CHAPTER 12

Tangent and Normal Lines

An example of a graph of a continuous function f is shown in Fig. 12-1(a). If P is a point of the graph having abscissa x, then the coordinates of P are $(x, f(x))$. Let Q be a nearby point having abscissa $x + \Delta x$. Then the coordinates of Q are $(x + \Delta x, f(x + \Delta x))$. The line PQ has slope $\frac{f(x+\Delta x)-f(x)}{\Delta x}$. As Q approaches P along the graph, the lines PQ get closer and closer to the tangent line $\mathscr{T}$ to the graph at P. (See Fig. 12-1 (b).) Hence, the slope of PQ approaches the slope of the tangent line. Thus, the slope of the tangent line is $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, which is the derivative $f'(x)$.

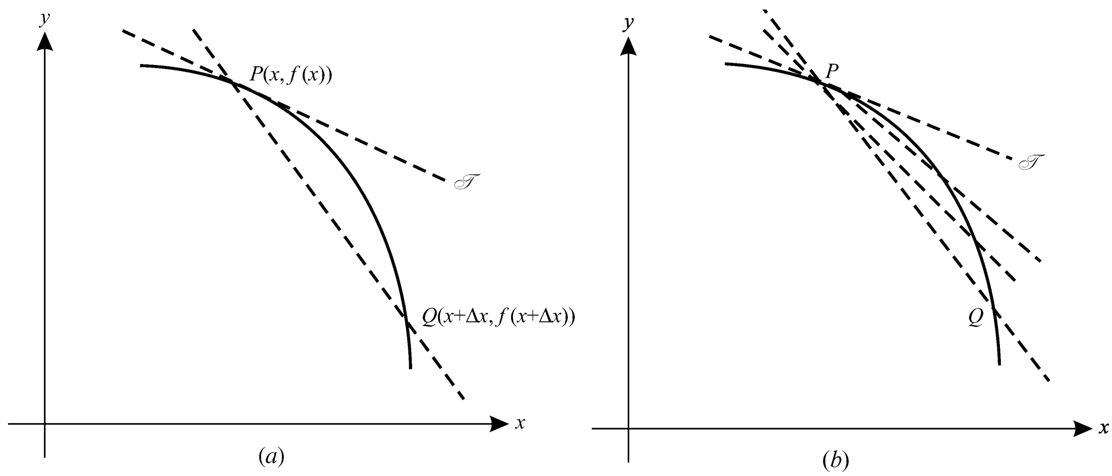

Fig. 12-1

If the slope m of the tangent line at a point of the curve $y = f(x)$ is zero, then the curve has a horizontal tangent line at that point, as at points A, C, and E of Fig. 12-2. In general, if the derivative of f is m at a point (x_0, y_0), then the point–slope equation of the tangent line is $y - y_0 = m(x - x_0)$. If f is continuous at x_0, but $\lim_{x \to x_0} f'(x) = \infty$, then the curve has a vertical tangent line at x_0, as at points B and D of Fig. 12-2.

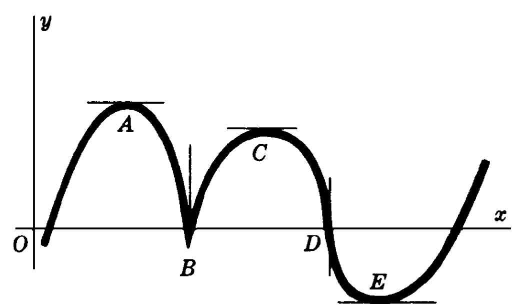

Fig. 12-2

The *normal line* to a curve at one of its points (x_0, y_0) is the line that passes through the point and is perpendicular to the tangent line at that point. Recall that a perpendicular to a line with nonzero slope m has slope $-1/m$. Hence, if $m \neq 0$ is the slope of the tangent line, then $y - y_0 = -(1/m)(x - x_0)$ is a point–slope equation of the normal line. If the tangent line is horizontal, then the normal line is vertical and has equation $x = x_0$. If the tangent line is vertical, then the normal line is horizontal and has equation $y = y_0$.

The Angles of Intersection

The angles of intersection of two curves are defined as the angles between the tangent lines to the curves at their point of intersection.

To determine the angles of intersection of the two curves:

1. Solve the equations of the curves simultaneously to find the points of intersection.
2. Find the slopes m_1 and m_2 of the tangent lines to the two curves at each point of intersection.
3. If $m_1 = m_2$, the angle of intersection is 0°, and if $m_1 = -1/m_2$, the angle of intersection is 90°; otherwise, the angle of intersection ϕ can be found from the formula

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

ϕ is the acute angle of intersection when $\tan\phi > 0$, and $180° - \phi$ is the acute angle of intersection when $\tan\phi < 0$.

SOLVED PROBLEMS

1. Find equations of the tangent and normal lines to $y = f(x) = x^3 - 2x^2 + 4$ at (2, 4).

$f'(x) = 3x^2 - 4x$. Thus, the slope of the tangent line at (2, 4) is $m = f'(2) = 4$, and an equation of the tangent line is $y - 4 = 4(x - 2)$. The slope–intercept equation is $y = 4x - 4$.

An equation of the normal line at (2, 4) is $y - 4 = -\frac{1}{4}(x - 2)$. Its slope–intercept equation is $y = -\frac{1}{4}x + \frac{9}{2}$.

2. Find equations of the tangent and normal lines to $x^2 + 3xy + y^2 = 5$ at (1, 1).

By implicit differentiation, $2x + 3xy' + 3y + 2yy' = 0$. So, $y' = -\dfrac{2x + 3y}{3x + 2y}$. Then the slope of the tangent line at (1, 1) is -1. An equation of the tangent line is $y - 1 = -(x - 1)$. Its slope–intercept equation is $y = -x + 2$. An equation of the normal line is $y - 1 = x - 1$, that is, $y = x$.

3. Find the equations of the tangent lines with slope $m = -\frac{2}{9}$ to the ellipse $4x^2 + 9y^2 = 40$.

By implicit differentiation, $y' = -4x/9y$. So, at a point of tangency (x_0, y_0), $m = -4x_0/9y_0 = -\frac{2}{9}$. Then $y_0 = 2x_0$.

Since the point is on the ellipse, $4x_0^2 + 9y_0^2 = 40$. So, $4x_0^2 + 9(2x_0)^2 = 40$. Therefore, $x_0^2 = 1$, and $x_0 = \pm 1$. The required points are (1, 2) and (–1, –2).

At (1, 2), an equation of the tangent line is $y - 2 = -\frac{2}{9}(x - 1)$.

At (–1, –2), an equation of the tangent line is $y + 2 = -\frac{2}{9}(x + 1)$.

4. Find an equation of the tangent lines to the hyperbola $x^2 - y^2 = 16$ that pass through the point (2, –2).

By implicit differentiation, $2x - 2yy' = 0$ and, therefore, $y' = x/y$. So, at a point of tangency (x_0, y_0), the slope of the tangent line must be x_0/y_0. On the other hand, since the tangent line must pass through (x_0, y_0) and (2, –2), the slope is $\dfrac{y_0 + 2}{x_0 - 2}$.

Thus, $\dfrac{x_0}{y_0} = \dfrac{y_0 + 2}{x_0 - 2}$. Hence, $x_0^2 - 2x_0 = y_0^2 + 2y_0$. Thus, $2x_0 + 2y_0 = x_0^2 - y_0^2 = 16$, yielding $x_0 + y_0 = 8$, and, therefore, $y_0 = 8 - x_0$.

If we substitute $8 - x_0$ for y_0 in $x_0^2 - y_0^2 = 16$ and solve for x_0, we get $x_0 = 5$. Then $y_0 = 3$. Hence, an equation of the tangent line is $y - 3 = \frac{5}{3}(x - 5)$.

5. Find the points of tangency of horizontal and vertical tangent lines to the curve $x^2 - xy + y^2 = 27$.

By implicit differentiation, $2x - xy' - y + 2yy' = 0$, whence $y' = \dfrac{y - 2x}{2y - x}$.

For horizontal tangent lines, the slope must be zero. So, the numerator $y - 2x$ of y' must be zero, yielding $y = 2x$. Substituting $2x$ for y in the equation of the curve, we get $x^2 = 9$. Hence, the points of tangency are (3, 6) and (–3, –6).

For vertical tangent lines, the slope must be infinite. So, the denominator $2y - x$ of y' must be zero, yielding $x = 2y$. Replacing x in the equation of the curve, we get $y^2 = 9$. Hence, the points of tangency are (6, 3) and (–6, –3).

6. Find equations of the vertical lines that meet the curves (a) $y = x^3 + 2x^2 - 4x + 5$ and (b) $3y = 2x^3 + 9x^2 - 3x - 3$ in points at which the tangent lines to the two curves are parallel.

Let $x = x_0$ be such a line. The tangent lines at x_0 have slopes:
For (a): $y' = 3x^2 + 4x - 4$; at x_0, $m_1 = 3x_0^2 + 4x_0 - 4$
For (b): $3y' = 6x^2 + 18x - 3$; at x_0, $m_2 = 2x_0^2 + 6x_0 - 1$

Since $m_1 = m_2$, $3x_0^2 + 4x_0 - 4 = 2x_0^2 + 6x_0 - 1$. Then $x_0^2 - 2x_0 - 3 = 0$, $(x_0 - 3)(x_0 + 1) = 0$. Hence, $x_0 = 3$ or $x_0 = -1$. Thus, the vertical lines are $x = 3$ and $x = -1$.

7. (a) Show that the slope–intercept equation of the tangent line of slope $m \neq 0$ to the parabola $y^2 = 4px$ is $y = mx + p/m$.
(b) Show that an equation of the tangent line to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at the point $P_0(x_0, y_0)$ on the ellipse is $b^2x_0x + a^2y_0y = a^2b^2$.

(a) $y' = 2p/y$. Let $P_0(x_0, y_0)$ be the point of tangency. Then $y_0^2 = 4px_0$ and $m = 2p/y_0$. Hence, $y_0 = 2p/m$ and $x_0 = \frac{1}{4}y_0^2/p = p/m^2$. The equation of the tangent line is then $y - 2p/m = m(x - p/m^2)$, which reduces to $y = mx + p/m$.

(b) $y' = -\dfrac{b^2x}{a^2y}$. At P_0, $m = -\dfrac{b^2x_0}{a^2y_0}$. An equation of the tangent line is $y - y_0 = -\dfrac{b^2x_0}{a^2y_0}(x - x_0)$, which reduces to $b^2x_0x + a^2y_0y = b^2x_0^2 + a^2y_0^2 = a^2b^2$ (since (x_0, y_0) satisfies the equation of the ellipse).

8. Show that at a point $P_0(x_0, y_0)$ on the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, the tangent line bisects the angle included between the focal radii of P_0.

At P_0 the slope of the tangent to the hyperbola is b^2x_0/a^2y_0 and the slopes of the focal radii P_0F' and P_0F (see Fig. 12-3) are $y_0/(x_0 + c)$ and $y_0/(x_0 - c)$, respectively. Now

$$\tan\alpha = \frac{\dfrac{b^2x_0}{a^2y_0} - \dfrac{y_0}{x_0 + c}}{1 + \dfrac{b^2x_0}{a^2y_0}\cdot\dfrac{y_0}{x_0 + c}} = \frac{(b^2x_0^2 - a^2y_0^2) + b^2cx_0}{(a^2 + b^2)x_0y_0 + a^2cy_0} = \frac{a^2b^2 + b^2cx_0}{c^2x_0y_0 + a^2cy_0} = \frac{b^2(a^2 + cx_0)}{cy_0(a^2 + cx_0)} = \frac{b^2}{cy_0}$$

since $b^2x_0^2 - a^2y_0^2 = a^2b^2$ and $a^2 + b^2 = c^2$, and

$$\tan\beta = \frac{\dfrac{y_0}{x_{0-c}} - \dfrac{b^2x_0}{a^2y_0} -}{1 + \dfrac{b^2x_0}{a^2x_0}\cdot\dfrac{y_0}{x_0 + c}} = \frac{b^2cx_0 - (b^2x_0^2 - a^2y_0^2)}{(a^2 + b^2)x_0y_0 - a^2cy_0} = \frac{b^2cx_0 - a^2b^2}{c^2x_0y_0 - a^2cy_0} = \frac{b^2}{cy_0}$$

Hence, $\alpha = \beta$ because $\tan\alpha = \tan\beta$.

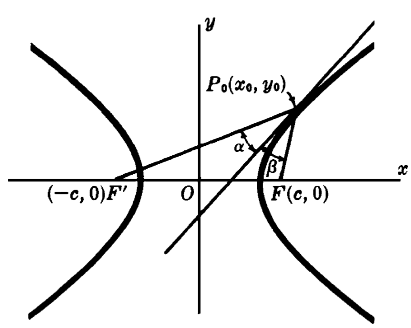

Fig. 12-3

9. One of the points of intersection of the curves (a) $y^2 = 4x$ and (b) $2x^2 = 12 - 5y$ is (1, 2). Find the acute angle of intersection of the curves at that point.

For (a), $y' = 2/y$. For (b), $y' = -4x/5$. Hence, at (1, 2), $m_1 = 1$ and $m_2 = -\frac{4}{5}$. So,

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1 + \frac{4}{5}}{1 - \frac{4}{5}} = 9$$

Then $\phi \approx 83°\ 40'$ is the acute angle of intersection.

10. Find the angles of intersection of the curves (a) $2x^2 + y^2 = 20$ and (b) $4y^2 - x^2 = 8$.

Solving simultaneously, we obtain $y^2 = 4$, $y = \pm 2$. Then the points of intersection are $(\pm 2\sqrt{2},\ 2)$ and $(\pm 2\sqrt{2},\ -2)$. For (a), $y' = -2x/y$, and for (b), $y' = x/4y$. At the point $(2\sqrt{2},\ 2)$, $m_1 = -2\sqrt{2}$ and $m_2 = \frac{1}{4}\sqrt{2}$. Since $m_1 m_2 = -1$, the angle of intersection is 90° (that is, the curves are *orthogonal*). By symmetry, the curves are orthogonal at each of their points of intersection.

11. A cable of a certain suspension bridge is attached to supporting pillars 250 ft apart. If it hangs in the form of a parabola with the lowest point 50 ft below the point of suspension, find the angle between the cable and the pillar.

Take the origin at the vertex of the parabola, as in Fig. 12-4. The equation of the parabola is $y = \frac{2}{625}x^2$ and $y' = 4x/625$.

At (125, 50), $m = 4(125)/625 = 0.8000$ and $\theta = 38°40'$. Hence, the required angle is $\phi = 90° - \theta = 51°\ 20'$.

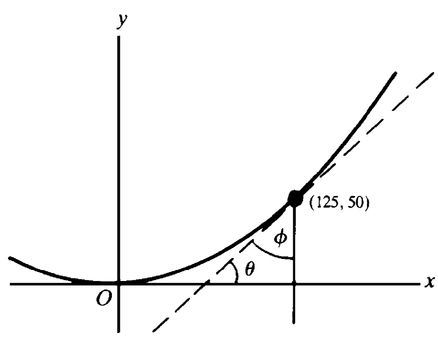

Fig. 12-4

SUPPLEMENTARY PROBLEMS

12. Examine $x^2 + 4xy + 16y^2 = 27$ for horizontal and vertical tangent lines.

Ans. Horizontal tangents at $(3,\ -\frac{3}{2})$ and $(-3,\ \frac{3}{2})$. Vertical tangents at $(6,\ -\frac{3}{4})$ and $(-6,\ -\frac{3}{4})$.

13. Find equations of the tangent and normal lines to $x^2 - y^2 = 7$ at the point (4, –3).

Ans. $4x + 3y = 7$ and $3x - 4y = 24$

14. At what points on the curve $y = x^3 + 5$ is its tangent line: (a) parallel to the line $12x - y = 17$; (b) perpendicular to the line $x + 3y = 2$?

Ans. (a) (2, 13), (–2, –3); (b) (1, 6), (–1, 4)

15. Find equations of the tangent lines to $9x^2 + 16y^2 = 52$ that are parallel to the line $9x - 8y = 1$.

Ans. $9x - 8y = \pm 26$

16. Find equations of the tangent lines to the hyperbola $xy = 1$ that pass through the point $(-1, 1)$.

Ans. $y = (2\sqrt{2} - 3)x + 2\sqrt{2} - 2$; $y = -(2\sqrt{2} + 3)x - 2\sqrt{2} - 2$

17. For the parabola $y^2 = 4px$, show that an equation of the tangent line at one of its points $P(x_0, y_0)$ is $y_0 y = 2p(x + x_0)$.

18. For the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, show that the equations of its tangent lines of slope m are $y = mx \pm \sqrt{a^2m^2 + b^2}$

19. For the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, show that (a) an equation of the tangent line at one of its points $P(x_0, y_0)$ is $b^2x_0x - a^2y_0y = a^2b^2$; and (b) the equations of its tangent lines of slope m are $y = mx \pm \sqrt{a^2m^2 - b^2}$.

20. Show that the normal line to a parabola at one of its points P bisects the angle included between the focal radius of P and the line through P parallel to the axis of the parabola.

21. Prove: Any tangent line to a parabola, except at the vertex, intersects the directrix and the latus rectum (produced if necessary) in points equidistant from the focus.

22. Prove: The chord joining the points of contact of the tangent lines to a parabola from any point on its directrix passes through the focus.

23. Prove: The normal line to an ellipse at any of its points P bisects the angle included between the focal radii of P.

24. Prove: (a) The sum of the intercepts on the coordinate axes of any tangent line to $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is a constant. (b) The sum of the squares of the intercepts on the coordinate axes of any tangent line to $x^{2/3} + y^{2/3} = a^{2/3}$ is a constant.

25. Find the acute angles of intersection of the circles $x^2 - 4x + y^2 = 0$ and $x^2 + y^2 = 8$.

Ans. 45°

26. Show that the curves $y = x^3 + 2$ and $y = 2x^2 + 2$ have a common tangent line at the point $(0, 2)$ and intersect at the point $(2, 10)$ at an angle ϕ such that $\tan\phi = \frac{4}{97}$.

27. Show that the ellipse $4x^2 + 9y^2 = 45$ and the hyperbola $x^2 - 4y^2 = 5$ are orthogonal (that is, intersect at a right angle).

28. Find equations of the tangent and normal lines to the parabola $y = 4x^2$ at the point $(-1, 4)$.

Ans. $y + 8x + 4 = 0$; $8y - x - 33 = 0$

29. At what points on the curve $y = 2x^3 + 13x^2 + 5x + 9$ does its tangent line pass through the origin?

Ans. $x = -3, \ -1, \ \frac{3}{4}$