CHAPTER 13

Law of the Mean. Increasing and Decreasing Functions

Relative Maximum and Minimum

A function f is said to have a *relative maximum* at x_0 if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0 (and for which $f(x)$ is defined). In other words, the value of f at x_0 is greater than or equal to all values of f at nearby points. Similarly, f is said to have a *relative minimum* at x_0 if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0 (and for which $f(x)$ is defined). In other words, the value of f at x_0 is less than or equal to all values of f at nearby points. By a *relative extremum* of f we mean either a relative maximum or a relative minimum of f.

Theorem 13.1: If f has a relative extremum at a point x_0 at which $f'(x_0)$ is defined, then $f'(x_0) = 0$.

Thus, if f is differentiable at a point at which it has a relative extremum, then the graph of f has a horizontal tangent line at that point. In Fig. 13-1, there are horizontal tangent lines at the points A and B where f attains a relative maximum value and a relative minimum value, respectively. See Problem 5 for a proof of Theorem 13.1.

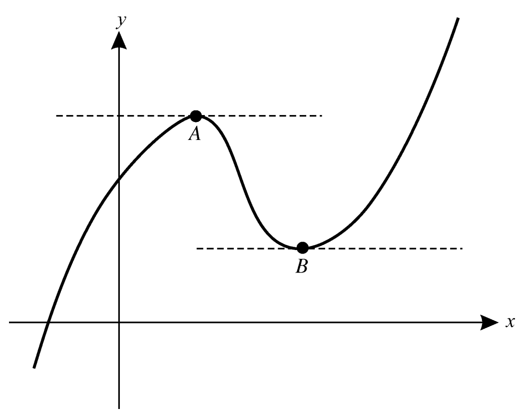

Fig. 13-1

Theorem 13.2 (Rolle's Theorem): Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Assume that $f(a) = f(b) = 0$. Then $f'(x_0) = 0$ for at least one point x_0 in (a, b).

This means that, if the graph of a continuous function intersects the x axis at $x = a$ and $x = b$, and the function is differentiable between a and b, then there is at least one point on the graph between a and b where the tangent line is horizontal. See Fig. 13-2, where there is one such point. For a proof of Rolle's Theorem, see Problem 6.

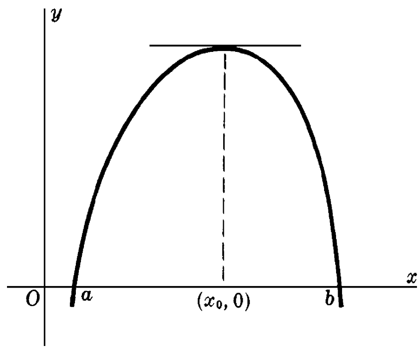

Fig. 13-2

Corollary 13.3 (Generalized Rolle's Theorem): Let g be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Assume that $g(a) = g(b)$. Then $g'(x_0) = 0$ for at least one point x_0 in (a, b).

See Fig. 13-3 for an example in which there is exactly one such point. Note that Corollary 13.3 follows from Rolle's Theorem if we let $f(x) = g(x) - g(a)$.

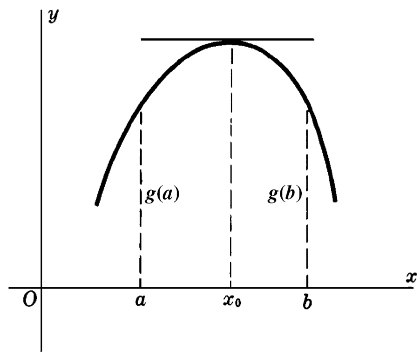

Fig. 13-3

Theorem 13.4 (Law of the Mean)[†]: Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point x_0 in (a, b) for which

$$\frac{f(b)-f(a)}{b-a} = f'(x_0)$$

See Fig. 13-4. For a proof, see Problem 7. Geometrically speaking, the conclusion says that there is some point inside the interval where the slope $f'(x_0)$ of the tangent line is equal to the slope $(f(b) - f(a))/(b - a)$ of the line P_1P_2 connecting the points $(a, f(a))$ and $(b, f(b))$ of the graph. At such a point, the tangent line is parallel to P_1P_2, since their slopes are equal.

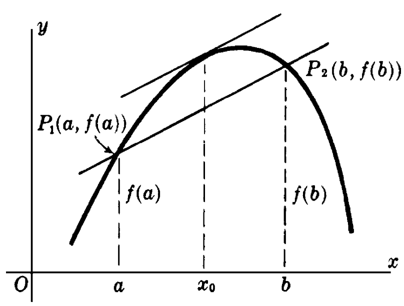

Fig. 13-4

[†] The Law of the Mean is also called the Mean-Value Theorem for Derivatives.

Theorem 13.5 (Extended Law of the Mean): Assume that $f(x)$ and $g(x)$ are continuous on $[a, b]$, and differentiable on (a, b). Assume also that $g'(x) \neq 0$ for all x in (a, b). Then there exists at least one point x_0 in (a, b) for which

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

For a proof, see Problem 13. Note that the Law of the Mean is the special case when $g(x) = x$.

Theorem 13.6 (Higher-Order Law of the Mean): If f and its first $n - 1$ derivatives are continuous on $[a, b]$ and $f^{(n)}(x)$ exists on (a, b), then there is at least one x_0 in (a, b) such that

$$\begin{aligned} f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots \\ + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(b-a)^n \end{aligned} \tag{1}$$

(For a proof, see Problem 14.)

When b is replaced by x, formula (1) becomes

$$\begin{aligned} f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(x-a)^n \end{aligned} \tag{2}$$

for some x_0 between a and x.

In the special case when $a = 0$, formula (2) becomes

$$\begin{aligned} f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \\ + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(x_0)}{n!}x^n \end{aligned} \tag{3}$$

for some x_0 between 0 and x.

Increasing and Decreasing Functions

A function f is said to be *increasing* on an interval if $u < v$ implies $f(u) < f(v)$ for all u and v in the interval. Similarly, f is said to be *decreasing* on an interval if $u < v$ implies $f(u) > f(v)$ for all u and v in the interval.

Theorem 13.7: (a) If f' is positive on an interval, then f is increasing on that interval. (b) If f' is negative on an interval, then f is decreasing on that interval.

For a proof, see Problem 9.

SOLVED PROBLEMS

1. Find the value of x_0 prescribed in Rolle's Theorem for $f(x) = x^3 - 12x$ on the interval $0 \le x \le 2\sqrt{3}$.

Note that $f(0) = f(2\sqrt{3}) = 0$. If $f'(x) = 3x^2 - 12 = 0$, then $x = \pm 2$. Then $x_0 = 2$ is the prescribed value.

2. Does Rolle's Theorem apply to the functions (a) $f(x) = \dfrac{x^2-4x}{x-2}$, and (b) $f(x) = \dfrac{x^2-4x}{x+2}$ on the interval (0, 4)?

(a) $f(x) = 0$ when $x = 0$ or $x = 4$. Since f has a discontinuity at $x = 2$, a point on [0, 4], the theorem does not apply.

(b) $f(x) = 0$ when $x = 0$ or $x = 4$. f has a discontinuity at $x = -2$, a point not on [0, 4]. In addition, $f'(x) = (x^2 + 4x - 8)/(x + 2)^2$ exists everywhere except at $x = -2$. So, the theorem applies and $x_0 = 2(\sqrt{3} - 1)$, the positive root of $x^2 + 4x - 8 = 0$.

3. Find the value of x_0 prescribed by the law of the mean when $f(x) = 3x^2 + 4x - 3$ and $a = 1$, $b = 3$.

$f(a) = f(1) = 4$, $f(b) = f(3) = 36$, $f'(x_0) = 6x_0 + 4$, and $b - a = 2$. So, $6x_0 + 4 = \frac{36-4}{2} = 16$. Then $x_0 = 2$.

4. Find a value x_0 prescribed by the extended law of the mean when $f(x) = 3x + 2$ and $g(x) = x^2 + 1$, on [1, 4].

We have to find x_0 so that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f(4)-f(1)}{g(4)-g(1)} = \frac{14-5}{17-2} = \frac{3}{5} = \frac{f'(x_0)}{g'(x_0)} = \frac{3}{2x_0}$$

Then $x_0 = \frac{5}{2}$.

5. Prove Theorem 13.1: If f has a relative extremum at a point x_0 at which $f'(x_0)$ is defined, then $f'(x_0) = 0$.

Consider the case of a relative maximum. Since f has a relative maximum at x_0, then, for sufficiently small $|\Delta x|$, $f(x_0 + \Delta x) < f(x_0)$, and so $f(x_0 + \Delta x) - f(x_0) < 0$. Thus, when $\Delta x < 0$, $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} > 0$. So,

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \ge 0 \end{aligned}$$

When $\Delta x > 0$, $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} < 0$. Hence,

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \le 0 \end{aligned}$$

Since $f'(x_0) \ge 0$ and $f'(x_0) \le 0$, it follows that $f'(x_0) = 0$.

6. Prove Rolle's Theorem (Theorem 13.2): If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b) = 0$, then $f'(x_0) = 0$ for some point x_0 in (a, b).

If $f(x) = 0$ throughout $[a, b]$, then $f'(x) = 0$ for all x in (a, b). On the other hand, if $f(x)$ is positive (negative) somewhere in (a, b), then, by the Extreme Value Theorem (Theorem 8.7), f has a maximum (minimum) value at some point x_0 on $[a, b]$. That maximum (minimum) value must be positive (negative), and, therefore, x_0 lies on (a, b), since $f(a) = f(b) = 0$. Hence, f has a relative maximum (minimum) at x_0. By Theorem 13.1, $f'(x_0) = 0$.

7. Prove the Law of the Mean (Theorem 13.4): Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point x_0 in (a, b) for which $(f(b) - f(a))/(b - a) = f'(x_0)$.

Let $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$.

Then $F(a) = 0 = F(b)$. So, Rolle's Theorem applies to F on $[a, b]$. Hence, for some x_0 in (a, b), $F'(x_0) = 0$.

But $F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Thus, $f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0$.

8. Show that, if g is increasing on an interval, then $-g$ is decreasing on that interval.

Assume $u < v$. Then $g(u) < g(v)$. Hence, $-g(u) > -g(v)$.

9. Prove Theorem 13.7: (a) If f' is positive on an interval, then f is increasing on that interval, (b) If f' is negative on an interval, then f is decreasing on that interval.

(a) Let a and b be any two points on the interval with $a < b$. By the Law of the Mean, $(f(b) - f(a))/(b - a) = f'(x_0)$ for some point x_0 in (a, b). Since x_0 is in the interval, $f'(x_0) > 0$. Thus, $(f(b) - f(a))/(b - a) > 0$. But, $a < b$ and, therefore, $b - a > 0$. Hence, $f(b) - f(a) > 0$. So, $f(a) < f(b)$.

(b) Let $g = -f$. So, g' is positive on the interval. By part (a), g is increasing on the interval. So, f is decreasing on the interval.

10. Show that $f(x) = x^5 + 20x - 6$ is an increasing function for all values of x.

$f'(x) = 5x^4 + 20 > 0$ for all x. Hence, by Theorem 13.7(a), f is increasing everywhere.

11. Show that $f(x) = 1 - x^3 - x^7$ is a decreasing function for all values of x.

$f'(x) = -3x^2 - 7x^6 < 0$ for all $x \neq 0$. Hence, by Theorem 13.7(b), f is decreasing on any interval not containing 0. Note that, if $x < 0$, $f(x) > 1 = f(0)$, and, if $x > 0$, $f(0) = 1 > f(x)$. So, f is decreasing for all real numbers.

12. Show that $f(x) = 4x^3 + x - 3 = 0$ has exactly one real solution.

$f(0) = -3$ and $f(1) = 2$. So, the intermediate value theorem tells us that $f(x) = 0$ has a solution in (0, 1). Since $f'(x) = 12x^2 + 1 > 0$, f is an increasing function. Therefore, there cannot be two values of x for which $f(x) = 0$.

13. Prove the Extended Law of the Mean (Theorem 13.5): If $f(x)$ and $g(x)$ are continuous on $[a, b]$, and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b), then there exists at least one point x_0 in (a, b) for which $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(x_0)}{g'(x_0)}$.

Suppose that $g(b) = g(a)$. Then, by the generalized Rolle's Theorem, $g'(x) = 0$ for some x in (a, b), contradicting our hypothesis. Hence, $g(b) \neq g(a)$.

Let $F(x) = f(x) - f(b) - \dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(b))$.

Then
$$F(a) = 0 = F(b) \quad \text{and} \quad F'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$$

By Rolle's Theorem, there exists x_0 in (a, b) for which $f'(x_0) - \dfrac{f(b)-f(a)}{g(b)-g(a)} g'(x_0) = 0$.

14. Prove the Higher-Order Law of the Mean (Theorem 13.6): If f and its first $n - 1$ derivatives are continuous on $[a, b]$ and $f^{(n)}(x)$ exists on (a, b), then there is at least one x_0 in (a, b) such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{(n-1)} + \frac{f^{(n)}(x_0)}{n!}(b-a)^n \tag{1}$$

Let a constant K be defined by

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{(n-1)} + K(b-a)^n \tag{2}$$

and consider

$$F(x) = f(x) - f(b) + \frac{f'(x)}{1!}(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1} + K(b-x)^n$$

Now $F(a) = 0$ by (2), and $F(b) = 0$. By Rolle's Theorem, there exists x_0 in (a, b) such that

$$F'(x_0) = f'(x_0) + [f''(x_0)(b - x_0) - f'(x_0)] + \left[\frac{f'''(x_0)}{2!}(b - x_0)^2 - f''(x_0)(b - x_0)\right]$$

$$+ \cdots + \left[\frac{f^{(n)}(x_0)}{(n-1)!}(b - x_0)^{n-1} - \frac{f^{(n-1)}(x_0)}{(n-2)!}(b - x_0)^{n-2}\right] - Kn(b - x_0)^{n-1}$$

$$= \frac{f^{(n)}(x_0)}{(n-1)!}(b - x_0)^{n-1} - Kn(b - x_0)^{n-1} = 0$$

Then $K = \dfrac{f^{(n)}(x_0)}{n!}$, and (2) becomes (1).

15. If $f'(x) = 0$ for all x on (a, b), then f is constant on (a, b).

Let u and v be any two points in (a, b) with $u < v$. By the Law of the Mean, there exists x_0 in (u, v) for which $\dfrac{f(v) - f(u)}{v - u} = f'(x_0)$. By hypothesis, $f'(x_0) = 0$. Hence, $f(v) - f(u) = 0$, and, therefore, $f(v) = f(u)$.

SUPPLEMENTARY PROBLEMS

16. If $f(x) = x^2 - 4x + 3$ on $[1, 3]$, find a value prescribed by Rolle's Theorem.

Ans. $x_0 = 2$

17. Find a value prescribed by the Law of the Mean, given:

(a) $y = x^3$ on $[0, 6]$ *Ans.* $x_0 = 2\sqrt{3}$

(b) $y = ax^2 + bx + c$ on $[x_1, x_2]$ *Ans.* $x_0 = \frac{1}{2}(x_1 + x_2)$

18. If $f'(x) = g'(x)$ for all x in (a, b), prove that there exists a constant K such that $f(x) = g(x) + K$ for all x in (a, b). (*Hint*: $D_x(f(x) - g(x)) = 0$ in (a, b). By Problem 15, there is a constant K such that $f(x) - g(x) = K$ in (a, b).)

19. Find a value x_0 precribed by the extended law of the mean when $f(x) = x^2 + 2x - 3$, $g(x) = x^2 - 4x + 6$ on the interval $[0, 1]$.

Ans. $\frac{1}{2}$

20. Show that $x^3 + px + q = 0$ has: (a) one real root if $p > 0$, and (b) three real roots if $4p^3 + 27q^2 < 0$.

21. Show that $f(x) = \dfrac{ax + b}{cx + d}$ has neither a relative maximum nor a relative minimum. (*Hint*: Use Theorem 13.1.)

22. Show that $f(x) = 5x^3 + 11x - 20 = 0$ has exactly one real solution.

23. (a) Where are the following functions (i)–(vii) increasing and where are they decreasing? Sketch the graphs.
(b) (GC) Check your answers to (*a*) by means of a graphing calculator.

(i) $f(x) = 3x + 5$ *Ans.* Increasing everywhere

(ii) $f(x) = -7x + 20$ *Ans.* Decreasing everywhere

(iii) $f(x) = x^2 + 6x - 11$ *Ans.* Decreasing on $(-\infty, -3)$, increasing on $(-3, +\infty)$

(iv) $f(x) = 5 + 8x - x^2$ *Ans.* Increasing on $(-\infty, 4)$, decreasing on $(4, +\infty)$

(v) $f(x) = \sqrt{4 - x^2}$ *Ans.* Increasing on $(-2, 0)$, decreasing on $(0, 2)$

(vi) $f(x) = |x - 2| + 3$ *Ans.* Decreasing on $(-\infty, 2)$, increasing on $(2, +\infty)$

(vii) $f(x) = \dfrac{x}{x^2 - 4}$ *Ans.* Decreasing on $(-\infty, -2)$, $(-2, 2)$, $(2, +\infty)$; never increasing

24. (GC) Use a graphing calculator to estimate the intervals on which $f(x) = x^5 + 2x^3 - 6x + 1$ is increasing, and the intervals on which it is decreasing.

25. For the following functions, determine whether Rolle's Theorem is applicable. If it is, find the prescribed values.

(a) $f(x) = x^{3/4} - 2$ on $[-3, 3]$ *Ans.* No. Not differentiable at $x = 0$.

(b) $f(x) = |x^2 - 4|$ on $[0, 8]$ *Ans.* No. Not differentiable at $x = 2$.

(c) $f(x) = |x^2 - 4|$ on $[0, 1]$ *Ans.* No. $f(0) \neq f(1)$

(d) $f(x) = \dfrac{x^2 - 3x - 4}{x - 5}$ on $[-1, 4]$ *Ans.* Yes. $x_0 = 5 - \sqrt{6}$