CHAPTER 24

The Fundamental Theorem of Calculus

Mean-Value Theorem for Integrals

Let f be continuous on $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_b^a f(x)dx = (b-a)f(c) \tag{24.1}$$

To see this, let m and M be the minimum and maximum values of f in $[a, b]$, and apply Problem 3(c) of Chapter 23 to obtain

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \qquad \text{and, therefore,} \qquad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

So, by the intermediate value theorm, $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$ for some c in $[a, b]$.

Average Value of a Function on a Closed Interval

Let f be defined on $[a, b]$. Since f may assume infinitely many values on $[a, b]$, we cannot talk about the average of all of the values of f. Instead, divide $[a, b]$ into n equal subintervals, each of $\Delta x = \frac{b-a}{n}$. Select an arbitrary point x_k^* in the kth subinterval. Then the average of the n values $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ is

$$\frac{f(x_1^*) + f(x_2^*) + \ldots + f(x_n^*)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k^*)$$

When n is large, this value is intuitively a good estimate of the "average value of f on $[a, b]$." However, since $\frac{1}{n} = \frac{1}{b-a}\Delta x$,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k^*) = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

As $n \to \infty$, the sum on the right approaches $\int_a^b f(x)\,dx$. This suggests the following definition.

Definition: The *average value* of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Let f be continuous on $[a, b]$. If x is in $[a, b]$, then $\int_a^x f(t)\,dt$ is a function of x, and:

$$D_x\left(\int_a^x f(t)\,dt\right) = f(x) \tag{24.2}$$

For a proof, see Problem 4.

Fundamental Theorem of Calculus

Let f be continuous on $[a, b]$, and let $F(x) = \int f(x)\,dx$, that is, F is an antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{24.3}$$

To see this, note that, by (24.2), $\int_a^x f(t)\,dt$ and $F(x)$ have the same derivative, $f(x)$. Hence, by Problem 18 of Chapter 13, there is a constant K such that $\int_a^x f(t)\,dt = F(x) + K$. When $x = a$, we get

$$F(a) + K = \int_a^a f(t)\,dt = 0 \quad \text{So,} \quad K = -F(a)$$

Hence, $\int_a^x f(t)\,dt = F(x) - F(a)$. When $x = b$, this yields

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Equation (24.3) provides a simple way of computing $\int_a^b f(x)\,dx$ when we can find an antiderivative F of f. The expression $F(b) - F(a)$ on the right side of (24.3) is often abbreviated as $F(x)\Big]_a^b$. Then the fundamental theorem of calculus can be written as follows:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

EXAMPLE 24.1:

(i) The complicated evaluation of $\int_a^b x\,dx$ in Example 23.3 of Chapter 23 can be replaced by the following simple one:

$$\int_a^b x\,dx = \tfrac{1}{2}x^2\Big]_a^b = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2 = \tfrac{1}{2}(b^2 - a^2)$$

(ii) The very tedious computation of $\int_0^1 x^2 dx$ in Problem 4 of Chapter 23 can be replaced by

$$\int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{3}1^3 - \tfrac{1}{3}0^3 = \tfrac{1}{3}$$

(iii) In general, $\displaystyle\int_a^b x^r dx = \frac{1}{r+1}x^{r+1}\Big]_a^b = \frac{1}{r+1}(b^{r+1} - a^{r+1}) \quad \text{for } r \neq -1$

Change of Variable in a Definite Integral

In the computation of a definite integral by the fundamental theorem, an antiderivative $\int f(x)\,dx$ is required. In Chapter 22, we saw that substitution of a new variable u is sometimes useful in finding $\int f(x)\,dx$. When the substitution also is made in the definite integral, the limits of integration must be replaced by the corresponding values of u.

EXAMPLE 24.2: Evaluate $\int_1^9 \sqrt{5x+4}\, dx$.

Let $u = 5x + 4$. Then $du = 5\, dx$. When $x = 1$, $u = 9$, and when $x = 9$, $u = 49$. Hence,

$$\begin{aligned}\int_1^9 \sqrt{5x+4}\, dx &= \int_9^{49} \sqrt{u}\, \tfrac{1}{5}\, du = \tfrac{1}{5}\int_9^{49} u^{1/2}\, du \\ &= \tfrac{1}{5}\left(\tfrac{2}{3}u^{3/2}\right)\Big]_9^{49} \quad \text{(by the fundamental theorem)} \\ &= \tfrac{2}{15}(49^{3/2} - 9^{3/2}) = \tfrac{2}{15}[(\sqrt{49})^3 - (\sqrt{9})^3] \\ &= \tfrac{2}{15}(7^3 - 3^3) = \tfrac{2}{15}(316) = \tfrac{632}{15}\end{aligned}$$

For justification of this method, see Problem 5.

SOLVED PROBLEMS

1. Evaluate $\int_0^{\pi/2} \sin^2 x \cos x\, dx$.

$\int \sin^2 x \cos x\, dx = \frac{1}{3}\sin^3 x$ by Quick Formula I. Hence, by the fundamental theorem,

$$\int_0^{\pi/2} \sin^2 x \cos x\, dx = \tfrac{1}{3}\sin^3 x\Big]_0^{\pi/2} = \tfrac{1}{3}\left[\left(\sin\frac{\pi}{2}\right)^3 - (\sin 0)^3\right] = \tfrac{1}{3}(1^3 - 0^3) = \tfrac{1}{3}$$

2. Find the area under the graph of $f(x) = \frac{1}{\sqrt{4-x^2}}$, above the x axis, and between 0 and 1.

The area is $\int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx = \sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}$.

3. Find the average value of $f(x) = 4 - x^2$ on [0, 2].

The average value is

$$\frac{1}{b-a}\int_a^b f(x)\, dx = \tfrac{1}{2}\int_0^2 (4 - x^2)\, dx = \tfrac{1}{2}\left(4x - \frac{x^3}{3}\right)\Big]_0^2 = \tfrac{1}{2}\left[\left(8 - \tfrac{8}{3}\right) - (0-0)\right] = \tfrac{8}{3}$$

4. Prove formula (24.2): $D_x\left(\int_a^x f(t)\, dt\right) = f(x)$

Let $h(x) = \int_a^x f(t)\, dt$. Then:

$$\begin{aligned}h(x+\Delta x) - h(x) &= \int_a^{x+\Delta x} f(t)\, dt - \int_a^x f(t)\, dt \\ &= \int_a^x f(t)\, dt + \int_x^{x+\Delta x} f(t)\, dt - \int_a^x f(t)\, dt \qquad \text{(by 23.7)} \\ &= \int_x^{x+\Delta x} f(t)\, dt \\ &= \Delta x \cdot f(x^*) \qquad \text{for some } x^* \text{ between } x \text{ and } x+\Delta x \text{ (by the mean value theorem for integrals)}\end{aligned}$$

Thus, $\dfrac{h(x+\Delta x) - h(x)}{\Delta x} = f(x^*)$ and therefore,

$$D_x\left(\int_a^x f(t)\, dt\right) = D_x(h(x)) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} = \lim_{\Delta x \to 0} f(x^*)$$

But, as $\Delta x \to 0$, $x + \Delta x \to x$ and so, $x^* \to x$ (since x^* is between x and $x + \Delta x$). Since f is continuous, $\lim_{\Delta x \to 0} f(x^*) = f(x)$.

5. Justify a change of variable in a definite integral in the following precise sense. Given $\int_a^b f(x)\,dx$, let $x = g(u)$ where, as x varies from a to b, u increases or decreases from c to d. (See Fig. 24-1 for the case where u is increasing.) Show that

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

(The right side is obtained by substituting $g(u)$ for x, $g'(u)\,du$ for dx, and changing the limits of integration from a and b to c and d.)

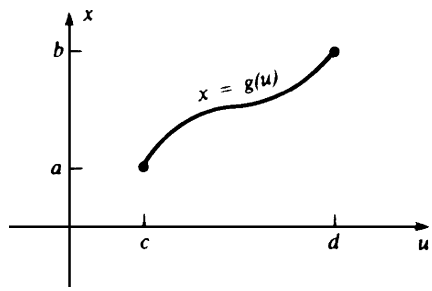

Fig. 24-1

Let $F(x) = \int f(x)\,dx$, that is, $F'(x) = f(x)$. By the Chain Rule,

$$D_u(F(g(u)) = F'(g(u)) \cdot g'(u) = f(g(u))g'(u) \qquad \text{Thus,} \qquad \int f(g(u))g'(u)\,du = F(g(u))$$

So, by the fundamental theorem,

$$\int_c^d f(g(u))g'(u)\,du = F(g(u))\Big]_c^d = F(g(d)) - F(g(c))$$

$$= F(b) - F(a) = \int_a^b f(x)\,dx$$

6. (a) If f is an even function, show that, for $a > 0$, $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is an odd function, show that, for $a > 0$, $\int_{-a}^a f(x)\,dx = 0$.

Let $u = -x$. Then $du = -dx$, and

$$\int_{-a}^0 f(x)\,dx = \int_a^0 f(-u)(-1)\,du = -\int_a^0 f(-u)\,du = \int_0^a f(-u)\,du$$

Rewriting u as x in the last integral, we have:

$$\int_{-a}^0 f(x)\,dx = \int_0^a f(-x)\,dx \qquad (*)$$

Thus,

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \quad \text{(by (23.7))}$$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad \text{(by (*))}$$

$$= \int_0^a f(-x) + f(x)\,dx \quad \text{(by 23.5))}$$

(a) If f is even, $f(-x) + f(x) = 2f(x)$, whence $\int_{-a}^a f(x)\,dx = \int_0^a 2f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, $f(-x) + f(x) = 0$, whence $\int_{-a}^a f(x)\,dx = \int_0^a 0\,dx = 0\int_0^a 1\,dx = 0$.

7. Trapezoidal Rule

(a) Let $f(x) \geq 0$ on $[a, b]$. Divide [a, b] into n equal parts, each of length $\Delta x = \dfrac{b-a}{n}$, by means of points $x_1, x_2, \ldots,$ x_{n-1}. (See Fig. 24-2(*a*).) Prove the following *trapezoidal rule*: $\displaystyle\int_a^b f(x)\,dx \sim \frac{\Delta x}{2}\left(f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right)$

(b) Use the trapezoidal rule with $n = 10$ to approximate $\displaystyle\int_0^1 x^2\,dx$.

(a) The area of the strip, over $[x_{k-1}, x_k]$, is approximately the area of trapezoid *ABCD* (in Fig. 24-2(*b*)):, $\frac{1}{2}\Delta x(f(x_{k-1}) + f(x_k))$† (Remember that $x_0 = a$ and $x_n = b$.) So, the area under the curve is approximated by the sum of the trapezoidal areas,

$$\frac{\Delta x}{2}\{[f(x_0) + f(x_1)] + [f(x_1) + f(x_2)] + \ldots + [f(x_{n-1}) + f(x_n)]\} = \frac{\Delta x}{2}[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)]$$

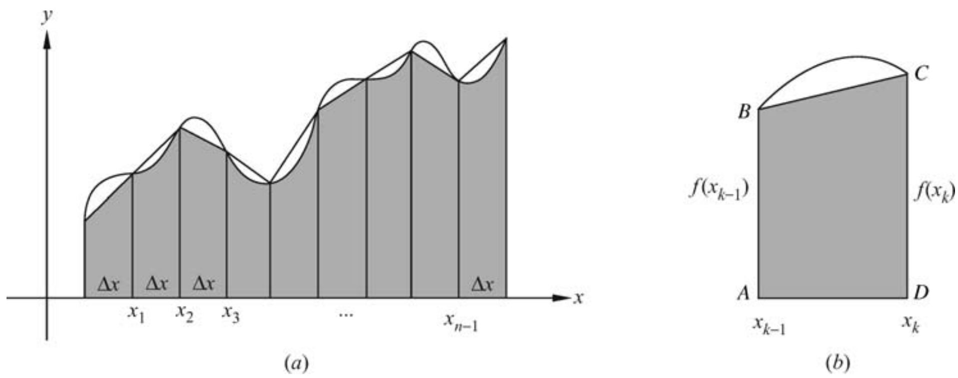

Fig. 24-2

(b) With $n = 10$, $a = 0$, $b = 1$, $\Delta x = \frac{1}{10}$ and $x_k = k/10$, we get

$$\int_0^1 x^2\,dx \sim \frac{1}{20}\left(0^2 + 2\sum_{k=1}^{9}\frac{k^2}{100} + 1^2\right) = \frac{1}{20}\left(\frac{2}{100}\sum_{k=1}^{9} k^2 + 1\right)$$
$$= \frac{1}{20}\left[\frac{2}{100}(285) + 1\right] \qquad \text{(by Problem 12 of Chapter 23)}$$
$$= 0.335$$

The exact value is $\frac{1}{3}$ (by Example 24.1 (ii)).

SUPPLEMENTARY PROBLEMS

In Problems 8–22, use the fundamental theorem of calculus to evaluate the definite integral.

8. $\displaystyle\int_{-1}^{1}(2x^2 - x^3)\,dx$ *Ans.* $\frac{4}{3}$

9. $\displaystyle\int_{-3}^{-1}\left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx$ *Ans.* $\frac{10}{9}$

10. $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}}$ *Ans.* 2

† Recall that the area of a trapezoid of height h and bases b_1 and b_2 is $\frac{1}{2}h(b_1 + b_2)$.

11. $\int_{\pi/2}^{3\pi/4} \sin x\, dx$ *Ans.* $\frac{\sqrt{2}}{2}$

12. $\int_0^2 (2+x)\, dx$ *Ans.* 6

13. $\int_0^2 (2-x)^2\, dx$ *Ans.* $\frac{8}{3}$

14. $\int_0^3 (3-2x+x^2)\, dx$ *Ans.* 9

15. $\int_{-1}^2 (1-t^2)t\, dt$ *Ans.* $-\frac{9}{4}$

16. $\int_1^4 (1-u)\sqrt{u}\, du$ *Ans.* $-\frac{116}{15}$

17. $\int_1^8 \sqrt{1+3x}\, dx$ *Ans.* 26

18. $\int_0^2 x^2(x^3+1)\, dx$ *Ans.* $\frac{40}{3}$

19. $\int_0^3 \frac{1}{\sqrt{1+x}}\, dx$ *Ans.* 2

20. $\int_0^1 x(1-\sqrt{x})^2\, dx$ *Ans.* $\frac{1}{30}$

21. $\int_4^8 \frac{x}{\sqrt{x^2-15}}\, dx$ *Ans.* 6

22. $\int_0^{2\pi} \sin\frac{t}{2}\, dt$ *Ans.* 4

In Problems 23–26, use Problem 6(*a, b*).

23. $\int_{-2}^2 \frac{dx}{x^2+4}\, dx$ *Ans.* $\frac{\pi}{4}$

24. $\int_{-2}^2 (x^3-x^5)\, dx$ *Ans.* 0

25. $\int_{-3}^3 \sin\frac{x}{5}\, dx$ *Ans.* 0

26. $\int_{-\pi/2}^{\pi/2} \cos x\, dx$ *Ans.* 2

27. Prove: $D_x\left(\int_x^b f(t)\, dt\right) = -f(x)$.

28. Prove $D_x\left(\int_{h(x)}^{g(x)} f(t)\, dt\right) = f(g(x))g'(x) - f(h(x))h'(x)$.

In Problems 29–32, use Problems 27–28 and (24.2) to find the given derivative.

29. $D_x\left(\int_1^x \sin t\, dt\right)$ *Ans.* $\sin x$

30. $D_x\left(\int_x^0 t^2\, dt\right)$ *Ans.* $-x^2$

31. $D_x\left(\int_0^{\sin x} t^3\,dt\right)$ *Ans.* $\sin^3 x \cos x$

32. $D_x\left(\int_{x^2}^{4x} \cos t\,dt\right)$ *Ans.* $4\cos 4x - 2x\cos x^2$

33. Compute the average value of the following functions on the indicated intervals.

(a) $f(x)=\sqrt[5]{x}$ on $[0, 1]$ *Ans.* $\frac{5}{6}$

(b) $f(x)=\sec^2 x$ on $\left[0,\frac{\pi}{3}\right]$ *Ans.* $\frac{3\sqrt{3}}{\pi}$

(c) $f(x)=3x^2-1$ on $[-1, 4]$ *Ans.* 12

(d) $f(x)=\sin x-\cos x$ on $[0,\pi]$ *Ans.* $\frac{2}{\pi}$

34. Use the change-of-variables method to find $\int_{1/2}^{3}\sqrt{2x+3}\;x\,dx$.

Ans. $\frac{58}{5}$

35. An object moves along the x axis for a period of time T. If its initial position is x_1 and its final position is x_2, show that its average velocity was $\frac{x_2-x_1}{T}$.

36. Let $f(x)=\begin{cases}\cos x & \text{for } x<0\\ 1-x & \text{for } x\geq 0\end{cases}$. Evaluate $\int_{-\pi/2}^{1} f(x)\,dx$.

Ans. $\frac{3}{2}$

37. Evaluate $\lim_{h\to 0}\frac{1}{h}\int_3^{3+h}\frac{5}{x^3+7}\,dx$.

Ans. $\frac{5}{34}$

38. (**Midpoint Rule**) In an approximating sum (23.1) $\sum_{k=1}^{n} f(x_k^*)\Delta_k x$, if we select x_k^* to be the midpoint of the kth subinterval, then the sum is said to be obtained by the *midpoint rule*. Apply the midpoint rule to approximate $\int_0^1 x^2\,dx$, using a division into five equal subintervals, and compare with the exact result of $\frac{1}{3}$.

Ans. 0.33

39. **(Simpson's Rule)** If we divide $[a, b]$ into n equal subintervals, where n is even, the following approximating sum for $\int_a^b f(x)\,dx$,

$$\frac{b-a}{3n}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+4f(x_{n-1})+f(x_n)]$$

is said to be obtained by *Simpson's rule*. Except for the first and last terms, the coefficients consist of alternating 4s and 2s. (The basic idea is to use parabolas as approximating arcs instead of line segments as in the trapezoidal rule. Simpson's rule is usually much more accurate than the midpoint or trapezoidal rule.)

Apply Simpson's rule to approximate (a) $\int_0^1 x^2\,dx$ and (b) $\int_0^{\pi}\sin x\,dx$ with $n = 4$, and compare the results with the answers obtained by the fundamental theorem.

Ans. (a) $\frac{1}{3}$, which is the exact answer; (b) $\frac{\pi}{6}(2\sqrt{2}+1)\sim 2.0046$ as compared to 2

40. Consider $\int_0^1 x^3\,dx$. (a) Show that the fundamental theorem yields the answer $\frac{1}{4}$. (b) (GC) With $n = 10$, approximate (to four decimal places) the integral by the trapezoidal, midpoint, and Simpson's rules.

Ans. Trapezoidal 0.2525; midpoint 0.2488; Simpson's 0.2500

41. Evaluate:

(a) $\lim_{n\to+\infty} \frac{1}{n}\left(\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n\pi}{n}\right)$

(b) $\lim_{n\to+\infty} \frac{\pi}{6n}\left[\sec^2\left(\frac{\pi}{6n}\right) + \sec^2\left(2\frac{\pi}{6n}\right) + \cdots + \sec^2\left((n-1)\frac{\pi}{6n}\right) + \frac{4}{3}\right]$

Ans. (a) $\frac{1}{\pi}\int_0^{\pi} \cos x\,dx = 0$; (b) $\int_0^{\pi/6} \sec^2 x\,dx = \frac{\sqrt{3}}{3}$

42. (a) Use a substitution to evaluate $\int_1^2 \frac{x}{\sqrt{x+1}}\,dx$ (to eight decimal places).

(b) (GC) Use a graphing calculator to estimate the integral of (a).

Ans. (a) $\frac{2}{3}\sqrt{2}$; (b) 0.39052429

43. (GC) Estimate $\int_0^{\pi/4} x\sin^3(\tan x)\,dx$ (to four decimal places).

Ans. 0.0710

44. (GC) Consider $\int_1^2 x\sqrt[3]{x^5 + 2x^2 - 1}\,dx$. Estimate (to six decimal places) its value using the trapezoidal and Simpson's rule (both with $n = 4$), and compare with the value given by a graphing calculator.

Ans. trapezoidal 3.599492; Simpson's 3.571557; graphing calculator 3.571639