1, -1)$, $B(\frac{1}{2}, \frac{9}{4}, 4)$, $C(2, 0, 1)$ are on this line?

From (50.19), the vector equation is

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k})$$

or

$$(x-1)\mathbf{i}+(y-2)\mathbf{j}+(z-3)\mathbf{k} = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \tag{1}$$

The rectangular equations are

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{-4} \tag{2}$$

Using (2), it is readily found that A and B are on the line while C is not.

In the vector equation (1), a point $P(x, y, z)$ on the line is found by giving k a value and comparing components. The point A is on the line because

$$(3-1)\mathbf{i}+(1-2)\mathbf{j}+(-1-3)\mathbf{k} = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k})$$

when $k = 1$. Similarly B is on the line because

$$-\tfrac{1}{2}\mathbf{i} + \tfrac{1}{4}\mathbf{j} + \mathbf{k} = k(2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$$

when $k = -\frac{1}{4}$. The point C is not on the line because

$$\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = k(2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$$

for no value of k.

16. Write the equation of the plane:

(a) Passing through $P_0(1, 2, 3)$ and parallel to $3x - 2y + 4z - 5 = 0$.

(b) Passing through $P_0(1, 2, 3)$ and $P_1(3, -2, 1)$, and perpendicular to the plane $3x - 2y + 4z - 5 = 0$.

(c) Through $P_0(1, 2, 3)$, $P_1(3, -2, 1)$ and $P_2(5, 0, -4)$.

Let $P(x, y, z)$ be a general point in the required plane.

(a) Here a $= 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ is normal to the given plane and to the required plane. The vector equation of the latter is $(\mathbf{r} - \mathbf{r}_0)\cdot\ \mathbf{a} = \mathbf{0}$ and the rectangular equation is

$$3(x - 1) - 2(y - 2) + 4(z - 3) = 0$$

or

$$3x - 2y + 4z - 11 = 0$$

(b) Here $\mathbf{r}_1 - \mathbf{r}_0 = 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are parallel to the required plane; thus, $(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{a}$ is normal to this plane. Its vector equation is $(\mathbf{r} - \mathbf{r}_0) \cdot [(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{a}] = 0$. The rectangular equation is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -2 \\ 3 & -2 & 4 \end{vmatrix} = [(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 3)\mathbf{k}] \cdot [-20\mathbf{i} - 14\mathbf{j} + 8\mathbf{k}]$$

$$= -20(x - 1) - 14(y - 2) + 8(z - 3) = 0$$

or $20x + 14y - 8z - 24 = 0$.

(c) Here $\mathbf{r}_1 - \mathbf{r}_0 = 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{r}_2 - \mathbf{r}_0 = 4\mathbf{i} = 2\mathbf{j} - 7\mathbf{k}$ are parallel to the required plane, so that $(\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)$ is normal to it. The vector equation is $(\mathbf{r} - \mathbf{r}_0) \cdot [(\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)] = \mathbf{0}$ and the rectangular equation is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -2 \\ 4 & -2 & -7 \end{vmatrix} = [(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 3)\mathbf{k}] \cdot [-24\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}]$$

$$= 24(x - 1) + 6(y - 2) + 12(z - 3) = 0$$

or $4x + y + 2z - 12 = 0$.

17. Find the shortest distance d between the point $P_0(1, 2, 3)$ and the plane Π given by the equation $3x - 2y + 5z - 10 = 0$.

A normal to the plane is $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$. Take $P_1(2, 3, 2)$ as a convenient point in Π. Then, apart from sign, d is the scalar projection of $\mathbf{P}_0\mathbf{P}_1$ on $\mathbf{a}$. Hence,

$$d = \left|\frac{(\mathbf{r}_1 - \mathbf{r}_0)\cdot\mathbf{a}}{|\mathbf{a}|}\right| = \left|\frac{(\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})}{\sqrt{38}}\right| = \frac{2}{19}\sqrt{38}$$

SUPPLEMENTARY PROBLEMS

18. Find the length of (a) the vector $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$; (b) the vector $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} + 9\mathbf{k}$; and (c) the vector **c**, joining $P_1(3, 4, 5)$ to $P_2(1, -2, 3)$.

Ans. (a) $\sqrt{14}$; (b) $\sqrt{115}$; (c) $2\sqrt{11}$

19. For the vectors of Problem 18:

(a) Show that **a** and **b** are perpendicular.
(b) Find the smaller angle between **a** and **c**, and that between **b** and **c**.
(c) Find the angles that **b** makes with the coordinate axes.

Ans. (b) 165°14′, 85°10′; (c) 73°45′, 117°47′, 32°56′

20. Prove: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.

21. Write a unit vector in the direction of **a** and a unit vector in the direction of **b** of Problem 18.

Ans. (a) $\frac{\sqrt{14}}{7}\mathbf{i} + \frac{3\sqrt{14}}{14}\mathbf{j} + \frac{\sqrt{14}}{14}\mathbf{k}$; (b) $\frac{3}{\sqrt{115}}\mathbf{i} - \frac{5}{\sqrt{115}}\mathbf{j} + \frac{9}{\sqrt{115}}\mathbf{k}$

22. Find the interior angles β and γ of the triangle of Problem 3.

Ans. $\beta = 22°12'$; $\gamma = 9°16'$

23. For the unit cube in Fig. 50-14, find (a) the angle between its diagonal and an edge, and (b) the angle between its diagonal and a diagonal of a face.

Ans. (a) 54°44′; (b) 35°16′

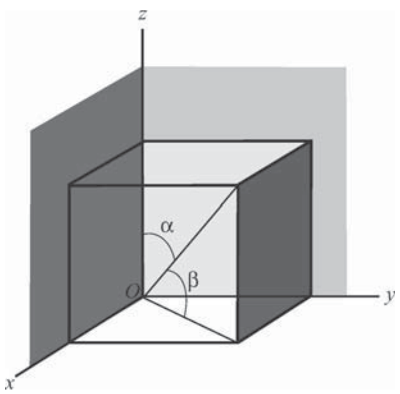

Fig. 50-14

24. Show that the scalar projection of **b** onto **a** is given by $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$.

25. Show that the vector **c** of (50.4) is perpendicular to both **a** and **b**.

26. Given $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, confirm the following equations:

(a) $\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ (b) $\mathbf{b} \times \mathbf{c} = 6\mathbf{i} - 8\mathbf{j} + 3\mathbf{k}$
(c) $\mathbf{c} \times \mathbf{a} = -4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ (d) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$
(e) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ (f) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -2$
(g) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 3\mathbf{i} - 3\mathbf{j} - 14\mathbf{k}$ (h) $\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -11\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$

27. Find the area of the triangle whose vertices are $A(1, 2, 3)$, $B(2, -1, 1)$, and $C(-2, 1, -1)$. (*Hint*: $|\mathbf{AB} \times \mathbf{AC}|$ = twice the area.)

Ans. $5\sqrt{3}$

28. Find the volume of the parallelepiped whose edges are OA, OB, and OC, for $A(1, 2, 3)$, $B(1, 1, 2)$, and $C(2, 1, 1)$.

Ans. 2

29. If $\mathbf{u} = \mathbf{a} \times \mathbf{b}$, $\mathbf{v} = \mathbf{b} \times \mathbf{c}$, $\mathbf{w} = \mathbf{c} \times \mathbf{a}$, show that:

(a) $\mathbf{u} \cdot \mathbf{c} = \mathbf{v} \cdot \mathbf{a} = \mathbf{w} \cdot \mathbf{b}$
(b) $\mathbf{a} \cdot \mathbf{u} = \mathbf{b} \cdot \mathbf{u} = 0$, $\mathbf{b} \cdot \mathbf{v} = \mathbf{c} \cdot \mathbf{v} = 0$, $\mathbf{c} \cdot \mathbf{w} = \mathbf{a} \cdot \mathbf{w} = 0$
(c) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$

30. Show that $(\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})] = 2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

31. Find the smaller angle of intersection of the planes $5x - 14y + 2z - 8 = 0$ and $10x - 11y + 2z + 15 = 0$. (*Hint:* Find the angle between their normals.)

Ans. 22°25′

32. Write the vector equation of the line of intersection of the planes $x + y - z - 5 = 0$ and $4x - y - z + 2 = 0$.

Ans. $(x - 1)\mathbf{i} + (y - 5)\mathbf{j} + (z - 1)\mathbf{k} = k(-2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$, where $P_0(1, 5, 1)$ is a point on the line.

33. Find the shortest distance between the line through $A(2, -1, -1)$ and $B(6, -8, 0)$ and the line through $C(2, 1, 2)$ and $D(0, 2, -1)$.

Ans. $\sqrt{6}/6$

34. Define a line through $P_0(x_0, y_0, z_0)$ as the locus of all points $P(x, y, z)$ such that $\mathbf{P_0P}$ and $\mathbf{OP_0}$ are perpendicular. Show that its vector equation is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{r}_0 = 0$.

35. Find the rectangular equations of the line through $P_0(2, -3, 5)$ and

(a) Perpendicular to $7x - 4y + 2z - 8 = 0$.
(b) Parallel to the line $x - y + 2z + 4 = 0$, $2x + 3y + 6z - 12 = 0$.
(c) Through $P_1(3, 6, -2)$.

Ans. (a) $\frac{x-2}{7} = \frac{y+3}{-4} = \frac{z-5}{2}$; (b) $\frac{x-2}{12} = \frac{y+3}{2} = \frac{z-5}{-5}$; (c) $\frac{x-2}{1} = \frac{y+3}{9} = \frac{z-5}{-7}$

36. Find the equation of the plane:

(a) Through $P_0(1, 2, 3)$ and parallel to $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.
(b) Through $P_0(2, -3, 2)$ and the line $6x + 4y + 3z + 5 = 0$, $2x + y + z - 2 = 0$.
(c) Through $P_0(2, -1, -1)$ and P_1 (1, 2, 3) and perpendicular to $2x + 3y - 5z - 6 = 0$.

Ans. (a) $4x + y + 9z - 33 = 0$; (b) $16x + 7y + 8z - 27 = 0$; (c) $9x - y + 3z - 16 = 0$

37. If $\mathbf{r}_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{r}_1 = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and $\mathbf{r}_2 = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$ are three position vectors, show that $\mathbf{r}_0 \times \mathbf{r}_1 + \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_0 = \mathbf{0}$. What can be said of the terminal points of these vectors?

Ans. They are collinear.

38. If P_0, P_1, and P_2 are three noncollinear points and $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are their position vectors, what is the position of $\mathbf{r}_0 \times \mathbf{r}_1 + \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_0$ with respect to the plane $P_0P_1P_2$?

Ans. normal

39. Prove: (a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$; (b) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$.

40. Prove: (a) The perpendiculars erected at the midpoints of the sides of a triangle meet in a point; (b) the perpendiculars dropped from the vertices to the opposite sides (produced if necessary) of a triangle meet in a point.

41. Let $A(1, 2, 3)$, $B(2, -1, 5)$, and $C(4, 1, 3)$ be three vertices of the parallelogram $ABCD$. Find (a) the coordinates of D; (b) the area of $ABCD$; and (c) the area of the orthogonal projection of $ABCD$ on each of the coordinate planes.

Ans. (a) $D(3, 4, 1)$; (b) $2\sqrt{26}$; (c) 8, 6, 2

42. Prove that the area of a parallogram in space is the square root of the sum of the squares of the areas of projections of the parallelogram on the coordinate planes.