2 Finite Geometries

We are now about to examine two systems of axioms that exemplify, in miniature, the structure and characteristics of many modern axiomatic systems. These systems, both of which were introduced at the turn of the twentieth century, have by now attained the stature of classics in the history of mathematics. Except for phrasing (and one omission—we have left out one axiom that will be added at the end of the chapter), they are the systems of J. W. Young and G. Fano. We will see that the two systems have much in common while being quite different. By analyzing, comparing, and modifying them, it is hoped that one will gain a better understanding of exactly what an axiomatic system is.[1]

2.1 Axiom Sets 1 and 2

In first stating the two sets of axioms, we present them in what might be called their "natural" or common language phrasing. Young's axiom set is listed first:

Axiom Set 1

1. There is exactly one line through any two points.
2. Every line contains at least three points.
3. There exists at least one line.
4. Not all points are on the same line.

[1] By an "axiomatic system," or "system of axioms," we mean the entire structure made up of axioms, theorems, and definitions. By an "axiom set" we mean merely "the set of axioms." It may not be good English to use "axiom" as an adjective, but it is common in mathematics.

point & line are undefined term

5. Through a point not on a given line, there is exactly one line which does not meet the given line.

Axiom Set 2

1. There exists at least one line.
2. Every line contains at least three points.
3. There is exactly one line through any two points.
4. Not all points are on the same line.
5. Any two lines have a point in common.

1-4 this true for great circle but 5 is not.

Even a cursory glance at the lists shows us that they have much in common. Indeed, except for the order in which they occur, the first four axioms are the same in both sets. And even at this stage of our presentation it should be obvious that a system of axioms is not essentially changed when the order in which the axioms are written down is rearranged. The sets differ, then, only in the fifth axiom; it shall be interesting to see how much this affects each system, how it introduces new theorems into each system; before attempting proofs, however, we would like to make some changes for the sake of clarity.

First, a word about the use of "point" and "line." In modern systems the words "point" and "line" are often not used at all, at least not in the first presentation. Instead, one might simply use "x's" and "y's"; another favorite substitute is such nonsense words as "abba" and "dabba." This is to stress the fact that the words "point" and "line" are truly undefined terms and that no other properties whatsoever can be assumed about these terms except what is given by the axioms.

An offshoot of this is that, whatever a "point" is, a "line" is going to be some kind of collection of "points" (because they are undefined terms, it could be the other way around). Thus, whatever "points" are, they can be considered as elements of some universal set; a "line" may then be regarded as some *undefined subset* of the universal set. With this approach, new words may be substituted for "point" and "line" respectively, as follows: "element of" and "l-set"; "bead" and "wire"; "man" and "committee"; and so forth. In effect, this sets up a relationship between the two undefined terms that could be introduced into the system as the following *axiom:* every line is a set of points. We are not going to adopt this procedure at the present time, but should point out that the statement is indeed an axiom and not merely a definition, as one might be tempted to think. To see that it is not a definition, one needs merely to observe that in most systems where this axiom could be used there will be other sets of points which are not lines, such as circles, triangles, and so on.

To summarize: in order to force one to refrain from using any familiar qualities of points and lines—qualities that may be actually false in a particular system or, at best, qualities which, while true, have not been formally introduced into the system—many modern systems use nonsense terms in place of "point" and "line". We shall not do so. For if one guards against using anything not explicitly given, it is easier for the beginner to form proofs if he has some interpretation of the undefined terms in mind He may even use diagrams, *but only if he guards against bringing in any hidden assumptions.*

It may surprise some readers to learn that nowhere, in the two axiom sets presented, is there an *explicit* guarantee that points exist. Statements such as that given above, that "every line is a set of points," and statements such as axioms 1, 2, and 5 in either set, *seem* to guarantee that points exist, but do not. This is because of the way in which such statements are translated into "if . . . then . . . ." form in modern logic. Let us look into this.

It is customary to equate several types of universal statements. Thus each of the following are interpreted as saying the same thing:

1. All lines have at least three points on them.
2. Any line has at least three points on it.
3. Each line has at least three points on it.
4. Every line has at least three points on it.

The last three are more precise, the first suggesting as it does that all lines (collectively) have at least three points on them. But generally they are accepted as saying the same thing, and mathematicians tend to equate them with still another form.

5. If l is any line, then it has at least three points on it.

In this form it is more apparent that the statement says nothing about the existence of l and hence nothing about the existence of three points. If, in addition to the above, it is known that there *exists* a line l, it could then be concluded:

Some line has three points on it.

or

There exists a line l with three points on it.

To sum up: universal statements (of the types 1–5, above) are not existential.

Now let us restate the two axiom sets and renumber them for future use.

Axiom Set 1

1a. If P and Q are any two points, then there exists at least one line containing both P and Q.

1b. If P and Q are any two points, then there exists at most one line containing both P and Q.
2. If l is a line, then there exist at least three points on it.
3. If l is a line, then there exists a point P not on it.
4. There exists at least one line.
5a. If l is a line and P a point not on it, then there exists at least one line m through P with no point in common with l.
5b. If l is a line and P a point not on it, then there exists at most one line m through P with no point in common with l.

Axiom Set 2

1a. If P and Q are any two points, then there exists at least one line containing both P and Q.
1b. If P and Q are any two points, then there exists at most one line containing both P and Q.
2. If l is a line, then there exist at least three points on it.
3. If l is a line, then there exists a point P not on it.
4. There exists at least one line.
5. If l and m are any two lines, then there exists at least one point P belonging to both l and m.

As is more apparent in this reworded version, only Axiom 4 is an existence statement. The others merely say "if . . . then . . . ," without guaranteeing the existence of the "if" part of the statement. Without Axiom 4, both systems would be empty of both "points" and "lines," because nothing can be assumed about these concepts, let alone something as essential as their existence.

It will pay the reader to study the restatements of these axioms. We shall, from time to time throughout the book, make a statement first in its "natural" form. Then, for the sake of clarity, and perhaps for ease of reference, we shall restate it. The restatement of Axiom 1 and Axiom 5 into two parts is an example of a device often used to prove theorems in mathematics. It depends on equating the phrase "there exists exactly one" with the conjunction of two statements: "there exists at least one" and "there exists at most one."

2.2 An Axiomatic System, 3

Now let us consider the set of *all* axioms that Axiom Set 1 and Axiom Set 2 have in common. We might have chosen any subset of *each* or *any* common subset, but there is a reason for our present choice. For future reference let us list them and call them Axiom Set 3.

Axiom Set 3

1a. If P and Q are any two points, then there exists at least one line containing both P and Q.

1b. If P and Q are any two points, then there exists at most one line containing both P and Q.

2. If l is a line, then there exist at least three points on it.
3. If l is a line, then there exists a point P not on it.
4. There exists at least one line.

Before theorems can be proved from these axioms—in fact, before the axioms can be understood—the following rules of language are needed:

Language Rule: *If a point P is an element of the line l, then we say variously:* l PASSES THROUGH P*:* l CONTAINS P*:* P IS ON l*;* P LIES ON l.

Language Rule: *If a point P is an element of more than one line, then we may say:* l MEETS m *in* P*;* l INTERSECTS m. *(ϕ not allowed).*

In a less simple system than the one being examined, many definitions and rules would be introduced. Such definitions might be avoided, but only at the expense of a great amount of wordiness and stilted phrasing.

We are now ready for some theorems. Proving theorems is difficult enough, the reader might be thinking; how does one go about *discovering* what theorems one is going to prove? In examining a system of axioms we must ask, first, why one chooses the axioms one does and, second, what theorems it is possible to prove from them. These questions are often ignored, not because they lack interest or importance, but because it is difficult, if not impossible, to present a workable answer. Choosing a useful set of axioms and discovering the theorems they imply is a creative act. It is as creative as anything in literature or painting, or any of the other recognized creative arts.

Our present undertaking, however, is more modest. We have observed that the existence of points in our system is not *explicitly* given; their existence is assured, however, by the following theorem:

Theorem 1. There exist at least three distinct points.

Proof: Follows immediately from Axiom 4 and Axiom 2. ●

This theorem suggests the following analogous theorem:

Theorem 2. There exist at least three distinct lines.

Proof: By Axiom 4 there exists a line l and by Axiom 2 it has at least three points, say P, Q, and R. By Axiom 3 there exists a point, say S, not on l, and by Axiom 1a and Axiom 1b there is exactly one line through each of the pairs: PS, QS, RS. Moreover, these lines are distinct. For suppose any two of them were equal, say $SR = SQ$; then there would be two lines containing QR, namely PQR and SQR, contradicting Axiom 1b. This would be true, similarly, if $SR = SP$ or $SQ = SP$. Hence, there exist at least three distinct lines. ●

An examination of the above proof not only shows that we have proved more than we set out to prove (that there exist four distinct lines), but shows that, in the process, we did not use Theorem 1. It therefore would have been possible to prove the second one first; the numbers of the theorems are not important. But should not Theorem 2 depend upon Theorem 1, and 3 upon 2 and 1, and so on? Not necessarily. In a very simple system, everything can be proved directly from the axioms. In fact, in any system one can in theory prove any theorem directly from the axioms (and definitions); however, the theorems are usually a convenient shortcut to proofs. And once one begins to cite theorems as reasons for steps, care must be taken about the ordering lest, in a proof, a theorem is used that in turn is proved by using the theorem that is in the process of being proved. This is circular reasoning; it is worse than circular definitions.

In the search for more theorems, the following questions are likely to occur after proving the first two theorems: How many points (lines) are there on a line (point)? Is there a finite number and, if so, can it be determined? How many points (lines) are there in the system? After exploring these questions, one discovers one can prove a related theorem:

Theorem 3. There exist at least seven points and seven lines.

Proof: By Axiom 4 there exists at least one line, which by Axiom 2 has at least three points on it. Let us designate points by A, B, C, D, . . . and lines by AB . . . (for this proof). Then, by Axiom 1a and Axiom 1b, there is a unique line containing the three points, say ABC; it is, for example, the only line containing both A and C.

Since by Axiom 3 not all points are on the same line, there exists a point D which is not on ABC. And now by Axiom 1a, D must lie on a line with A, a line which by Axiom 2 must have at least three points on it, and which by Axiom 1b cannot contain either B or C. Hence, there exists a line ADE and, furthermore, A, B, C, D, and E are distinct. By similar arguing there exists a line BDF and A, B, C, D, E, and F are distinct.

So far we have three lines, ABC, ADE, and BDF and six points. By Axiom 1a every pair of points must have a line containing them. Point A now lies on a line with B, C, D, and E and hence must lie on a line with F; but this line must have at least one more point, and by Axiom 1b it cannot be B, C, D, or E, so there must exist another point G such that AFG.

Thus, there exist at least seven points.

Continuing in this manner, we have ABC, ADE, BDF, and AFG and at least seven points. By Axiom 1a the following possible lines must be considered: $BE \ldots$, $BG \ldots$, $EG \ldots$, $CD \ldots$, $CG \ldots$, $DG \ldots$, $CE \ldots$, $CF \ldots$, $EF \ldots$. Consider line $BE \ldots$. By Axiom 1b it can contain only G from the given choice of points, and by Axiom 2 it must; hence we have a new line, BEG. (Axiom 1b now rules out $BG \ldots$ and $EG \ldots$).

Now consider $CD \ldots$. By Axiom 1b and Axiom 2 it must contain G; so we have a new line CDG. (And Axiom 1b rules out $CG \ldots$ and $DG \ldots$). By Axiom 1b and Axiom 2, $CE \ldots$ must contain F, giving us CEF. (And Axiom 1b rules out $CF \ldots$, and $EF \ldots$). Hence we have generated distinct lines: ABC, ADE, BDF, AFG, BEG, CDG, and CEF.

Thus, there exist at least seven lines. ● (Can we prove this theorem for $n > 7$? See Exercises 2.2)

Continuing in this manner, and keeping in mind that we are dealing with undefined terms, we might ask other questions that we ordinarily might not think of. We might, for example, ask the number of points which two lines may have in common; must they have any? If so, how many?

If we set out to answer the question, "must two lines have any points in common?", we will arrive at an interesting answer. We need merely observe that the fifth axioms of Axiom Sets 1 and 2 answer this question both negatively and affirmatively. It would seem likely, therefore, that this question cannot be answered at all from Axiom Set 3, unless there is reason to believe that one of the axioms numbered "5" can be proved as a theorem and is therefore not an axiom. If one can show that none of the axioms in either Set 1 or Set 2 can be proved as a theorem; specifically, if it can be shown that Axioms 5a and 5b of Set I are not implied by Axiom Set 3; and if it can further be shown that Axiom 5 of Set 2 cannot be derived from Axiom Set 3, then one is left with an inescapable conclusion: a question has been found phraseable in the terms of our system that is unanswerable in our system. It cannot be proved that any two lines have a point in common; contrary to this, it cannot be proved that there exist parallel lines. We shall return to this interesting problem in the next chapter.

If we now turn to the question, "what is the *most* number of points two lines may have in common?", the answer is not so difficult. It will be given after the next section.

EXERCISES 2.2

Using Axiom Set 3, try to prove the following:

1. There exist at least 8 points.
2. There exist at least 4 points on a line.
3. There exists at least one pair of nonintersecting lines.

2.3 Direct and Indirect Proofs

We have just seen some examples of proofs. To better understand them, let us take a look at their skeletal forms. First, we shall look at direct proofs, which for our present purposes can be regarded as falling simply into two patterns:

(a)
If P then Q
P
———
Q

(b)
If P then Q
not-Q
———
not-P

These are called *valid-argument* patterns. They say that if the two statements above the line are granted, the statement below the line is an inescapable conclusion or, alternatively, that it *follows from* the first two. Stated in such terms, there are overtones of mental processes at work. So instead, we shall say, "If P then Q, and P" implies "Q," and "If P then Q and not-Q" implies "not-P." Keeping in mind that "implies" is to be regarded as an undefined relation, it is useful to point out that these two patterns illustrate one of the properties the relation is to have: that true statements cannot imply false statements.

A fallacy occurs when the valid patterns (a) and (b) are confused with the following invalid ones:

(c)
If P then Q
Q
———
P

(d)
If P then Q
not-P
———
not-Q

The easiest way to commit this error is to confuse a conditional statement with its converse.

The patterns (a) and (b) are by no means the only valid direct argument patterns but they are the only ones we shall introduce for now. It is surprising how far we can go with just these and equivalent variations of them when we introduce a property of implication that permits us to build long chains of such patterns. This property of implication is called *transitivity:* if A implies B, and B implies C, then A implies C.

Directing our attention now to indirect proofs, we shall see that this type of proof is based on a property of implication just mentioned: it is impossible for a true statement to imply a false statement. Let us see how this works.

Suppose one wishes to prove the statement:

(a) If P then Q.

One starts by assuming its contradictory; specifically, assume:

(b) P and not-Q.

If statement (b) implies a statement known to be false, then statement (b) must *be* a false statement—for it is impossible for a true statement to imply a false statement. But, if the statement (b) is false, then statement (a), its contradictory, must be true; hence, in this manner, "If P then Q" has been shown to be true.

The phrase "if statement (b) implies a statement known to be false" used in the preceding explanation requires some comment. First, rarely in mathematics does a single statement imply another. Usually, when it is said that a statement implies another, this means that a statement, together with others (assumed or previously proved), imply the second statement. Secondly, a contradiction may be reached in several ways. As soon as the truth of "P and not-Q" is assumed, it follows from the definition of conjunction that both "P" and "not-Q" hold; therefore, a contradiction is obtained in any one of the following ways: (1) "*P and not-Q*" *implies* "*not-P*"; (2) "*P and not-Q*" *implies* "*Q*"; or, (3) "*P and not-Q*" *implies* "*R and not-R.*" In any one of these cases the assumption "P and not-Q" must be false.

The most common error one makes in using an *indirect proof* (which we are using synonymously with *proof by contradiction*) is to use *any* denial of the universal statement rather than its contradictory. Suppose, for example, we are trying to prove that two line segments $\overline{AB}$ and $\overline{CD}$ are equal, and suppose we assume that $\overline{AB}$ is greater than $\overline{CD}$, giving us a contradiction; this does not then entitle us to claim that the two segments are equal, for it is still possible that $\overline{AB}$ might be less than $\overline{CD}$. Whenever the contradiction of a statement is not used in a proof, one must be careful to take into account all the possible cases.

Universal statements in various forms have already been discussed.

For illustrative purposes, consider "Every line has at least three points on it." The contradiction of this is not "No line has three points on it"—nor can it be stated as "There exists a line with only one point on it." Both of these are contraries rather than contradictories. The contradictory would be "There exists a line which does not have at least three points on it" or "There exists a line with at most two points on it."

One final remark is in order. Our comments so far have been concerned with attempts to prove a universal statement true. In order to prove one false, one need only find a single *counterexample*, a single instance of its falsity.

Before going on to prove more theorems, let us return to Theorem 2 for a brief analysis. A brief look at the proof seems to show us that it is both direct and indirect. The first half is direct and the second half appears to be indirect. But this is only apparently so. We shall consider this a direct proof. The only proof which we will call *indirect* is one which begins immediately by contradicting (or denying and considering all cases) the statement we are attempting to prove. Let us analyze the proof step by step.

STEP 1: Axiom 2 together with Axiom 4 is an instance of argument form (a).

STEP 2: Axiom 3 together with Axiom 4 is another instance of pattern (a).

STEP 3: Axioms 1a and 1b together with the results of Steps 1 and 2 are another instance of pattern (a).

It now remains to prove that the three lines are distinct. Instead of incorporating this into the proof, suppose we take it aside and prove a helping theorem, called a *lemma*.

Lemma 1: *Suppose there exist three points P, Q, R on a line, a point S not on the line, and three lines PS, QS, and RS joining these points; then these lines are distinct.*

Proof: Same as given.

We can now give the *direct* proof of Theorem 2: Steps 1, 2, 3 and Lemma 1.

2.4 Further Proofs in Axiomatic System 3

We cannot easily continue our heuristic approach to geometry without creating a mass of confusion. This type of unordered thinking—asking about possible theorems, hunting and finding them by trial and error, and trying to discover proofs for them—goes into creating a system, but once the discoveries have been made we must present the theorems in an orderly manner. Let us attempt to do that now.

Theorem 4. Two lines have at most one point in common.

Restatement: *If l and m are any two lines, then there exists at most one point P in their intersection.*
Proof: Suppose the statement is false. Then there exist two lines l and m which have at least two points, say P and Q, in common. But this immediately contradicts Axiom 1b, for l contains P and Q, and m contains P and Q, and Axiom 1b says that at most one line contains two given points. The assumption that our statement is false leads to a contradiction, so we conclude: two lines have at most one point in common. ●

(NOTE: We have been using an unstated convention. Whenever we speak of points P, Q, or lines m, n, it is possible they may be the same; but whenever we speak of the *two* points P, Q, or two lines m, n, we mean them to be distinct.)

To further illustrate the differences in techniques of direct and indirect proofs, we shall prove the next theorem both directly and indirectly.

Theorem 5. Not all lines pass through the same point.

Restatement: *If P is a point, then there exists at least one line not containing P.*
Direct Proof: Let P be a point. This may be said because by Theorem 1 points are known to exist. By the same theorem there exists a second point Q. Now by Axiom 1a there exists a line l containing P and Q. By Axiom 3 there exists a point R not on l and by Axiom 1a there exists a line m containing Q and R. Because R is not on l, l and m are distinct. Now if P belongs to m, then there exist distinct lines with two points in common, contradicting Theorem 4. Hence, m is a line not containing P. ●

As stated previously, we call this a direct proof despite the contradiction used to prove that P does not belong to m. We are calling any proof direct if it does not rest on assuming that the statement being proved is false.

Indirect Proof: Suppose that every line passes through the same point P. By Theorem 2 there exist at least two lines. If they are to pass through P and be distinct, they must contain points not in common, so let us say that l contains P and Q and m contains P and R. By Axiom 1a there exists a line, say k, containing Q and R; by assumption, it must contain P. But this contradicts Theorem 4. Hence, our assumption that every line passes through P must be false. Hence, there exists at least one line not through P. ●

Theorem 6. There exist at least three lines through any point.

Restatement: *If P is any point, there exist at least three lines through P.*
Proof: Suppose that there exists a point P with at most two lines through it, call them l (containing P and Q) and m (containing P and R). By Theorem 5 there exists a line not through P. If this line does not contain Q and R, then it contains three other points by Axiom 2, and by Axiom 1a there exist at least three more lines containing these points and P. If, on the other hand, the line not containing P does contain Q or R, then by Axiom 2 it contains at least one other point S and, once again by Axiom 1a, there exists another line containing P and S. In either case our assumption is contradicted. Hence there exist at least three lines through any point. ●

Note that while our proof classifies as indirect in this case, we are actually obtaining our contradiction by showing the existence of a third line. This suggests that a direct proof is certainly possible and may even be simpler, but this is left to be solved as an exercise. Instead, we will show how a slight change in the proof of Theorem 6 gives us another indirect proof.

Theorem 6 (***Alternative***)
Proof: Suppose that there exists a point P with at most two lines through it, call them l (containing P and Q) and m (containing P and R). By Theorem 5 there exists a line not through P. Whether this line contains Q and R or not, there exists a new point on it, call it S. But by our assumption there cannot be another line containing P, and this now contradicts Axiom 1a. Hence our assumption is false and there exist at least three lines through any point. ●

EXERCISES 2.4

1. Can you prove from Axiom Set 3 that there exist at most seven points?
2. Can you prove from Axiom Set 3 that there exist at most three points on a line?
3. Can you prove from Axiom Set 3 that any two lines must intersect?
4. Give a direct proof of Theorem 6.

5. Analyze each of the indirect proofs given in this section. Which of the three types listed in Section 2.3 is each?

6. What is wrong with the following direct "proof" of Theorem 6?

Proof: By Axiom 4 there exists a line; by Axiom 2 it has three points on it, call them P, Q, R; by Axiom 3 there exists a point S not on the line and by Axiom 1a there exist three lines PS, QS, and RS through S.

7. Can the "proof" in exercise 6 be revised to correct it?

2.5 Axiomatic System 1

In the exercises in the preceding sections it was suggested that one try to prove that there exists at least one pair of nonintersecting lines and that there exist at least eight points. We trust that attempts to prove these statements were not successful.

Let us now convert Axiom Set 3 back into Axiom Set 1 by adding Axioms 5a and 5b in order to see what new theorems we can prove. Using Axiom Set 1 we may derive the following:

Theorem 7. There exist at least nine points and at least twelve lines.

Proof: Left as an exercise.

Definition. Two lines l and m are called *parallel* if l and m have no point in common.

Theorem 8. There exist at least two lines parallel to a given line.

Proof: By Axiom 4 there exists at least one line l; by Axiom 3 there exists a point P not on l; by Axiom 5a there exists at least one line m parallel to l through P. By Axiom 2 there exists at least one point Q on l and by Axiom 1a there exists a line PQ, which by Axiom 2 has a third point S on it. By Axiom 5a there exists a line k through S, parallel to l. These lines must be distinct. (Why?) ●

Theorem 9. If a line, distinct from two parallel lines, intersects one of two parallels, it must intersect the other.

Restatement: *If k and l are two parallels, and m intersects k at a point P, then m intersects l at some point.*

Proof: Suppose not. Then through P there would be two lines m and k not intersecting l. But this contradicts Axiom 5b. Hence the assumption must be false and thus, if a line intersects one of two parallels, it intersects the other. ●

Theorem 10. Two lines parallel to the same line are parallel to each other.

Restatement: *If k and l are parallel and l and m are parallel, then k and m are parallel.*
Proof: Suppose not. If k is not parallel to m, then it intersects m and by Theorem 9 must then also intersect l, contradicting our assumption that it is parallel to l. Hence our assumption must be false. So two lines parallel to the same line must be parallel to each other. ●

Before attempting a proof of the next theorem, let us take time to look back at some of the preceding proofs. It is likely that those who attempted to follow proofs of Theorems 7 and 8 very closely had to resort to some kind of visual aid. We have purposely omitted diagrams up to now in order to force the reader to fend for himself. In a proof as difficult as the next one, however, it is *almost* necessary to use a diagram to construct the proof and to follow it. We say "almost" because it is never necessary; a diagram is merely an aid to one's intuition and not a part of a proof. As an aid, it is perfectly all right to use a diagram at any time, but observe the warning mentioned earlier and do not, in using a diagram, bring in any hidden assumptions. In a proof such as the following it is easy to transgress on this rule without even being aware of it.

Because the proof is a long one, we shall break it up into two lemmas. We have already indicated that a *lemma* is a theorem proved primarily to help prove some other theorem. Generally, it is called a *lemma* when its use is more or less restricted to the theorem in question; if it had other uses, one would usually call it a theorem and give it an appropriate number. (We assume knowledge of the properties of the natural numbers.) The reader may refer to Figure 2.1.

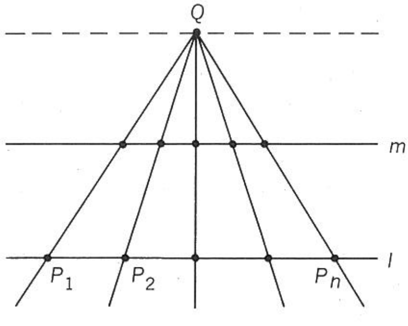

FIGURE 2.1

Lemma 1: *If one line l contains exactly n points, then any line parallel to l contains exactly n points.*

Proof: Suppose l is a line containing exactly n points $P_1, P_2, \cdots, P_n$. Since by Theorem 8 there exist other lines parallel to l, we may let m be any line parallel to l. Once again by Theorem 8 there exists still another line, parallel to both m and l, and by Axiom 2 it has a point Q on it, which by the definition of parallel cannot be on either m or l. Now by Axiom 1a there exist lines $QP_1, QP_2, \cdots, QP_n$, which by Axiom 1b are distinct. By Theorem 9 these lines intersect m and by Theorem 4 it must be in n distinct points. Hence there exist at least n distinct points on m.

Suppose there exists another point on m, say P_{n+1}. Then there must be a line connecting Q with P_{n+1}, and by Theorem 9 and Theorem 4 it must intersect l in some point other than $P_1, P_2, \cdots, P_n$, contradicting the assumption that l has exactly n points. Hence there exist at most n distinct points on m.

The lemma now follows. ●

The reader should fill in the steps, mostly justifications, that have been left out of the above proof, and should convince himself that the locations of Q, m, and l in the diagram are immaterial. We could choose Q as shown in Figure 2.2, or any other of the possible arrangements of Q, m and l. It

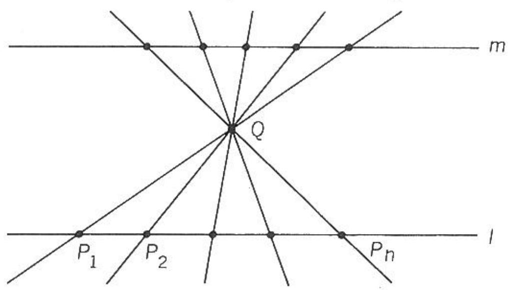

FIGURE 2.2

is the power of Theorem 9 that assures us that we shall obtain our n points on m regardless of where Q is located. To follow the next proof refer to Figure 2.3.

Lemma 2: *If any line l contains exactly n points, then there exist exactly $n - 1$ lines parallel to l.*

Proof: Let l be a line with exactly n points $P_1, P_2, \cdots, P_n$ on it and let Q be a point not on l. Then there exists a line QP_1 and another line QP_i (where P_i is any of the n points other than P_1). By Axioms 5a and 5b there exist exactly $n - 1$ lines through $P_2, \cdots, P_n$ that are parallel to

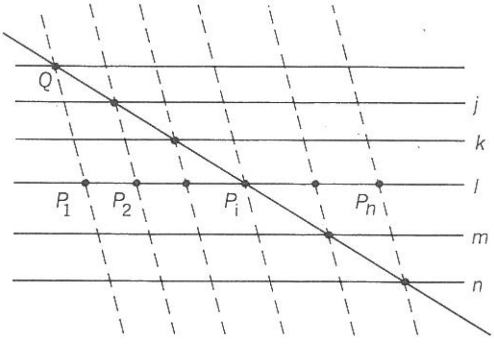

FIGURE 2.3

QP_1 (one of which will pass through P_i). By Theorem 9 and Theorem 4, QP_i, which intersects QP_1 and the line parallel to it through P_i, must intersect each of the other lines in exactly one point. Hence, there exist exactly n points on line QP_i.

By Axioms 5a and 5b and Theorem 4 there exist *exactly* $n - 1$ parallels to l (Why?). ●

Theorem 11. If one line contains exactly n points, then every line contains exactly n points.

Proof: Given a line l with exactly n points on it, any line either intersects l or does not. If it does not, then by Lemma 1 it has exactly n points on it. If it does intersect l, then by Theorem 9 it intersects the $n - 1$ lines parallel to l, lines that exist by Lemma 2; by Theorem 4 it intersects in exactly n points. Hence, if one line contains exactly n points, every line does. ●

EXERCISES 2.5

1. Prove Theorem 7.
2. Can you prove that there exist *at most* nine points?
3. Complete the proofs of the two lemmas.
4. Prove: If there exists one line with exactly n points, then every point has exactly $n + 1$ lines through it.

Using Axiom Set 2, prove the following:

5. If there exists one line with exactly n points on it, then every line contains exactly n points.
6. If there exists one line with exactly n points on it, then every point has exactly n lines through it.

2.6 The Systems of Young and Fano

If to Axiom Set 1 we add the following axiom we get the original system credited to Young.

Axiom 6. If l is a line, there exist at most three points on it.

We shall refer to this system as the System of Young. The addition of this last axiom has a profound effect on the system; it makes it a finite geometry, as is illustrated by the next three theorems.

Theorem 12Y. There exist exactly nine points in the system.

Proof: Left as an exercise.

Theorem 13Y. There exist exactly twelve lines in the system.

Proof: Left as an exercise.

Theorem 14Y. There exist exactly four lines through any point.

Proof: Left as an exercise.

If now the same Axiom 6 is added to Axiom Set 2, the system originally credited to Fano is obtained. We shall refer to this system as the System of Fano. It, too, is a finite system but, as one might expect, of a different nature than the System of Young.

Using now the System of Fano, we can prove:

Theorem 12F. There exist exactly seven points.

Proof: Left as an exercise.

Theorem 13F. There exist exactly seven lines.

Proof: Left as an exercise.

Theorem 14F. There exist exactly three lines through any point.

Proof: Left as an exercise.

EXERCISES 2.6

1–6. Prove Theorems 12Y, 13Y, 14Y, 12F, 13F, and 14F.

7. Suppose one adds Axiom 6 to Axiom Set 3; at most, how many points and lines do you suppose one can prove to exist?