CHAPTER 14

Maximum and Minimum Values

Critical Numbers

A number x_0 in the domain of f such that either $f'(x_0) = 0$ or $f'(x_0)$ is not defined is called a *critical number* of f.

Recall (Theorem 13.1) that, if f has a relative extremum at x_0 and $f'(x_0)$ is defined, then $f'(x_0) = 0$ and, therefore, x_0 is a critical number of f. Observe, however, that the condition that $f'(x_0) = 0$ does not guarantee that f has a relative extremum at x_0. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$, and therefore, 0 is a critical number of f; but f has neither a relative maximum nor a relative minimum at 0. (See Fig. 5-5).

EXAMPLE 14.1:

(a) Let $f(x) = 7x^2 - 3x + 5$. Then $f'(x) = 14x - 3$. Set $f'(x) = 0$ and solve. The only critical number of f is $\frac{3}{14}$.
(b) Let $f(x) = x^3 - 2x^2 + x + 1$. Then $f'(x) = 3x^2 - 4x + 1$. Solving $f'(x) = 0$, we find that the critical numbers are 1 and $\frac{1}{3}$.
(c) Let $f(x) = x^{2/3}$. Then $f'(x) = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$. Since $f'(0)$ is not defined, 0 is the only critical number of f.

We shall find some conditions under which we can conclude that a function f has a relative maximum or a relative minimum at a given critical number.

Second Derivative Test for Relative Extrema

Assume that $f'(x_0) = 0$ and that $f''(x_0)$ exists. Then:

(i) if $f''(x_0) < 0$, then f has a relative maximum at x_0;
(ii) if $f''(x_0) > 0$, then f has a relative minimum at x_0;
(iii) if $f''(x_0) = 0$, we do not know what is happening at x_0.

A proof is given in Problem 9. To see that (iii) holds, consider the three functions $f(x) = x^4$, $g(x) = -x^4$, and $h(x) = x^3$. Since $f'(x) = 4x^3$, $g'(x) = -4x^3$, and $h'(x) = 3x^2$, 0 is a critical number of all three functions. Since $f''(x) = 12x^2$, $g''(x) = -12x^2$, and $h''(x) = 6x$, the second derivative of all three functions is 0 at 0. However, f has a relative minimum at 0, g has a relative maximum at 0, and h has neither a relative maximum nor a relative minimum at 0.

EXAMPLE 14.2:

(a) Consider the function $f(x) = 7x^2 - 3x + 5$ of Example 1(*a*). The only critical number was $\frac{3}{14}$. Since $f''(x) = 14$, $f''(\frac{3}{14}) = 14 > 0$. So, the second derivative test tells us that f has a relative minimum at $\frac{3}{14}$.
(b) Consider the function $f(x) = x^3 - 2x^2 + x + 1$ of Example 1(*b*). Note that $f''(x) = 6x - 4$. At the critical numbers 1 and $\frac{1}{3}$, $f''(1) = 2 > 0$ and $f''(\frac{1}{3}) = -2 < 0$. Hence f has a relative minimum at 1 and a relative maximum at $\frac{1}{3}$.
(c) In Example 1(*c*), $f(x) = x^{2/3}$ and $f'(x) = \frac{2}{3}x^{-1/3}$. The only critical number is 0, where f' is not defined. Hence, $f''(0)$ is not defined and the second derivative test is not applicable.

If the second derivative test is not usable or convenient, either because the second derivative is 0, or does not exist, or is difficult to compute, then the following test can be applied. Recall that $f'(x)$ is the slope of the tangent line to the graph of f at x.

First Derivative Test

Assume $f'(x_0) = 0$.

Case {+, −}

If f' is positive in an open interval immediately to the left of x_0, and negative in an open interval immediately to the right of x_0, then f has a relative maximum at x_0. (See Fig. 14-1(*a*).)

Case {−, +}

If f' is negative in an open interval immediately to the left of x_0, and positive in an open interval immediately to the right of x_0, then f has a relative minimum at x_0. (See Fig. 14-1(*b*).)

Cases {+, +} and {−, −}

If f' has the same sign in open intervals immediately to the left and to the right of x_0, then f has neither a relative maximum nor a relative minimum at x_0. (See Fig. 14-1(*c, d*).)

For a proof of the first derivative test, see Problem 8.

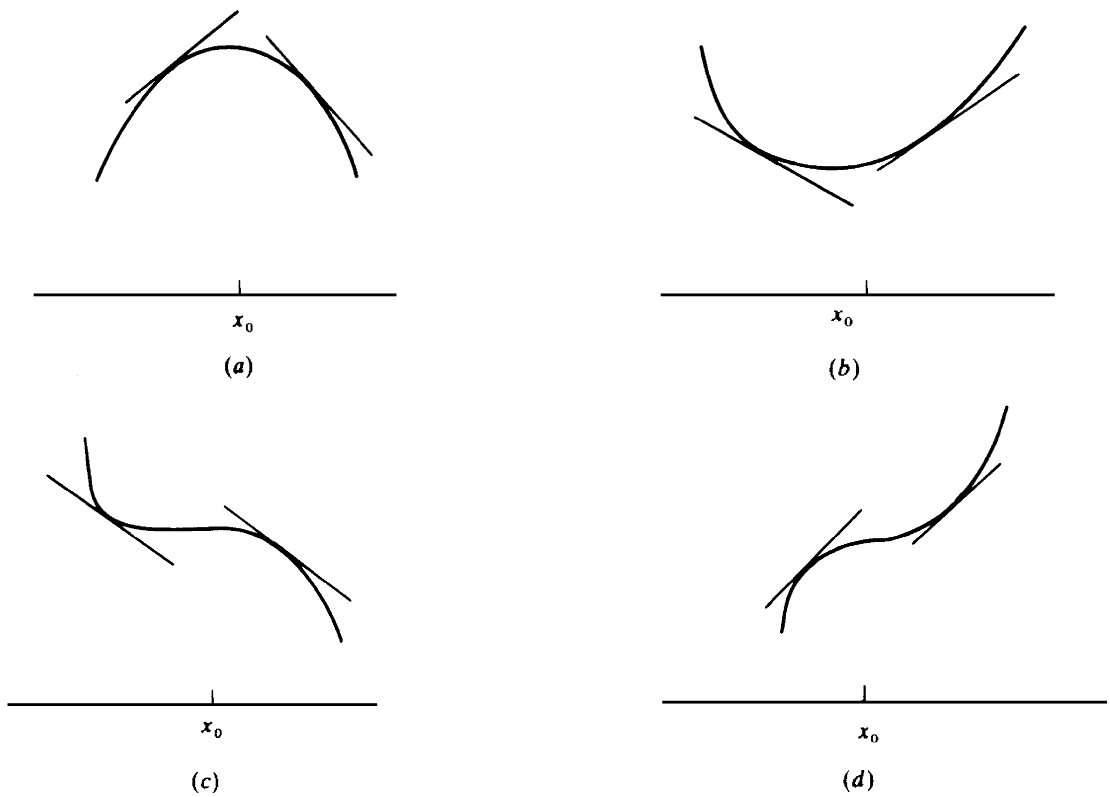

Fig. 14-1

EXAMPLE 14.3: Consider the three functions $f(x) = x^4$, $g(x) = -x^4$, and $h(x) = x^3$ discussed above. At their critical number 0, the second derivative test was not applicable because the second derivative was 0. Let us try the first derivative test.

(a) $f'(x) = 4x^3$. To the left of 0, $x < 0$, and so, $f'(x) < 0$. To the right of 0, $x > 0$, and so, $f'(x) > 0$. Thus, we have the case {−, +} and f must have a relative minimum at 0.
(b) $g'(x) = -4x^3$. To the left of 0, $x < 0$, and so, $g'(x) > 0$. To the right of 0, $x > 0$, and so, $g'(x) < 0$. Thus, we have the case {+, −} and g must have a relative maximum at 0.
(c) $h'(x) = 3x^2$. $h'(x) > 0$ on both sides of 0. Thus, we have the case {+, +} and h has neither a relative maximum nor a relative minimum at 0. There is an *inflection point* at $x = 0$.

These results can be verified by looking at the graphs of the functions.

Absolute Maximum and Minimum

An *absolute maximum* of a function f on a set S occurs at x_0 in S if $f(x) \le f(x_0)$ for all x in S. An *absolute minimum* of a function f on a set S occurs at x_0 in S if $f(x) \ge f(x_0)$ for all x in S.

Tabular Method for Finding the Absolute Maximum and Minimum

Let f be continuous on $[a, b]$ and differentiable on (a, b). By the Extreme Value Theorem, we know that f has an absolute maximum and minimum on $[a, b]$. Here is a tabular method for determining what they are and where they occur. (See Fig. 14-2.)

x	$f(x)$
c_1	$f(c_1)$
c_2	$f(c_2)$
....	....
....	....
c_n	$f(c_n)$
a	$f(a)$
b	$f(b)$

Fig. 14-2

First, find the critical numbers (if any) $c_1, c_2, \ldots$ of f in (a, b). Second, list these numbers in a table, along with the endpoints a and b of the interval. Third, calculate the value of f for all the numbers in the table.

Then:

1. The largest of these values is the absolute maximum of f on $[a, b]$.
2. The smallest of these values is the absolute minimum of f on $[a, b]$.

EXAMPLE 14.4: Let us find the absolute maximum and minimum of $f(x) = x^3 - x^2 - x + 2$ on $[0, 2]$.

$f'(x) = 3x^2 - 2x - 1 = (3x + 1)(x - 1)$. Hence, the critical numbers are $x = -\frac{1}{3}$ and $x = 1$. The only critical number in $[0, 2]$ is 1. From the table in Fig. 14-3, we see that the maximum value of f on $[0, 2]$ is 4, which is attained at the right endpoint 2, and the minimum value is 1, attained at 1.

x	$f(x)$
1	1
0	2
2	4

Fig. 14-3

Let us see why the method works. By the Extreme Value Theorem, f achieves maximum and minimum values on the closed interval $[a, b]$. If either of those values occurs at an endpoint, that value will appear in the table and, since it is actually a maximum or minimum, it will show up as the largest or smallest value. If the maximum or minimum is assumed at a point x_0 inside the interval, f has a relative maximum or minimum at x_0 and, therefore, by Theorem 13.1, $f'(x_0) = 0$. Thus, x_0 will be a critical number and will be listed in the table, so that the corresponding maximum or minimum value $f(x_0)$ will be the largest or smallest value in the right-hand column.

Theorem 14.1: Assume that f is a continuous function defined on an interval J. The interval J can be a finite or infinite interval. If f has a unique relative extremum within J, then that relative extremum is also an absolute extremum on J.

To see why this is so, look at Fig. 14-4, where f is assumed to have a unique extremum, a relative maximum at c. Consider any other number d in J. The graph moves downward on both sides of c. So, if $f(d)$

were greater than $f(c)$, then, by the Extreme Value Theorem for the closed interval with endpoints c and d, f would have an absolute minimum at some point u between c and d. (u could not be equal to c or d.) Then f would have a relative minimum at u, contradicting our hypothesis that f has a relative extremum only at c. We can extend this argument to the case where f has a relative minimum at c by applying the result we have just obtained to $-f$.

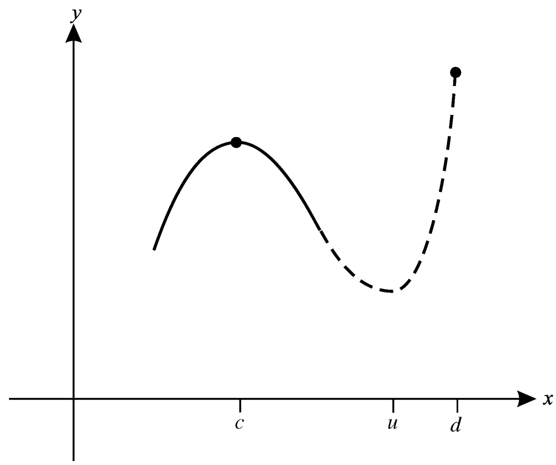

Fig. 14-4

SOLVED PROBLEMS

1. Locate the absolute maximum or minimum of the following functions on their domains:

(a) $y = -x^2$; (b) $y = (x - 3)^2$; (c) $y = \sqrt{25 - 4x^2}$; (d) $y = \sqrt{x - 4}$.

(a) $y = -x^2$ has an absolute maximum (namely, 0) when $x = 0$, since $y < 0$ when $x \neq 0$. It has no relative minimum, since its range is $(-\infty, 0)$. The graph is a parabola opening downward, with vertex at (0, 0).

(b) $y = (x - 3)^2$ has an absolute minimum, 0, when $x = 3$, since $y > 0$ when $x \neq 3$. It has no absolute maximum, since its range is $(0, +\infty)$. The graph is a parabola opening upward, with vertex at (3, 0).

(c) $y = \sqrt{25 - 4x^2}$ has 5 as its absolute maximum, when $x = 0$, since $25 - 4x^2 < 25$ when $x \neq 0$. It has 0 as its absolute minimum, when $x = \frac{5}{2}$. The graph is the upper half of an ellipse.

(d) $y = \sqrt{x - 4}$ has 0 as its absolute minimum when $x = 4$. It has no absolute maximum. Its graph is the upper half of a parabola with vertex at (4, 0) and the x axis as its axis of symmetry.

2. Let $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$. Find: (a) the critical numbers of f; (b) the points at which f has a relative maximum or minimum; (c) the intervals on which f is increasing or decreasing.

(a) $f'(x) = x^2 + x - 6 = (x + 3)(x - 2)$. Solving $f'(x) = 0$ yields the critical numbers -3 and 2.

(b) $f''(x) = 2x + 1$. Thus, $f''(-3) = -5 < 0$ and $f''(2) = 5$. Hence, by the second derivative test, f has a relative maximum at $x = -3$, where $f(-3) = \frac{43}{2}$. By the second derivative test, f has a relative minimum at $x = 2$, where $f(2) = \frac{2}{3}$.

(c) Look at $f'(x) = (x + 3)(x - 2)$. When $x > 2$, $f'(x) > 0$. For $-3 < x < 2$, $f'(x) < 0$. For $x < -3$, $f'(x) > 0$. Thus, by Theorem 13.7, f is increasing for $x < -3$ and $2 < x$, and decreasing for $-3 < x < 2$.

A sketch of part of the graph of f is shown in Fig. 14-5. Note that f has neither absolute maximum nor absolute minimum.

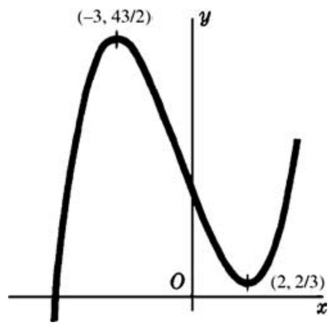

Fig. 14-5

3. Let $f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$. Find: (a) the critical numbers of f; (b) the points at which f has a relative extremum; (c) the intervals on which f is increasing or decreasing.

(a) $f'(x) = 4x^3 + 6x^2 - 6x - 4$. It is clear that $x = 1$ is a zero of $f'(x)$. Dividing $f'(x)$ by $x - 1$ yields $4x^2 + 10x + 4$, which factors into $2(2x^2 + 5x + 2) = 2(2x + 1)(x + 2)$. Thus, $f'(x) = 2(x - 1)(2x + 1)(x + 2)$, and the critical numbers are 1, $-\frac{1}{2}$, and -2.

(b) $f''(x) = 12x^2 + 12x - 6 = 6(2x^2 + 2x - 1)$. Using the second derivative test, we find: (i) at $x = 1, f''(1) = 18 > 0$, and there is a relative minimum; (ii) at $x = -\frac{1}{2}$, $f''(-\frac{1}{2}) = -9 < 0$, so that there is a relative maximum; (iii) at $x = -2, f''(-2) = 18 > 0$, so that there is a relative minimum.

(c) $f'(x) > 0$ when $x > 1, f'(x) < 0$ when $-\frac{1}{2} < x < 1$, $f'(x) > 0$ when $-2 < x < -\frac{1}{2}$, and $f'(x) < 0$ when $x < -2$. Hence, f is increasing when $x > 1$ or $-2 < x < -\frac{1}{2}$, and decreasing when $-\frac{1}{2} < x < 1$ or $x < -2$.

The graph is sketched in Fig. 14-6.

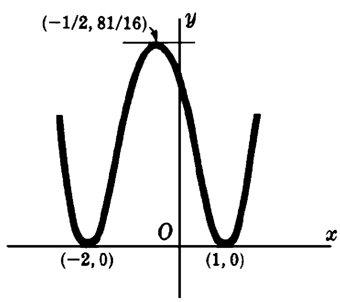

Fig. 14-6

4. Examine $f(x) = \dfrac{1}{x-2}$ for relative extrema, and find the intervals on which f is increasing or decreasing.

$f(x) = (x - 2)^{-1}$, so that $f'(x) = -(x-2)^{-2} = -\dfrac{1}{(x-2)^2}$. Thus, f' is never 0, and the only number where f' is not defined is the number 2, which is not in the domain of f. Hence, f has no critical numbers. So, f has no relative extrema. Note that $f'(x) < 0$ for $x \neq 2$. Hence, f is decreasing for $x < 2$ and for $x > 2$. There is a nonremovable discontinuity at $x = 2$. The graph is shown in Fig. 14-7.

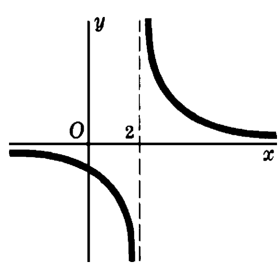

Fig. 14-7

5. Locate the relative extrema of $f(x) = 2 + x^{2/3}$ and the intervals on which f is increasing or decreasing.

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$. Then $x = 0$ is a critical number, since $f'(0)$ is not defined (but 0 is in the domain of f). Note that $f'(x)$ approaches ∞ as x approaches 0. When $x < 0$, $f'(x)$ is negative and, therefore, f is decreasing. When $x > 0$, $f'(x)$ is positive and, therefore, f is increasing. The graph is sketched in Fig. 14-8. f has an absolute minimum at $x = 0$.

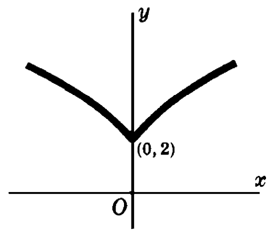

Fig. 14-8

6. Use the second derivative test to examine the relative extrema of the following functions: (a) $f(x) = x(12 - 2x)^2$; (b) $f(x) = x^2 + \frac{250}{x}$.

(a) $f'(x) = x(2)(12 - 2x)(-2) + (12 - 2x)^2 = (12 - 2x)(12 - 6x) = 12(x - 6)(x - 2)$. So, 6 and 2 are the critical numbers. $f''(x) = 12(2x - 8) = 24(x - 4)$. So, $f''(6) = 48 > 0$, and $f''(2) = -48 < 0$. Hence, f has a relative minimum at $x = 6$ and a relative maximum at $x = 2$.

(b) $f'(x) = 2x - \frac{250}{x^2} = 2\left(\frac{x^3 - 125}{x^2}\right)$. So, the only critical number is 5 (where $x^3 - 125 = 0$). $f''(x) = 2 + 500/x^3$. Since $f''(5) = 6 > 0$, f has a relative minimum at $x = 5$.

7. Determine the relative extrema of $f(x) = (x - 2)^{2/3}$.

$f'(x) = \frac{2}{3(x-2)^{2/3}}$. So, 2 is the only critical number. Since $f'(2)$ is not defined, $f''(2)$ will be undefined. Hence, we shall try the first derivative test. For $x < 2$, $f'(x) < 0$, and, for $x > 2$, $f'(x) > 0$. Thus, we have the case $\{-, +\}$ of the first derivative test, and f has a relative minimum at $x = 2$.

8. Prove the first derivative test.

Assume $f'(x_0) = 0$. Consider the case $\{+, -\}$: If f' is positive in an open interval immediately to the left of x_0, and negative in an open interval immediately to the right of x_0, then f has a relative maximum at x_0. To see this, notice that, by Theorem 13.8, since f' is positive in an open interval immediately to the left of x_0, f is increasing in that interval, and, since f' is negative in an open interval immediately to the right of x_0, f is decreasing in that interval. Hence, f has a relative maximum at x_0. The case $\{-, +\}$ follows from the case $\{+, -\}$ applied to $-f$. In the case $\{+, +\}$, f will be increasing in an interval around x_0, and, in the case $\{-, -\}$, f will be decreasing in an interval around x_0. So, in both cases, f has neither a relative maximum nor minimum at x_0.

9. Prove the second derivative test: If $f(x)$ is differentiable on an open interval containing a critical value x_0 of f, and $f''(x_0)$ exists and $f''(x_0)$ is positive (negative), then f has a relative minimum (maximum) at x_0.

Assume $f''(x_0) > 0$. Then, by Theorem 13.8, f' is increasing at x_0. Since $f'(x_0) = 0$, this implies that f' is negative nearby and to the left of x_0, and f' is positive nearby and to the right of x_0. Thus, we have the case $\{-, +\}$ of the first derivative test and, therefore, f has a relative minimum at x_0. In the opposite situation, where $f''(x_0) < 0$, the result we have just proved is applicable to the function $g(x) = -f(x)$. Then g has a relative minimum at x_0 and, therefore, f has a relative maximum at x_0.

10. Among those positive real numbers u and v whose sum is 50, find that choice of u and v that makes their product P as large as possible.

$P = u(50 - u)$. Here, u is any positive number less than 50. But we also can allow u to be 0 or 50, since, in those cases, $P = 0$, which will certainly not be the largest possible value. So, P is a continuous function $u(50 - u)$, defined on [0, 50]. $P = 50u - u^2$ is also differentiable everywhere, and $dP/du = 50 - 2u$. Setting $dP/du = 0$ yields a unique critical number $u = 25$. By the tabular method (Fig. 14-9), we see that the maximum value of P is 625, when $u = 25$ (and, therefore, $v = 50 - u = 25$).

u	P
25	625
0	0
50	0

Fig. 14-9

11. Divide the number 120 into two parts such that the product P of one part and the square of the other is a maximum.

Let x be one part and $120 - x$ the other part. Then $P = (120 - x)x^2$ and $0 \le x \le 120$. Since $dP/dx = 3x(80 - x)$, the critical numbers are 0 and 80. Using the tabular method, we find $P(0) = 0$, $P(80) = 256{,}000$ and $P(120) = 0$. So, the maximum value occurs when $x = 80$, and the required parts are 80 and 40.

12. A sheet of paper for a poster is to be 18 ft^2 in area. The margins at the top and bottom are to be 9 inches, and the margins at the sides 6 inches. What should be the dimensions of the sheet to maximize the printed area?

Let x be one dimension, measured in feet. Then $18/x$ is the other dimension. (See Fig. 14-10.) The only restriction on x is that $x > 0$. The printed area in square feet is $A = (x-1)\left(\frac{18}{x} - \frac{3}{2}\right)$, and $\frac{dA}{dx} = \frac{18}{x^2} - \frac{3}{2}$.

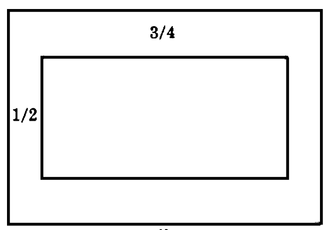

Fig. 14-10

Solving $dA/dx = 0$ yields the critical number $x = 2\sqrt{3}$. Since $d^2A/dx^2 = -36/x^3$ is negative when $x = 2\sqrt{3}$, the second derivative test tells us that A has a relative maximum at $x = 2\sqrt{3}$. Since $2\sqrt{3}$ is the only critical number in the interval $(0, +\infty)$, Theorem 14.1 tells us that A has an absolute maximum at $x = 2\sqrt{3}$. Thus, one side is $2\sqrt{3}$ ft and the other side is $18/(2\sqrt{3}) = 3\sqrt{3}$ ft.

13. At 9 A.M., ship B is 65 miles due east of another ship A. Ship B is then sailing due west at 10 mi/h, and A is sailing due south at 15 mi/h. If they continue on their respective courses, when will they be nearest one another, and how near? (See Fig. 14–11.)

Let A_0 and B_0 be the positions of the ships at 9 A.M., and A_t and B_t their positions t hours later. The distance covered in t hours by A is $15t$ miles; by B, $10t$ miles. The distance D between the ships is determined by $D^2 = (15t)^2 + (65 - 10t)^2$. Then

$$2D\frac{dD}{dt} = 2(15t)(15) + 2(65 - 10t)(-10); \text{ hence, } \frac{dD}{dt} = \frac{325t - 650}{D}$$

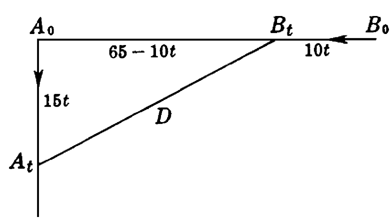

Fig. 14-11

Solving $dD/dt = 0$, yields the critical number $t = 2$. Since $D > 0$ and $325t - 650$ is negative to the left of 2 and positive to the right of 2, the case (–, +) of the first derivative test tells us that $t = 2$ yields a relative minimum for D. Since $t = 2$ is the only critical number, Theorem 14.1 implies that there is an absolute minimum at $t = 2$.

Setting $t = 2$ in $D^2 = (15t)^2 + (65 - 10t)^2$ yields $D = 15\sqrt{13}$ miles. Hence, the ships are nearest at 11 A.M., at which time they are $15\sqrt{13}$ miles apart.

14. A cylindrical container with circular base is to hold 64 in³. Find its dimensions so that the amount (surface area) of metal required is a minimum when the container is (a) an open can and (b) a closed can.

Let r and h be, respectively, the radius of the base and the height in inches, A the amount of metal, and V the volume of the container.

(a) Here $V = \pi r^2 h = 64$, and $A = 2\pi rh + \pi r^2$. To express A as a function of one variable, we solve for h in the first relation (because it is easier) and substitute in the second, obtaining

$$A = 2\pi r\frac{64}{\pi r^2} + \pi r^2 = \frac{128}{r} + \pi r^2 \quad \text{and} \quad \frac{dA}{dr} = -\frac{128}{r^2} + 2\pi r = \frac{2(\pi r^3 - 64)}{r^2}$$

and the critical number is $r = 4/\sqrt[3]{\pi}$. Then $h = 64/\pi r^2 = 4/\sqrt[3]{\pi}$. Thus, $r = h = 4/\sqrt[3]{\pi}$ in.

Now $dA/dr > 0$ to the right of the critical number, and $dA/dr < 0$ to the left of the critical number. So, by the first derivative test, we have a relative minimum. Since there is no other critical number, that relative minimum is an absolute minimum.

(b) Here again $V = \pi r^2 h = 64$, but $A = 2\pi rh + 2\pi r^2 = 2\pi r(64/\pi r^2) + 2\pi r^2 = 128/r + 2\pi r^2$. Hence,

$$\frac{dA}{dr} = -\frac{128}{r^2} + 4\pi r = \frac{4(\pi r^3 - 32)}{r^2}$$

and the critical number is $r = 2\sqrt[3]{4/\pi}$. Then $h = 64/\pi r^2 = 4\sqrt[3]{4/\pi}$. Thus, $h = 2r = 4\sqrt[3]{4/\pi}$ in. That we have found an absolute minimum can be shown as in part (a).

15. The total cost of producing x radio sets per day is $\$(\frac{1}{4}x^2 + 35x + 25)$, and the price per set at which they may be sold is $\$(50 - \frac{1}{2}x)$.

(a) What should be the daily output to obtain a maximum total profit?
(b) Show that the cost of producing a set is a relative minimum at that output.

(a) The profit on the sale of x sets per day is $P = x(50 - \frac{1}{2}x) - (\frac{1}{4}x^2 + 35x + 25)$. Then $dP/dx = 15 - 3x/2$; solving $dP/dx = 0$ gives the critical number $x = 10$.

Since $d^2P/dx^2 = -\frac{3}{2} < 0$, the second derivative test shows that we have found a relative maximum. Since $x = 10$ is the only critical number, the relative maximum is an absolute maximum. Thus, the daily output that maximizes profit is 10 sets per day.

(b) The cost of producing a set is $C = \dfrac{\frac{1}{4}x^2 + 35x + 25}{x} = \dfrac{1}{4}x + 35 + \dfrac{25}{x}$. Then $\dfrac{dC}{dx} = \dfrac{1}{4} - \dfrac{25}{x^2}$; solving $dC/dx = 0$ gives the critical number $x = 10$.

Since $d^2C/dx^2 = 50/x^3 > 0$ when $x = 10$, we have found a relative minimum. Since there is only one critical number, this must be an absolute minimum.

16. The cost of fuel to run a locomotive is proportional to the square of the speed and \$25 per hour for a speed of 25 miles per hour. Other costs amount to \$100 per hour, regardless of the speed. Find the speed that minimizes the cost per mile.

Let v be the required speed, and let C be the total cost per mile. The fuel cost per hour is kv^2, where k is a constant to be determined. When $v = 25$ mi/h, $kv^2 = 625k = 25$; hence, $k = 1/25$.

$$C = \frac{\text{cost in \$/h}}{\text{speed in mi/h}} = \frac{v^2/25 + 100}{v} = \frac{v}{25} + \frac{100}{v}$$

Then $$\frac{dC}{dv} = \frac{1}{25} - \frac{100}{v^2} = \frac{(v-50)(v+50)}{25v^2}$$

Since $v > 0$, the only relevant critical number is $v = 50$. Since $d^2C/dv^2 = 200/v^3 > 0$ when $v = 50$, the second derivative test tells us that C has a relative minimum at $v = 50$. Since $v = 50$ is the only critical number in $(0, +\infty)$, Theorem 14.1 tells us that C has an absolute minimum at $v = 50$. Thus, the most economical speed is 50 mi/h.

17. A man in a rowboat at P in Fig. 14-12, 5 miles from the nearest point A on a straight shore, wishes to reach a point B, 6 miles from A along the shore, in the shortest time. Where should he land if he can row 2 mi/h and walk 4 mi/h?

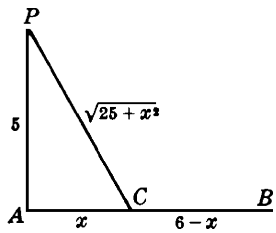

Fig. 14-12

Let C be the point between A and B at which the man lands, and let $AC = x$. The distance rowed is $PC = \sqrt{25+x^2}$, and the rowing time required is $t_1 = \frac{\text{distance}}{\text{speed}} = \frac{\sqrt{25+x^2}}{2}$. The distance walked is $CB = 6 - x$, and the walking time required is $t_2 = (6 - x)/4$. Hence, the total time required is

$$t = t_1 + t_2 = \frac{\sqrt{25+x^2}}{2} + \frac{6-x}{4} \qquad \text{Then } \frac{dt}{dx} = \frac{x}{2\sqrt{25+x^2}} - \frac{1}{4} = \frac{2x - \sqrt{25+x^2}}{4\sqrt{25+x^2}}.$$

The critical number obtained from setting $2x - \sqrt{25+x^2} = 0$ is $x = \frac{5}{3}\sqrt{3} \sim 2.89$. Thus, he should land at a point about 2.89 miles from A toward B. (How do we know that this point yields the *shortest* time?)

18. A given rectangular area is to be fenced off in a field that lies along a straight river. If no fencing is needed along the river, show that the least amount of fencing will be required when the length of the field is twice its width.

Let x be the length of the field, and y its width. The area of the field is $A = xy$. The fencing required is $F = x + 2y$, and $dF/dx = 1 + 2\,dy/dx$. When $dF/dx = 0$, $dy/dx = -\frac{1}{2}$.

Also, $dA/dx = 0 = y + x\,dy/dx$. Then $y - \frac{1}{2}x = 0$, and $x = 2y$ as required.

To see that F has been minimized, note that $dy/dx = -y^2/A$ and

$$\frac{d^2F}{dx^2} = 2\frac{d^2y}{dx^2} = 2\left(-2\frac{y}{A}\frac{dy}{dx}\right) = -4\frac{y}{A}\left(-\frac{1}{2}\right) = 2\frac{y}{A} > 0 \quad \text{when} \quad \frac{dy}{dx} = -\frac{1}{2}$$

Now use the second derivative test and the uniqueness of the critical number.

19. Find the dimensions of the right circular cone of minimum volume V that can be circumscribed about a sphere of radius 8 inches.

Let x be the radius of the base of the cone, and $y + 8$ the height of the cone. (See Fig. 14-13.) From the similar right triangles ABC and AED, we have

$$\frac{x}{8} = \frac{y+8}{\sqrt{y^2-64}} \quad \text{and therefore} \quad x^2 = \frac{64(y+8)^2}{y^2-64}.$$

Also, $$V = \frac{\pi x^2(y+8)}{3} = \frac{64\pi(y+8)^2}{3(y-8)}. \qquad \text{So,} \quad \frac{dV}{dy} = \frac{64\pi(y+8)(y-24)}{3(y-8)^2}.$$

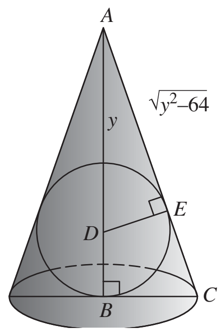

Fig. 14-13

The relevant critical number is $y = 24$. Then the height of the cone is $y + 8 = 32$ inches, and the radius of the base is $8\sqrt{2}$ inches. (How do we know that the volume has been minimized?)

20. Find the dimensions of the rectangle of maximum area A that can be inscribed in the portion of the parabola $y^2 = 4px$ intercepted by the line $x = a$.

Let $PBB'P'$ in Fig. 14-14 be the rectangle, and (x, y) the coordinates of P. Then

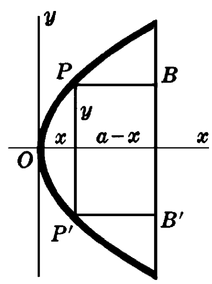

Fig. 14-14

$$A = 2y(a-x) = 2y\left(a - \frac{y^2}{4p}\right) = 2ay - \frac{y^3}{2p} \qquad \text{and} \qquad \frac{dA}{dy} = 2a - \frac{3y^2}{2p}$$

Solving $dA/dy = 0$ yields the critical number $y = \sqrt{4ap/3}$. The dimensions of the rectangle are $2y = \frac{4}{3}\sqrt{3ap}$ and $a - x = a - (y^2/4p) = 2a/3$.

Since $d^2A/dy^2 = -3y/p < 0$, the second derivative test and the uniqueness of the critical number ensure that we have found the maximum area.

21. Find the height of the right circular cylinder of maximum volume V that can be inscribed in a sphere of radius R. (See Fig. 14-15.)

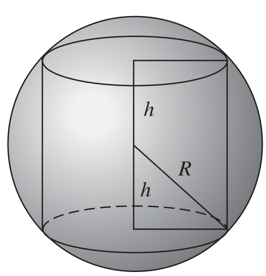

Fig. 14-15

Let r be the radius of the base, and $2h$ the height of the cylinder. From the geometry, $V = 2\pi r^2 h$ and $r^2 + h^2 = R^2$. Then

$$\frac{dV}{dr} = 2\pi\left(r^2\frac{dh}{dr} + 2rh\right) \quad \text{and} \quad 2r + 2h\frac{dh}{dr} = 0$$

From the last relation, $\frac{dh}{dr} = -\frac{r}{h}$, so $\frac{dV}{dr} = 2\pi\left(-\frac{r^3}{h} + 2rh\right)$. When V is a maximum, $\frac{dV}{dr} = 0$, from which $r^2 = 2h^2$.

Then $R^2 = r^2 + h^2 = 2h^2 + h^2$, so that $h = R/\sqrt{3}$ and the height of the cylinder is $2h = 2R/\sqrt{3}$. The second-derivative test can be used to verify that we have found a maximum value of V.

22. A wall of a building is to be braced by a beam that must pass over a parallel wall 10 ft high and 8 ft from the building. Find the length L of the shortest beam that can be used.

See Fig. 14-16. Let x be the distance from the foot of the beam to the foot of the parallel wall, and let y be the distance (in feet) from the ground to the top of the beam. Then $L = \sqrt{(x+8)^2 + y^2}$.

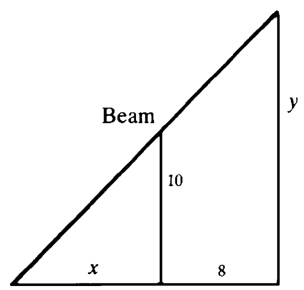

Fig. 14-16

Also, from similar triangles, $\frac{y}{10} = \frac{x+8}{x}$ and, therefore, $y = \frac{10(x+8)}{x}$. Hence,

$$L = \sqrt{(x+8)^2 + \frac{100(x+8)^2}{x^2}} = \frac{x+8}{x}\sqrt{x^2+100}$$

$$\frac{dL}{dx} = \frac{x[(x^2+100)^{1/2} + x(x+8)(x^2+100)^{-1/2}] - (x+8)(x^2+100)^{1/2}}{x^2} = \frac{x^3 - 800}{x^2\sqrt{x^2+100}}$$

The relevant critical number is $x = 2\sqrt[3]{100}$. The length of the shortest beam is

$$\frac{2\sqrt[3]{100}+8}{2\sqrt[3]{100}}\sqrt{4\sqrt[3]{10{,}000}+100} = (\sqrt[3]{100}+4)^{3/2} \text{ ft}$$

The first derivative test and Theorem 14.1 guarantee that we really have found the shortest length.

SUPPLEMENTARY PROBLEMS

23. Examine each of the following for relative maximum and minimum values, using the first derivative test.

(a) $f(x) = x^2 + 2x - 3$ *Ans.* $x = -1$ yields relative minimum -4

(b) $f(x) = 3 + 2x - x^2$ *Ans.* $x = 1$ yields relative maximum 4

(c) $f(x) = x^3 + 2x^2 - 4x - 8$ *Ans.* $x = \frac{2}{3}$ yields relative minimum $-\frac{256}{27}$; $x = -2$ yields relative maximum 0

(d) $f(x) = x^3 - 6x^2 + 9x - 8$ *Ans.* $x = 1$ yields relative maximum -4; $x = 3$ yields relative minimum -8

(e) $f(x) = (2 - x)^3$ *Ans.* neither relative maximum nor relative minimum

(f) $f(x) = (x^2 - 4)^2$ *Ans.* $x = 0$ yields relative maximum 16; $x = \pm 2$ yields relative minimum 0

(g) $f(x) = (x - 4)^4(x + 3)^3$ *Ans.* $x = 0$ yields relative maximum 6912; $x = 4$ yields relative minimum 0; $x = -3$ yields neither

(h) $f(x) = x^3 + 48/x$ *Ans.* $x = -2$ yields relative maximum -32; $x = 2$ yields relative minimum 32

(i) $f(x) = (x - 1)^{1/3}(x + 2)^{2/3}$ *Ans.* $x = -2$ yields relative maximum 0; $x = 0$ yields relative minimum $-\sqrt[3]{4}$; $x = 1$ yields neither

24. Examine the functions of Problem 23 ($a - f$) for relative maximum and minimum values, using the second derivative test.

25. Show that $y = (a_1 - x)^2 + (a_2 - x)^2 + \cdots + (a_n - x)^2$ has an absolute minimum when $x = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.

26. Examine the following for absolute maximum and minimum values on the given interval.

(a) $y = -x^2$ on $-2 < x < 2$ *Ans.* maximum (= 0) at $x = 0$

(b) $y = (x - 3)^2$ on $0 \le x \le 4$ *Ans.* maximum (= 9) at $x = 0$; minimum (= 0) at $x = 3$

(c) $y = \sqrt{25 - 4x^2}$ on $-2 \le x \le 2$ *Ans.* maximum (= 5) at $x = 0$; minimum (= 3) at $x = \pm 2$

(d) $y = \sqrt{x - 4}$ on $4 \le x \le 29$ *Ans.* maximum (= 5) at $x = 29$; minimum (= 0) at $x = 4$

27. The sum of two positive numbers is 20. Find the numbers if: (a) their product is a maximum; (b) the sum of their squares is a minimum; (c) the product of the square of one and the cube of the other is a maximum.

Ans. (a) 10, 10; (b) 10, 10; (c) 8, 12

28. The product of two positive numbers is 16. Find the numbers if: (a) their sum is least; (b) the sum of one and the square of the other is least.

Ans. (a) 4, 4; (b) 8, 2

29. An open rectangular box with square ends is to be built to hold 6400 ft^3 at a cost of \$0.75/ft^2 for the base and \$0.25/ft^2 for the sides. Find the most economical dimensions.

Ans. $20 \times 20 \times 16$

30. A wall 8 ft high is $3\frac{3}{8}$ ft from a house. Find the shortest ladder that will reach from the ground to the house when leaning over the wall.

Ans. $15\frac{5}{8}$ ft

31. A company offers the following schedule of charges: \$30 per thousand for orders of 50,000 or less, with the charge decreased by $37\frac{1}{2}$¢ for each thousand above 50,000. Find the order size that makes the company's receipts a maximum.

Ans. 65,000

32. Find an equation of the line through the point (3, 4) that cuts from the first quadrant a triangle of minimum area.

Ans. $4x + 3y - 24 = 0$

33. At what point in the first quadrant on the parabola $y = 4 - x^2$ does the tangent line, together with the coordinate axes, determine a triangle of minimum area?

Ans. $(2\sqrt{3}/3, 8/3)$

34. Find the minimum distance from the point (4, 2) to the parabola $y^2 = 8x$.

Ans. $2\sqrt{2}$

35. (a) Examine $2x^2 - 4xy + 3y^2 - 8x + 8y - 1 = 0$ for maximum and minimum values of y. (b) (GC) Check your answer to (a) on a graphing calculator.

Ans. (a) Maximum at (5, 3); (b) minimum at (−1, −3)

36. (GC) Find the absolute maximum and minimum of $f(x) = x^5 - 3x^2 - 8x - 3$ on [−1, 2] to three-decimal-place accuracy.

Ans. Maximum 1.191 at $x = -0.866$; minimum −14.786 at $x = 1.338$

37. An electric current, when flowing in a circular coil of radius r, exerts a force $F = \frac{kx}{(x^2 + r^2)^{5/2}}$ on a small magnet located at a distance x above the center of the coil. Show that F is greatest when $x = \frac{1}{2}r$.

38. The work done by a voltaic cell of constant electromotive force E and constant internal resistance r in passing a steady current through an external resistance R is proportional to $E^2R/(r + R)^2$. Show that the work done is greatest when $R = r$.

39. A tangent line is drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ so that the part intercepted by the coordinate axes is a minimum. Show that its length is 9.

40. A rectangle is inscribed in the ellipse $\frac{x^2}{400} + \frac{y^2}{225} = 1$ with its sides parallel to the axes of the ellipse. Find the dimensions of the rectangle of (a) maximum area and (b) maximum perimeter that can be so inscribed.

Ans. (a) $20\sqrt{2} \times 15\sqrt{2}$; (b) 32×18

41. Find the radius R of the right circular cone of maximum volume that can be inscribed in a sphere of radius r. (Recall that the volume of a right circular cone of radius R and height h is $\frac{1}{3}\pi R^2 h$.)

Ans. $R = \frac{2}{3}r\sqrt{2}$

42. A right circular cylinder is inscribed in a right circular cone of radius r. Find the radius R of the cylinder if: (a) its volume is a maximum; (b) its lateral area is a maximum. (Recall that the volume of a right circular cylinder of radius R and height h is $\pi R^2 h$, and its lateral area is $2\pi Rh$.)

Ans. (a) $R = \frac{2}{3}r$; (b) $R = \frac{1}{2}r$

43. Show that a conical tent of given volume will require the least amount of material when its height h is $\sqrt{2}$ times the radius r of the base. [Note first that the surface area $A = \pi(r^2 + h^2)$.]

44. Show that the equilateral triangle of altitude $3r$ is the isosceles triangle of least area circumscribing a circle of radius r.

45. Determine the dimensions of the right circular cylinder of maximum lateral surface area that can be inscribed in a sphere of radius 8.

Ans. $h = 2r = 8\sqrt{2}$

46. Investigate the possibility of inscribing a right circular cylinder of maximum total area (including its top and bottom) in a right circular cone of radius r and height h.

Ans. If $h > 2r$, radius of cylinder $= \frac{1}{2}\left(\frac{hr}{h-r}\right)$