CHAPTER 10

Rules for Differentiating Functions

Differentiation

Recall that a function f is said to be *differentiable* at x_0 if the derivative $f'(x_0)$ exists. A function is said to be differentiable on a set if the function is differentiable at every point of the set. If we say that a function is differentiable, we mean that it is differentiable at every real number. The process of finding the derivative of a function is called *differentiation*.

Theorem 10.1 (Differentiation Formulas): In the following formulas, it is assumed that u, v, and w are functions that are differentiable at x; c and m are assumed to be constants.

(1) $\frac{d}{dx}(c) = 0$ (The derivative of a constant function is zero.)

(2) $\frac{d}{dx}(x) = 1$ (The derivative of the identity function is 1.)

(3) $\frac{d}{dx}(cu) = c\frac{du}{dx}$

(4) $\frac{d}{dx}(u + v + \ldots) = \frac{du}{dx} + \frac{dv}{dx} + \ldots$ (Sum Rule)

(5) $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$ (Difference Rule)

(6) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (Product Rule)

(7) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ provided that $v \neq 0$ (Quotient Rule)

(8) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ provided that $x \neq 0$

(9) $\frac{d}{dx}\left(x^m\right) = mx^{m-1}$ (Power Rule)

Note that formula (8) is a special case of formula (9) when $m = -1$. For proofs, see Problems 1–4.

EXAMPLE 10.1: $D_x(x^3 + 7x + 5) = D_x(x^3) + D_x(7x) + D_x(5)$ (Sum Rule)

$= 3x^2 + 7D_x(x) + 0$ (Power Rule, formulas (3) and (1))

$= 3x^2 + 7$ (formula (2))

Every polynomial is differentiable, and its derivative can be computed by using the Sum Rule, Power Rule, and formulas (1) and (3).

Composite Functions. The Chain Rule

The *composite function* $f \circ g$ of functions g and f is defined as follows: $(f \circ g)(x) = f(g(x))$. The function g is applied first and then $f \cdot g$ is called the *inner function*, and f is called the *outer function*. $f \circ g$ is called the *composition* of g and f.

EXAMPLE 10.2: Let $f(x) = x^2$ and $g(x) = x + 1$. Then:

$$(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$$

Thus, in this case, $f \circ g \neq g \circ f$.

When f and g are differentiable, then so is their composition $f \circ g$. There are two procedures for finding the derivative of $f \circ g$. The first method is to compute an explicit formula for $f(g(x))$ and differentiate.

EXAMPLE 10.3: If $f(x) = x^2 + 3$ and $g(x) = 2x + 1$, then

$$y = f(g(x)) = f(2x+1) = (2x+1)^2 + 3 = 4x^2 + 4x + 4 \quad \text{and} \quad \frac{dy}{dx} = 8x + 4$$

Thus, $D_x(f \circ g) = 8x + 4$.

The second method of computing the derivative of a composite function is based on the following rule.

Chain Rule

$$D_x(f(g(x)) = f'(g(x)) \cdot g'(x)$$

Thus, the derivative of $f \circ g$ is the product of the derivative of the outer function f (evaluated at $g(x)$) and the derivative of the inner function (evaluated at x). It is assumed that g is differentiable at x and that f is differentiable at $g(x)$.

EXAMPLE 10.4: In Example 10.3, $f'(x) = 2x$ and $g'(x) = 2$. Hence, by the Chain Rule,

$$D_x(f(g(x)) = f'(g(x)) \cdot g'(x) = 2g(x) \cdot 2 = 4g(x) = 4(2x+1) = 8x + 4$$

Alternative Formulation of the Chain Rule

Let $u = g(x)$ and $y = f(u)$. Then the composite function of g and f is $y = f(u) = f(g(x))$, and we have the formula:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{(Chain Rule)}$$

EXAMPLE 10.5: Let $y = u^3$ and $u = 4x^2 - 2x + 5$. Then the composite function $y = (4x^2 - 2x + 5)^3$ has the derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(8x-2) = 3(4x^2 - 2x + 5)^2(8x - 2)$$

Warning. In the Alternative Formulation of the Chain Rule, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, the y on the left denotes the composite function of x, whereas the y on the right denotes the original function of u. Likewise, the two occurrences of u have different meanings. This notational confusion is made up for by the simplicity of the alternative formulation.

Inverse Functions

Two functions f and g such that $g(f(x)) = x$ and $f(g(y)) = y$ are said to be *inverse functions*. Inverse functions reverse the effect of each other. Given an equation $y = f(x)$, we can find a formula for the inverse of f by solving the equation for x in terms of y.

EXAMPLE 10.6:

(a) Let $f(x) = x + 1$. Solving the equation $y = x + 1$ for x, we obtain $x = y - 1$. Then the inverse g of f is given by the formula $g(y) = y - 1$. Note that g reverses the effect of f and f reverses the effect of g.

(b) Let $f(x) = -x$. Solving $y = -x$ for x, we obtain $x = -y$. Hence, $g(y) = -y$ is the inverse of f. In this case, the inverse of f is the same function as f.

(c) Let $f(x) = \sqrt{x}$. f is defined only for nonnegative numbers, and its range is the set of nonnegative numbers. Solving $y = \sqrt{x}$ for x, we get $x = y^2$, so that $g(y) = y^2$. Note that, since g is the inverse of f, g is only defined for nonnegative numbers, since the values of f are the nonnegative numbers. (Since $y = f(g(y))$, then, if we allowed g to be defined for negative numbers, we would have $-1 = f(g(-1)) = f(1) = 1$, a contradiction.)

(d) The inverse of $f(x) = 2x - 1$ is the function $g(y) = \frac{y+1}{2}$.

Notation

The inverse of f will be denoted f^{-1}.

Do not confuse this with the exponential notation for raising a number to the power -1. The context will usually tell us which meaning is intended.

Not every function has an inverse function. For example, the function $f(x) = x^2$ does not possess an inverse. Since $f(1) = 1 = f(-1)$, an inverse function g would have to satisfy g(1) = 1 and g(1) = −1, which is impossible. (However, if we restricted the function $f(x) = x^2$ to the domain $x \geq 0$, then the function $g(y) = \sqrt{y}$ would be an inverse function of f.)

The condition that a function f must satisfy in order to have an inverse is that f is *one-to-one*, that is, for any x_1 and x_2, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. Equivalently, f is one-to-one if and only if, for any x_1 and x_2, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

EXAMPLE 10.7: Let us show that the function $f(x) = 3x + 2$ is one-to-one. Assume $f(x_1) = f(x_2)$. Then $3x_1 + 2 = 3x_2 + 2$, $3x_1 = 3x_2$, $x_1 = x_2$. Hence, f is one-to-one. To find the inverse, solve $y = 3x + 2$ for x, obtaining $x = \frac{y-2}{3}$. Thus, $f^{-1}(y) = \frac{y-2}{3}$. (In general, if we can solve $y = f(x)$ for x in terms of y, then we know that f is one-to-one.)

Theorem 10.2 (Differentiation Formula for Inverse Functions): Let f be one-to-one and continuous on an interval (a, b). Then:

(a) The range of f is an interval I (possibly infinite) and f is either increasing or decreasing. Moreover, f^{-1} is continuous on I.

(b) If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.

The latter equation is sometimes written

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

where $x = f^{-11}(y)$.

For the proof, see Problem 69.

EXAMPLE 10.8:

(a) Let $y = f(x) = x^2$ for $x > 0$. Then $x = f^{-1}(y) = \sqrt{y}$. Since $\frac{dy}{dx} = 2x$, $\frac{dx}{dy} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$. Thus, $D_y(\sqrt{y}) = \frac{1}{2\sqrt{y}}$. (Note that this is a special case of Theorem 8.1(9) when $m = \frac{1}{2}$.)

(b) Let $y = f(x) = x^3$ for all x. Then $x = f^{-1}(y) = \sqrt[3]{y} = y^{1/3}$ for all y. Since $\frac{dy}{dx} = 3x^2$, $\frac{dx}{dy} = \frac{1}{3x^2} = \frac{1}{3y^{2/3}}$. This holds for all $y \neq 0$. (Note that $f^{-1}(0) = 0$ and $f'(0) = 3(0)^2 = 0$.)

Higher Derivatives

If $y = f(x)$ is differentiable, its derivative y' is also called the *first derivative* of f. If y' is differentiable, its derivative is called the *second derivative* of f. If this second derivative is differentiable, then its derivative is called the *third derivative* of f, and so on.

Notation

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$D_x y$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$D_x^2 y$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$D_x^3 y$
n^{th} derivative:	$y^{(n)}$,	$f^{(n)}$,	$\frac{d^ny}{dx^n}$,	$D_x^n y$

SOLVED PROBLEMS

1. Prove Theorem 10.1, (1)–(3): (1) $\frac{d}{dx}(c) = 0$; (2) $\frac{d}{dx}(x) = 1$; (3) $\frac{d}{dx}(cu) = c\frac{du}{dx}$.

Remember that $\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

(1) $\frac{d}{dx} c = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$

(2) $\frac{d}{dx}(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$

(3)
$$\frac{d}{dx}(cu) = \lim_{\Delta x \to 0} \frac{cu(x+\Delta x) - cu(x)}{\Delta x} = \lim_{\Delta x \to 0} c\frac{u(x+\Delta x) - u(x)}{\Delta x}$$
$$= c\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} = c\frac{du}{dx}$$

2. Prove Theorem 10.1, (4), (6), (7):

(4) $\frac{d}{dx}(u + v + \cdots) = \frac{du}{dx} + \frac{dv}{dx} + \cdots$

(6) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(7) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ provided that $v \neq 0$

(4) It suffice to prove this for just two summands, u and v. Let $f(x) = u + v$. Then

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x}$$
$$= \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x}$$

Taking the limit as $\Delta x \to 0$ yields $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$.

(6) Let $f(x) = uv$. Then

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{u(x+\Delta x)v(x+\Delta x)-u(x)v(x)}{\Delta x}$$

$$= \frac{[u(x+\Delta x)v(x+\Delta x)-v(x)u(x+\Delta x)]+[v(x)u(x+\Delta x)-u(x)v(x)]}{\Delta x}$$

$$= u(x+\Delta x)\frac{v(x+\Delta x)-v(x)}{\Delta x} + v(x)\frac{u(x+\Delta x)-u(x)}{\Delta x}$$

Taking the limit as $\Delta x \to 0$ yields

$$\frac{d}{dx}(uv) = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Note that $\lim\limits_{\Delta x\to 0} u(x+\Delta x) = u(x)$ because the differentiability of u implies its continuity.

(7) Set $f(x) = \dfrac{u}{v} = \dfrac{u(x)}{v(x)}$, then

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\dfrac{u(x+\Delta x)}{v(x+\Delta x)} - \dfrac{u(x)}{v(x)}}{\Delta x} = \frac{u(x+\Delta x)v(x)-u(x)v(x+\Delta x)}{\Delta x\{v(x)v(x+\Delta x)\}}$$

$$= \frac{[u(x+\Delta x)v(x)-u(x)v(x)]-[u(x)v(x+\Delta x)-u(x)v(x)]}{\Delta x[v(x)v(x+\Delta x)]}$$

$$= \frac{v(x)\dfrac{u(x+\Delta x)-u(x)}{\Delta x} - u(x)\dfrac{v(x+\Delta x)-v(x)}{\Delta x}}{v(x)v(x+\Delta x)}$$

and for $\Delta x \to 0$, $\dfrac{d}{dx}f(x) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(x)\dfrac{d}{dx}u(x) - u(x)\dfrac{d}{dx}v(x)}{[v(x)]^2} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

3. Prove Theorem 10.1(9): $D_x(x^m) = mx^{m-1}$, when m is a nonnegative integer.
Use mathematical induction. When $m = 0$,

$$D_x(x^m) = D_x(x^0) = D_x(1) = 0 = 0\cdot x^{-1} = mx^{m-1}$$

Assume the formula is true for m. Then, by the Product Rule,

$$D_x(x^{m+1}) = D_x(x^m\cdot x) = x^m D_x(x) + xD_x(x^m) = x^m\cdot 1 + x\cdot mx^{m-1}$$

$$= x^m + mx^m = (m+1)x^m$$

Thus, the formula holds for $m + 1$.

4. Prove Theorem 10.1(9): $D_x(x^m) = mx^{m-1}$, when m is a negative integer.
Let $m = -k$, where k is a positive integer. Then, by the Quotient Rule and Problem 3,

$$D_x(x^m) = D_x(x^{-k}) = D_x\left(\frac{1}{x^k}\right)$$

$$= \frac{x^k D_x(1) - 1\cdot D_x(x^k)}{(x^k)^2} = \frac{x^k\cdot 0 - kx^{k-1}}{x^{2k}}$$

$$= -k\frac{x^{k-1}}{x^{2k}} = -kx^{-k-1} = mx^{m-1}$$

5. Differentiate $y = 4 + 2x - 3x^2 - 5x^3 - 8x^4 + 9x^5$.

$$\frac{dy}{dx} = 0 + 2(1) - 3(2x) - 5(3x^2) - 8(4x^3) + 9(5x^4) = 2 - 6x - 15x^2 - 32x^3 + 45x^4$$

6. Differentiate $y = \frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3} = x^{-1} + 3x^{-2} + 2x^{-3}$.

$$\frac{dy}{dx} = -x^{-2} + 3(-2x^{-3}) + 2(-3x^{-4}) = -x^{-2} - 6x^{-3} - 6x^{-4} = -\frac{1}{x^2} - \frac{6}{x^3} - \frac{6}{x^4}$$

7. Differentiate $y = 2x^{1/2} + 6x^{1/3} - 2x^{3/2}$.

$$\frac{dy}{dx} = 2\left(\frac{1}{2}x^{-1/2}\right) + 6\left(\frac{1}{3}x^{-2/3}\right) - 2\left(\frac{3}{2}x^{1/2}\right) = x^{-1/2} + 2x^{-2/3} - 3x^{1/2} = \frac{1}{x^{1/2}} + \frac{2}{x^{2/3}} - 3x^{1/2}$$

8. Differentiate $y = \frac{2}{x^{1/2}} + \frac{6}{x^{1/3}} - \frac{2}{x^{3/2}} - \frac{4}{x^{3/4}} = 2x^{-1/2} + 6x^{-1/3} - 2x^{-3/2} - 4x^{-3/4}$.

$$\frac{dy}{dx} = 2\left(-\frac{1}{2}x^{-3/2}\right) + 6\left(-\frac{1}{3}x^{-4/3}\right) - 2\left(-\frac{3}{2}x^{-5/2}\right) - 4\left(-\frac{3}{4}x^{-7/4}\right)$$

$$= -x^{-3/2} - 2x^{-4/3} + 3x^{-5/2} + 3x^{-7/4} = -\frac{1}{x^{3/2}} - \frac{2}{x^{4/3}} + \frac{3}{x^{5/2}} + \frac{3}{x^{7/4}}$$

9. Differentiate $y = \sqrt[3]{3x^2} - \frac{1}{\sqrt{5x}} = (3x^2)^{1/3} - (5x)^{-1/2}$.

$$\frac{dy}{dx} = \frac{1}{3}(3x^2)^{-2/3}(6x) - \left(-\frac{1}{2}\right)(5x)^{-3/2}(5) = \frac{2x}{(9x^4)^{1/3}} + \frac{5}{2(5x)(5x)^{1/2}} = \frac{2}{\sqrt[3]{9x}} + \frac{1}{2x\sqrt{5x}}$$

10. Prove the Power Chain Rule: $D_x(y^m) = my^{m-1}D_x y$.
This is simply the Chain Rule, where the outer function is $f(x) = x^m$ and the inner function is y.

11. Differentiate $s = (t^2 - 3)^4$.
By the Power Chain Rule, $\frac{ds}{dt} = 4(t^2 - 3)^3(2t) = 8t(t^2 - 3)^3$.

12. Differentiate (a) $z = \frac{3}{(a^2 - y^2)^2} = 3(a^2 - y^2)^{-2}$; (b) $f(x) = \sqrt{x^2 + 6x + 3} = (x^2 + 6x + 3)^{1/2}$.

(a) $\frac{dz}{dy} = 3(-2)(a^2 - y^2)^{-3}\frac{d}{dy}(a^2 - y^2) = 3(-2)(a^2 - y^2)^{-3}(-2y) = \frac{12y}{(a^2 - y^2)^3}$

(b) $f'(x) = \frac{1}{2}(x^2 + 6x + 3)^{-1/2}\frac{d}{dx}(x^2 + 6x + 3) = \frac{1}{2}(x^2 + 6x + 3)^{-1/2}(2x + 6) = \frac{x + 3}{\sqrt{x^2 + 6x + 3)}}$

13. Differentiate $y = (x^2 + 4)^2(2x^3 - 1)^3$.
Use the Product Rule and the Power Chain Rule:

$$y' = (x^2 + 4)^2\frac{d}{dx}(2x^3 - 1)^3 + (2x^3 - 1)^3\frac{d}{dx}(x^2 + 4)^2$$

$$= (x^2 + 4)^2(3)(2x^3 - 1)^2\frac{d}{dx}(2x^3 - 1) + (2x^3 - 1)^3(2)(x^2 + 4)\frac{d}{dx}(x^2 + 4)$$

$$= (x^2 + 4)^2(3)(2x^3 - 1)^2(6x^2) + (2x^3 - 1)^3(2)(x^2 + 4)(2x)$$

$$= 2x(x^2 + 4)(2x^3 - 1)^2(13x^3 + 36x - 2)$$

14. Differentiate $y = \frac{3-2x}{3+2x}$.

Use the Quotient Rule:

$$y' = \frac{(3+2x)\frac{d}{dx}(3-2x) - (3-2x)\frac{d}{dx}(3+2x)}{(3+2x)^2} = \frac{(3+2x)(-2)-(3-2x)(2)}{(3+2x)^2} = \frac{-12}{(3+2x)^2}$$

15. Differentiate $y = \frac{x^2}{\sqrt{4-x^2}} = \frac{x^2}{(4-x^2)^{1/2}}$.

$$\frac{dy}{dx} = \frac{(4-x^2)^{1/2}\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(4-x^2)^{1/2}}{4-x^2} = \frac{(4-x^2)^{1/2}(2x)-(x^2)(\frac{1}{2})(4-x^2)^{-1/2}(-2x)}{4-x^2}$$

$$= \frac{(4-x^2)^{1/2}(2x)+x^3(4-x^2)^{-1/2}}{4-x^2}\,\frac{(4-x^2)^{1/2}}{(4-x^2)^{1/2}} = \frac{2x(4-x^2)+x^3}{(4-x^2)^{3/2}} = \frac{8x-x^3}{(4-x^2)^{3/2}}$$

16. Find $\frac{dy}{dx}$, given $x = y\sqrt{1-y^2}$.

By the Product Rule,

$$\frac{dx}{dy} = y\cdot\frac{1}{2}(1-y^2)^{-1/2}(-2y)+(1-y^2)^{1/2} = \frac{1-2y^2}{\sqrt{1-y^2}}$$

By Theorem 10.2,

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{\sqrt{1-y^2}}{1-2y^2}$$

17. Find the slope of the tangent line to the curve $x = y^2 - 4y$ at the points where the curve crosses the y axis.

The intersection points are (0, 0) and (0, 4). We have $\frac{dx}{dy} = 2y-4$ and so $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{2y-4}$. At (0, 0) the slope is $-\frac{1}{4}$, and at (0, 4) the slope is $\frac{1}{4}$.

18. Derive the Chain Rule: $D_x(f(g(x)) = f'(g(x))\cdot g'(x))$.

Let $H = f\circ g$. Let $y = g(x)$ and $K = g(x+h) - g(x)$. Also, let $F(t) = \frac{f(y+t)-f(y)}{t} - f'(y)$ for $t \neq 0$. Since $\lim_{t\to 0} F(t) = 0$, let $F(0) = 0$. Then $f(y+t) - f(y) = t(F(t) + f'(y))$ for all t. When $t = K$,

$$f(y+K) - f(y) = K(F(K) + f'(y))$$

$$f(g(x+h)) - f(g(x)) = K(F(K)+f'(y))$$

Hence,

$$\frac{H(x+h)-H(x)}{h} = \frac{K}{h}(F(K)+f'(y))$$

Now,

$$\lim_{h\to 0}\frac{K}{h} = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x)$$

Since $\lim_{h\to 0} K = 0$, $\lim_{h\to 0} F(K) = 0$. Hence,

$$H'(x) = f'(y)g'(x) = f'(g(x))g'(x).$$

19. Find $\frac{dy}{dx}$, given $y = \frac{u^2-1}{u^2+1}$ and $u = \sqrt[3]{x^2+2}$.

$$\frac{dy}{du} = \frac{4u}{(u^2+1)^2} \quad \text{and} \quad \frac{du}{dx} = \frac{2x}{3(x^2+2)^{2/3}} = \frac{2x}{3u^2}$$

Then $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{4u}{(u^2+1)^2}\frac{2x}{3u^2} = \frac{8x}{3u(u^2+1)^2}$$

20. A point moves along the curve $y = x^3 - 3x + 5$ so that $x = \frac{1}{2}\sqrt{t} + 3$, where t is time. At what rate is y changing when $t = 4$?

We must find the value of dy/dt when $t = 4$. First, $dy/dx = 3(x^2 - 1)$ and $dx/dt = 1/(4\sqrt{t})$. Hence,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{3(x^2-1)}{4\sqrt{t}}$$

When $t = 4$, $x = \frac{1}{2}\sqrt{4} + 3 = 4$, and $\frac{dy}{dt} = \frac{3(16-1)}{4(2)} = \frac{45}{8}$ units per unit of time.

21. A point moves in the plane according to equations $x = t^2 + 2t$ and $y = 2t^3 - 6t$. Find dy/dx when $t = 0$, 2, and 5.

Since the first equation may be solved for t and this result substituted for t in the second equation, y is a function of x. We have $dy/dt = 6t^2 - 6$. Since $dx/dt = 2t + 2$, Theorem 8.2 gives us $dt/dx = 1/(2t + 2)$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = 6(t^2-1)\frac{1}{2(t+1)} = 3(t-1).$$

The required values of dy/dx are -3 at $t = 0$, 3 at $t = 2$, and 12 at $t = 5$.

22. If $y = x^2 - 4x$ and $x = \sqrt{2t^2+1}$, find dy/dt when $t = \sqrt{2}$.

$$\frac{dy}{dx} = 2(x-2) \qquad \text{and} \qquad \frac{dx}{dt} = \frac{2t}{(2t^2+1)^{1/2}}$$

So $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{4t(x-2)}{(2t^2+1)^{1/2}}$$

When $t = \sqrt{2}$, $x = \sqrt{5}$ and $\frac{dy}{dt} = \frac{4\sqrt{2}(\sqrt{5}-2)}{\sqrt{5}} = \frac{4\sqrt{2}}{5}(5-2\sqrt{5}).$

23. Show that the function $f(x) = x^3 + 3x^2 - 8x + 2$ has derivatives of all orders and find them.

$f'(x) = 3x^2 + 6x - 8$, $f''(x) = 6x + 6$, $f'''(x) = 6$, and all derivatives of higher order are zero.

24. Investigate the successive derivatives of $f(x) = x^{4/3}$ at $x = 0$.

$$f'(x) = \tfrac{4}{3}x^{1/3} \qquad \text{and} \qquad f'(0) = 0$$

$$f''(x) = \tfrac{4}{9}x^{-2/3} = \frac{4}{9x^{2/3}} \qquad \text{and} \qquad f''(0) \text{ does not exist}$$

$f^{(n)}(0)$ does not exist for $n \geq 2$.

25. If $f(x) = \frac{2}{1-x} = 2(1-x)^{-1}$, find a formula for $f^{(n)}(x)$.

$$f'(x) = 2(-1)(1-x)^{-2}(-1) = 2(1-x)^{-2} = 2(1!)(1-x)^{-2}$$

$$f''(x) = 2(1!)(-2)(1-x)^{-3}(-1) = 2(2!)(1-x)^{-3}$$

$$f'''(x) = 2(2!)(-3)(1-x)^{-4}(-1) = 2(3!)(1-x)^{-4}$$

which suggest $f^{(n)}(x) = 2(n!)(1-x)^{-(n+1)}$. This result may be established by mathematical induction by showing that if $f^{(k)}(x) = 2(k!)(1-x)^{-(k+1)}$, then

$$f^{(k+1)}(x) = -2(k!)(k+1)(1-x)^{-(k+2)}(-1) = 2[(k+1)!](1-x)^{-(k+2)}$$

SUPPLEMENTARY PROBLEMS

26. Prove Theorem 10.1 (5): $D_x(u - v) = D_x u - D_x v$.

Ans. $D_x(u - v) = D_x(u + (-v)) = D_x u + D_x(-v) = D_x u + D_x((-1)v) = D_x u + (-1)D_x v = D_x u - D_x v$ by Theorem 8.1(4, 3)

In Problems 27 to 45, find the derivative.

27. $y = x^5 + 5x^4 - 10x^2 + 6$ *Ans.* $\dfrac{dy}{dx} = 5x(x^3 + 4x^2 - 4)$

28. $y = 3x^{1/2} - x^{3/2} + 2x^{-1/2}$ *Ans.* $\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{x}} - \frac{3}{2}\sqrt{x} - 1/x^{3/2}$

29. $y = \dfrac{1}{2x^2} + \dfrac{4}{\sqrt{x}} = \dfrac{1}{2}x^{-2} + 4x^{-1/2}$ *Ans.* $\dfrac{dy}{dx} = -\dfrac{1}{x^3} - \dfrac{2}{x^{3/2}}$

30. $y = \sqrt{2x} + 2\sqrt{x}$ *Ans.* $y' = (1+\sqrt{2})/\sqrt{2x}$

31. $f(t) = \dfrac{2}{\sqrt{t}} + \dfrac{6}{\sqrt[3]{t}}$ *Ans.* $f'(t) = -\dfrac{t^{1/2} + 2t^{2/3}}{t^2}$

32. $y = (1 - 5x)^6$ *Ans.* $y' = -30(1 - 5x)^5$

33. $f(x) = (3x - x^3 + 1)^4$ *Ans.* $f'(x) = 12(1 - x^2)(3x - x^3 + 1)^3$

34. $y = (3 + 4x - x^2)^{1/2}$ *Ans.* $y' = (2 - x)/y$

35. $\theta = \dfrac{3r+2}{2r+3}$ *Ans.* $\dfrac{d\theta}{dr} = \dfrac{5}{(2r+3)^2}$

36. $y = \left(\dfrac{x}{1+x}\right)^5$ *Ans.* $y' = \dfrac{5x^4}{(1+x)^6}$

37. $y = 2x^2\sqrt{2-x}$ *Ans.* $y' = \dfrac{x(8-5x)}{\sqrt{2-x}}$

38. $f(x) = x\sqrt{3-2x^2}$ *Ans.* $f'(x) = \dfrac{3-4x^2}{\sqrt{3-2x^2}}$

39. $y = (x-1)\sqrt{x^2 - 2x + 2}$ *Ans.* $\dfrac{dy}{dx} = \dfrac{2x^2 - 4x + 3}{\sqrt{x^2 - 2x + 2}}$

40. $z = \dfrac{w}{\sqrt{1-4w^2}}$ *Ans.* $\dfrac{dz}{dw} = \dfrac{1}{(1-4w^2)^{3/2}}$

41. $y = \sqrt{1+\sqrt{x}}$ *Ans.* $y' = \dfrac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$

42. $f(x) = \sqrt{\dfrac{x-1}{x+1}}$ *Ans.* $f'(x) = \dfrac{1}{(x+1)\sqrt{x^2-1}}$

43. $y = (x^2 + 3)^4(2x^3 - 5)^3$ *Ans.* $y' = 2x(x^2 + 3)^3(2x^3 - 5)^2(17x^3 + 27x - 20)$

44. $s = \dfrac{t^2+2}{3-t^2}$ *Ans.* $\dfrac{ds}{dt} = \dfrac{10t}{(3-t^2)^2}$

45. $y = \left(\dfrac{x^2-1}{2x^3+1}\right)^4$ *Ans.* $y' = \dfrac{8x(1+3x-x^3)(x^2-1)^3}{(2x^3+1)^5}$

46. For each of the following, compute dy/dx by two different methods and check that the results are the same: (a) $x = (1 + 2y)^3$ (b) $x = \frac{1}{2+y}$.

In Problems 47 to 50, use the Chain Rule to find $\frac{dy}{dx}$.

47. $y = \frac{u-1}{u+1}$, $u = \sqrt{x}$ *Ans.* $\frac{dy}{dx} = \frac{1}{\sqrt{x}(1+\sqrt{x})^2}$

48. $y = u^3 + 4$, $u = x^2 + 2x$ *Ans.* $\frac{dy}{dx} = 6x^2(x+2)^2(x+1)$

49. $y = \sqrt{1+u}, u = \sqrt{x}$ *Ans.* See Problem 42.

50. $y = \sqrt{u}$, $u = v(3 - 2v)$, $v = x^2$ *Ans.* See Problem 39.
$\left(\textit{Hint}: \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.\right)$

In Problems 51 to 54, find the indicated derivative.

51. $y = 3x^4 - 2x^2 + x - 5;\ y'''$ *Ans.* $y''' = 72x$

52. $y = \frac{1}{\sqrt{x}};\ y^{(4)}$ *Ans.* $y^{(4)} = \frac{105}{16x^{9/2}}$

53. $f(x) = \sqrt{2-3x^2};\ f''(x)$ *Ans.* $f''(x) = -\frac{6}{(2-3x^2)^{3/2}}$

54. $y = \frac{x}{\sqrt{x-1}};\ y''$ $y'' = \frac{4-x}{4(x-1)^{5/2}}$

In Problems 55 and 56, find a formula for the nth derivative.

55. $y = \frac{1}{x^2}$ *Ans.* $y^{(n)} = \frac{(-1)^n[(n+1)!]}{x^{n+2}}$

56. $f(x) = \frac{1}{3x+2}$ *Ans.* $f^{(n)}(x) = (-1)^n \frac{3^n(n!)}{(3x+2)^{n+1}}$

57. If $y = f(u)$ and $u = g(x)$, show that

(a) $\frac{d^2y}{dx^2} = \frac{dy}{du}\cdot\frac{d^2u}{dx^2} + \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2$ (b) $\frac{d^3y}{dx^3} = \frac{dy}{du}\cdot\frac{d^3u}{dx^3} + 3\frac{d^2y}{du^2}\cdot\frac{d^2u}{dx^2}\cdot\frac{du}{dx} + \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3$

58. From $\frac{dx}{dy} = \frac{1}{y'}$, derive $\frac{d^2x}{dy^2} = -\frac{y''}{(y')^3}$ and $\frac{d^3x}{dy^3} = \frac{3(y'')^2 - y'y'''}{(y')^5}$.

In Problems 59 to 64, determine whether the given function has an inverse; if it does, find a formula for the inverse f^{-1} and calculate its derivative.

59. $f(x) = 1/x$ *Ans.* $x = f^{-1}(y) = 1/y$; $dx/dy = -x^2 = -1/y^2$

60. $f(x) = \frac{1}{3}x + 4$ *Ans.* $x = f^{-1}(y) = 3y - 12$; $dx/dy = 3$.

61. $f(x) = \sqrt{x-5}$ *Ans.* $x = f^{-1}(y) = y^2 + 5;\ dx/dy = 2y = 2\sqrt{x-5}$

62. $f(x) = x^2 + 2$ *Ans.* no inverse function

63. $f(x) = x^3$ *Ans.* $x = f^{-1}(y) = \sqrt[3]{y};\ \frac{dx}{dy} = \frac{1}{3x^2} = \frac{1}{3}y^{-2/3}$

64. $f(x) = \frac{2x-1}{x+2}$ *Ans.* $x = f^{-1}(y) = -\frac{2y+1}{y-2};\ \frac{dx}{dy} = \frac{5}{(y-2)^2}$

65. Find the points at which the function $f(x) = |x + 2|$ is differentiable.

Ans. All points except $x = -2$

66. (GC) Use a graphing calculator to draw the graph of the parabola $y = x^2 - 2x$ and the curve $y = |x^2 - 2x|$. Find all points of discontinuity of the latter curve.

Ans. $x = 0$ and $x = 2$

67. Find a formula for the nth derivative of the following functions: (a) $f(x) = \frac{x}{x+2}$; (b) $f(x) = \sqrt{x}$.

Ans. (a) $f^{(n)}(x) = (-1)^{n+1}\frac{2n!}{(x+2)^{n+1}}$

(b) $f^{(n)}(x) = (-1)^{n+1}\frac{3\cdot 5\cdot 7\cdot\ \cdots\ \cdot(2n-3)}{2n}x^{-(2n-1)/2}$

68. Find the second derivatives of the following functions:

(a) $f(x) = 2x - 7$ (b) $f(x) = 3x^2 + 5x - 10$

(c) $f(x) = \frac{1}{x+4}$ (d) $f(x) = \sqrt{7-x}$

Ans. (a) 0; (b) 6; (c) $\frac{2}{(x+4)^3}$; (d) $-\frac{1}{4}\frac{1}{(7-x)^{3/2}}$

69. Prove Theorem 10.2.

Ans. *Hints*: (a) Use the intermediate value theorem to show that the range is an interval. That f is increasing or decreasing follows by an argument that uses the extreme value and intermediate value theorems. The continuity of f^{-1} is then derived easily.

(b) $$\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \frac{1}{\dfrac{f(f^{-1}(y))-f(f^{-1}(y_0))}{f^{-1}(y)-f^{-1}(y_0)}} = \frac{1}{\dfrac{f(x)-f(x_0)}{x-x_0}}$$

By the continuity of f^{-1}, as $y \to y_0$, $x \to x_0$, and we get $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.