CHAPTER 6

Functions

We say that a quantity y is a *function* of some other quantity x if the value of y is determined by the value of x. If f denotes the function, then we indicate the dependence of y on x by means of the formula $y = f(x)$. The letter x is called the *independent variable*, and the letter y is called the *dependent variable*. The independent variable is also called the *argument* of the function, and the dependent variable is called the *value* of the function.

For example, the area A of a square is a function of the length s of a side of the square, and that function can be expressed by the formula $A = s^2$. Here, s is the independent variable and A is the dependent variable.

The *domain* of a function is the set of numbers to which the function can be applied, that is, the set of numbers that are assigned to the independent variable. The *range* of a function is the set of numbers that the function associates with the numbers in the domain.

EXAMPLE 6.1: The formula $f(x) = x^2$ determines a function f that assigns to each real number x its square. The domain consists of all real numbers. The range can be seen to consist of all nonnegative real numbers. (In fact, each value x^2 is nonnegative. Conversely, if r is any nonnegative real number, then r appears as a value when the function is applied to $\sqrt{r}$, since $r = (\sqrt{r})^2$.)

EXAMPLE 6.2: Let g be the function defined by the formula $g(x) = x^2 - 4x + 2$ for all real numbers. Thus,

$$g(1) = (1)^2 - 4(1) + 2 = 1 - 4 + 2 = -1$$

and

$$g(-2) = (-2)^2 - 4(-2) + 2 = 4 + 8 + 2 = 14$$

Also, for any number a, $g(a + 1) = (a + 1)^2 - 4(a + 1) + 2 = a^2 + 2a + 1 - 4a - 4 + 2 = a^2 - 2a - 1$.

EXAMPLE 6.3: (a) Let the function $h(x) = 18x - 3x^2$ be defined for all real numbers x. Thus, the domain is the set of all real numbers. (b) Let the area A of a certain rectangle, one of whose sides has length x, be given by $A = 18x - 3x^2$. Both x and A must be positive. Now, by completing the square, we obtain

$$A = -3(x^2 - 6x) = -3[(x - 3)^2 - 9] = 27 - 3(x - 3)^2$$

Since $A > 0$, $3(x - 3)^2 < 27$, $(x - 3)^2 < 9$, $|x - 3| < 3$. Hence, $-3 < x - 3 < 3$, $0 < x < 6$. Thus, the function determining A has the open interval $(0, 6)$ as its domain. The graph of $A = 27 - 3(x - 3)^2$ is the parabola shown in Fig. 6-1. From the graph, we see that the range of the function is the half-open interval $(0, 27)$.

Notice that the function of part (b) is given by the same formula as the function of part (a), but the domain of the former is a proper subset of the domain of the latter.

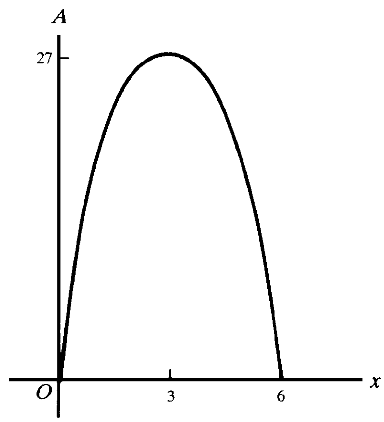

Fig. 6-1

The graph of a function f is defined to be the graph of the equation $y = f(x)$.

EXAMPLE 6.4: (a) Consider the function $f(x) = |x|$. Its graph is the graph of the equation $y = |x|$, and is indicated in Fig. 6-2. Notice that $f(x) = x$ when $x \geq 0$, whereas $f(x) = -x$ when $x \leq 0$. The domain of f consists of all real numbers. (*In general, if a function is given by means of a formula, then, if nothing is said to the contrary, we shall assume that the domain consists of all numbers for which the formula is defined.*) From the graph in Fig. 6-2, we see that the range of the function consists of all nonnegative real numbers. (*In general, the range of a function is the set of y coordinates of all points in the graph of the function.*) (b) The formula $g(x) = 2x + 3$ defines a function g. The graph of this function is the graph of the equation $y = 2x + 3$, which is the straight line with slope 2 and y intercept 3. The set of all real numbers is both the domain and range of g.

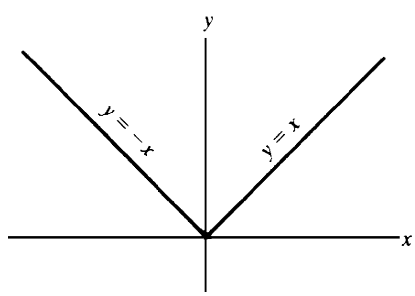

Fig. 6-2

EXAMPLE 6.5: Let a function g be defined as follows:

$$g(x) = \begin{cases} x^2 & \text{if } 2 \leq x \leq 4 \\ x + 1 & \text{if } 1 \leq x < 2 \end{cases}$$

A function defined in this way is said to be *defined by cases*. Notice that the domain of g is the closed interval [1, 4].

In a rigorous development of mathematics, a function f is defined to be a set of ordered pairs such that, if (x, y) and (x, z) are in the set f, then $y = z$. However, such a definition obscures the intuitive meaning of the notion of function.

SOLVED PROBLEMS

1. Given $f(x) = \dfrac{x-1}{x^2+2}$, find (a) $f(0)$; (b) $f(-1)$; (c) $f(2a)$; (d) $f(1/x)$; (e) $f(x+h)$.

(a) $f(0) = \dfrac{0-1}{0+2} = -\dfrac{1}{2}$ (b) $f(-1) = \dfrac{-1-1}{1+2} = -\dfrac{2}{3}$ (c) $f(2a) = \dfrac{2a-1}{4a^2+2}$

(d) $f(1/x) = \dfrac{1/x - 1}{1/x^2 + 2} = \dfrac{x - x^2}{1+2x^2}$ (e) $f(x+h) = \dfrac{x+h-1}{(x+h)^2+2} = \dfrac{x+h-1}{x^2+2hx+h^2+2}$

2. If $f(x) = 2^x$, show that (a) $f(x+3) - f(x-1) = \frac{15}{2} f(x)$ and (b) $\dfrac{f(x+3)}{f(x-1)} = f(4)$.

(a) $f(x+3) - f(x-1) = 2^{x+3} - 2^{x-1} = 2^x(2^3 - \frac{1}{2}) = \frac{15}{2} f(x)$ (b) $\dfrac{f(x+3)}{f(x-1)} = \dfrac{2^{x+3}}{2^{x-1}} = 2^4 = f(4)$

3. Determine the domains of the functions

(a) $y = \sqrt{4-x^2}$ (b) $y = \sqrt{x^2-16}$ (c) $y = \dfrac{1}{x-2}$

(d) $y = \dfrac{1}{x^2-9}$ (e) $y = \dfrac{x}{x^2+4}$

(a) Since y must be real, $4 - x^2 \geq 0$, or $x^2 \leq 4$. The domain is the interval $-2 \leq x \leq 2$.
(b) Here, $x^2 - 16 \geq 0$, or $x^2 \geq 16$. The domain consists of the intervals $x \leq -4$ and $x \geq 4$.
(c) The function is defined for every value of x except 2.
(d) The function is defined for $x \neq \pm 3$.
(e) Since $x^2 + 4 \neq 0$ for all x, the domain is the set of all real numbers.

4. Sketch the graph of the function defined as follows:

$$f(x) = 5 \text{ when } 0 < x \leq 1 \qquad f(x) = 10 \text{ when } 1 < x \leq 2$$
$$f(x) = 15 \text{ when } 2 < x \leq 3 \qquad f(x) = 20 \text{ when } 3 < x \leq 4 \qquad \text{etc.}$$

Determine the domain and range of the function.
The graph is shown in Fig. 6-3. The domain is the set of all positive real numbers, and the range is the set of integers, 5, 10, 15, 20, ...

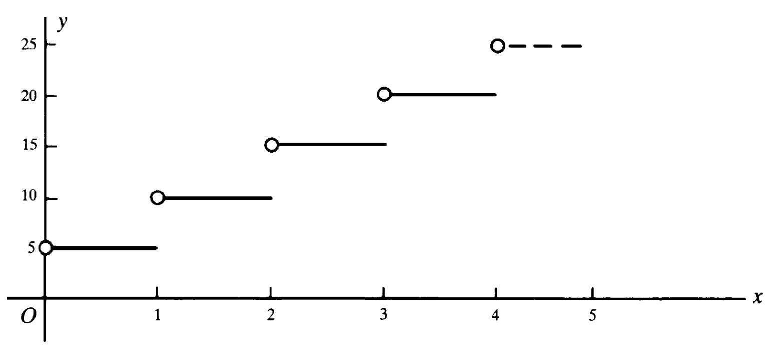

Fig. 6-3

5. A rectangular plot requires 2000 ft of fencing to enclose it. If one of its dimensions is x (in feet), express its area y (in square feet) as a function of x, and determine the domain of the function.

Since one dimension is x, the other is $\frac{1}{2}(2000 - 2x) = 1000 - x$. The area is then $y = x(1000 - x)$, and the domain of this function is $0 < x < 1000$.

6. Express the length l of a chord of a circle of radius 8 as a function of its distance x from the center of the circle. Determine the domain of the function.

From Fig. 6-4 we see that $\frac{1}{2}l = \sqrt{64 - x^2}$, so that $l = 2\sqrt{64 - x^2}$. The domain is the interval $0 \le x < 8$.

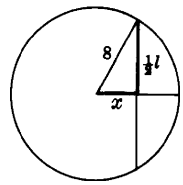

Fig. 6-4

7. From each corner of a square of tin, 12 inches on a side, small squares of side x (in inches) are removed, and the edges are turned up to form an open box (Fig. 6-5). Express the volume V of the box (in cubic inches) as a function of x, and determine the domain of the function.

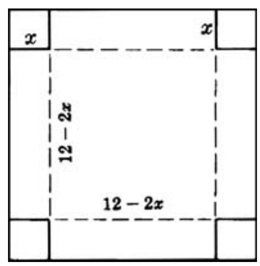

Fig. 6-5

The box has a square base of side $12 - 2x$ and a height of x. The volume of the box is then $V = x(12 - 2x)^2 = 4x(6 - x)^2$. The domain is the interval $0 < x < 6$.

As x increases over its domain, V increases for a time and then decreases thereafter. Thus, among such boxes that may be constructed, there is one of greatest volume, say M. To determine M, it is necessary to locate the precise value of x at which V ceases to increase. This problem will be studied in a later chapter.

8. If $f(x) = x^2 + 2x$, find $\frac{f(a+h) - f(a)}{h}$ and interpret the result.

$$\frac{f(a+h) - f(a)}{h} = \frac{[(a+h)^2 + 2(a+h)] - (a^2 + 2a)}{h} = 2a + 2 + h$$

On the graph of the function (Fig. 6-6), locate points P and Q whose respective abscissas are a and $a + h$. The ordinate of P is $f(a)$, and that of Q is $f(a + h)$. Then

$$\frac{f(a+h) - f(a)}{h} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \text{slope of } PQ$$

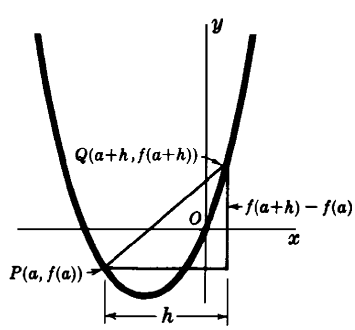

Fig. 6-6

9. Let $f(x) = x^2 - 2x + 3$. Evaluate (a) $f(3)$; (b) $f(-3)$; (c) $f(-x)$; (d) $f(x+2)$; (e) $f(x-2)$; (f) $f(x+h)$; (g) $f(x+h) - f(x)$; (h) $\dfrac{f(x+h)-f(x)}{h}$.

(a) $f(3) = 3^2 - 2(3) + 3 = 9 - 6 + 3 = 6$
(b) $f(-3) = (-3)^2 - 2(-3) + 3 = 9 + 6 + 3 = 18$
(c) $f(-x) = (-x)^2 - 2(-x) + 3 = x^2 + 2x + 3$
(d) $f(x+2) = (x+2)^2 - 2(x+2) + 3 = x^2 + 4x + 4 - 2x - 4 + 3 = x^2 + 2x + 3$
(e) $f(x-2) = (x-2)^2 - 2(x-2) + 3 = x^2 - 4x + 4 - 2x + 4 + 3 = x^2 - 6x + 11$
(f) $f(x+h) = (x+h)^2 - 2(x+h) + 3 = x^2 + 2hx + h^2 - 2x - 2h + 3 = x^2 + (2h-2)x + (h^2 - 2h + 3)$
(g) $f(x+h) - f(x) - [x^2 + (2h-2)x + (h^2 - 2h + 3)] - (x^2 - 2x + 3) = 2hx + h^2 - 2h = h(2x + h - 2)$
(h) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{h(2x+h-2)}{h} = 2x + h - 2$

10. Draw the graph of the function $f(x) = \sqrt{4 - x^2}$, and find the domain and range of the function.

The graph of f is the graph of the equation $y = \sqrt{4 - x^2}$. For points on this graph, $y^2 = 4 - x^2$; that is, $x^2 + y^2 = 4$. The graph of the last equation is the circle with center at the origin and radius 2. Since $y = \sqrt{4 - x^2} \geq 0$, the desired graph is the upper half of that circle. Fig. 6-7 shows that the domain is the interval $-2 \leq x \leq 2$, and the range is the interval $0 \leq y \leq 2$.

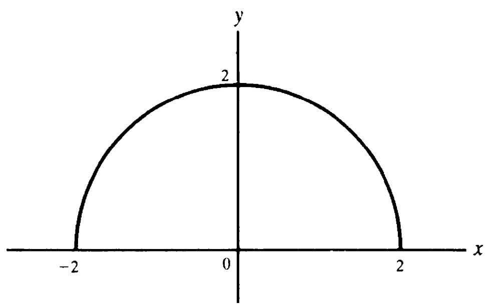

Fig. 6-7

SUPPLEMENTARY PROBLEMS

11. If $f(x) = x^2 - 4x + 6$, find (a) $f(0)$; (b) $f(3)$; (c) $f(-2)$. Show that $f(\frac{1}{2}) = f(\frac{7}{2})$ and $f(2-h) = f(2+h)$.

Ans. (a) −6; (b) 3; (c) 18

12. If $f(x) = \dfrac{x-1}{x+1}$, find (a) $f(0)$; (b) $f(l)$; (c) $f(-2)$. Show that $f\left(\dfrac{1}{x}\right) = -f(x)$ and $f\left(-\dfrac{1}{x}\right) = -\dfrac{1}{f(x)}$.

Ans. (a) −1; (b) 0; (c) 3

13. If $f(x) = x^2 - x$, show that $f(x+1) = f(-x)$.

14. If $f(x) = 1/x$, show that $f(a) - f(b) = f\left(\dfrac{ab}{b-a}\right)$.

15. If $y = f(x) = \dfrac{5x+3}{4x-5}$, show that $x = f(y)$.

16. Determine the domain of each of the following functions:

(a) $y = x^2 + 4$ (b) $y = \sqrt{x^2 + 4}$ (c) $y = \sqrt{x^2 - 4}$ (d) $y = \frac{x}{x+3}$

(e) $y = \frac{2x}{(x-2)(x+1)}$ (f) $y = \frac{1}{\sqrt{9 - x^2}}$ (g) $y = \frac{x^2 - 1}{x^2 + 1}$ (h) $y = \sqrt{\frac{x}{2-x}}$

Ans. (a), (b), (g) all values of x; (c) $|x| \geq 2$; (d) $x \neq -3$; (e) $x \neq -1, 2$; (f) $-3 < x < 3$; (h) $0 \leq x < 2$

17. Compute $\frac{f(a+h) - f(a)}{h}$ in the following cases:

(a) $f(x) = \frac{1}{x-2}$ when $a \neq 2$ and $a + h \neq 2$

(b) $f(x) = \sqrt{x-4}$ when $a \geq 4$ and $a + h \geq 4$

(c) $f(x) = \frac{x}{x+1}$ when $a \neq -1$ and $a + h \neq -1$

Ans. (a) $\frac{-1}{(a-2)(a+h-2)}$; (b) $\frac{1}{\sqrt{a+h-4} + \sqrt{a-4}}$; (c) $\frac{1}{(a+1)(a+h+1)}$

18. Draw the graphs of the following functions, and find their domains and ranges:

(a) $f(x) = -x^2 + 1$ (b) $f(x) = \begin{cases} x - 1 & \text{if } 0 < x < 1 \\ 2x & \text{if } 1 \leq x \end{cases}$

(c) $f(x) = [x]$ = the greatest integer less than or equal to x

(d) $f(x) = \frac{x^2 - 4}{x - 2}$ (e) $f(x) = 5 - x^2$ (f) $f(x) = -4\sqrt{x}$

(g) $f(x) = |x - 3|$ (h) $f(x) = 4/x$ (i) $f(x) = |x|/x$

(j) $f(x) = x - |x|$ (k) $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$

Ans. (a) domain, all numbers; range, $y \leq 1$
(b) domain, $x > 0$; range, $-1 < y < 0$ or $y \geq 2$
(c) domain, all numbers; range, all integers
(d) domain, $x \neq 2$; range, $y \neq 4$
(e) domain, all numbers; range, $y \leq 5$
(f) domain, $x \geq 0$; range, $y \leq 0$
(g) domain, all numbers; range, $y \geq 0$
(h) domain, $x \neq 0$; range, $y \neq 0$
(i) domain, $x \neq 0$; range, $\{-1, 1\}$
(j) domain, all numbers; range, $y \leq 0$
(k) domain, all numbers; range, $y \geq 0$

19. (GC) Use a graphing calculator to verify your answers to Problem 18.

20. Evaluate the expression $\frac{f(x+h) - f(x)}{h}$ for the following functions f:

(a) $f(x) = 3x - x^2$ (b) $f(x) = \sqrt{2x}$
(c) $f(x) = 3x - 5$ (d) $f(x) = x^3 - 2$

Ans. (a) $3 - 2x - h$ (b) $\frac{2}{\sqrt{2(x+h)} + \sqrt{2x}}$ (c) 3 (d) $3x^2 + 3xh + h^2$

21. Find a formula for the function f whose graph consists of all points satisfying each of the following equations. (In plain language, solve each equation for y.)

(a) $x^5y + 4x - 2 = 0$ (b) $x = \frac{2+y}{2-y}$ (c) $4x^2 - 4xy + y^2 = 0$

Ans. (a) $f(x) = \frac{2-4x}{x^5}$; (b) $f(x) = \frac{2(x-1)}{x+1}$; (c) $f(x) = 2x$

22. Graph the following functions and find their domain and range:

(a) $f(x)=\begin{cases} x+2 & \text{if } -1<x<0 \\ x & \text{if } 0\le x<1 \end{cases}$ (b) $g(x)=\begin{cases} 2-x & \text{if } 0<x<2 \\ x-1 & \text{if } 3\le x<4 \end{cases}$ (c) $h(x)=\begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x\ne 2 \\ 4 & \text{if } x=2 \end{cases}$

Ans. (a) domain = (−1, 1], range = [0, 2)
(b) domain = union of (0, 2) and [3, 4), range = (0, 3)
(c) domain and range = set of all real numbers

23. (GC) Verify your answers to Problem 22 by means of a graphing calculator.

24. In each of the following cases, define a function that has the given set $\mathscr{D}$ as its domain and the given set $\mathscr{R}$ as its range: (a) $\mathscr{D}=(0, 2)$ and $\mathscr{R}=(1, 7)$; (b) $\mathscr{D}=(0, 1)$ and $\mathscr{R}=(1, \infty)$.

Ans. (a) One such function is $f(x)=3x+1$. (b) One such function is $f(x)=\dfrac{1}{1-x}$.

25. (a) Prove the vertical line test: A set of points in the xy plane is the graph of a function if and only if the set intersects every vertical line in at most one point.

(b) Determine whether each set of points in Fig. 6-8 is the graph of a function.

Ans. Only (b) is the graph of a function.

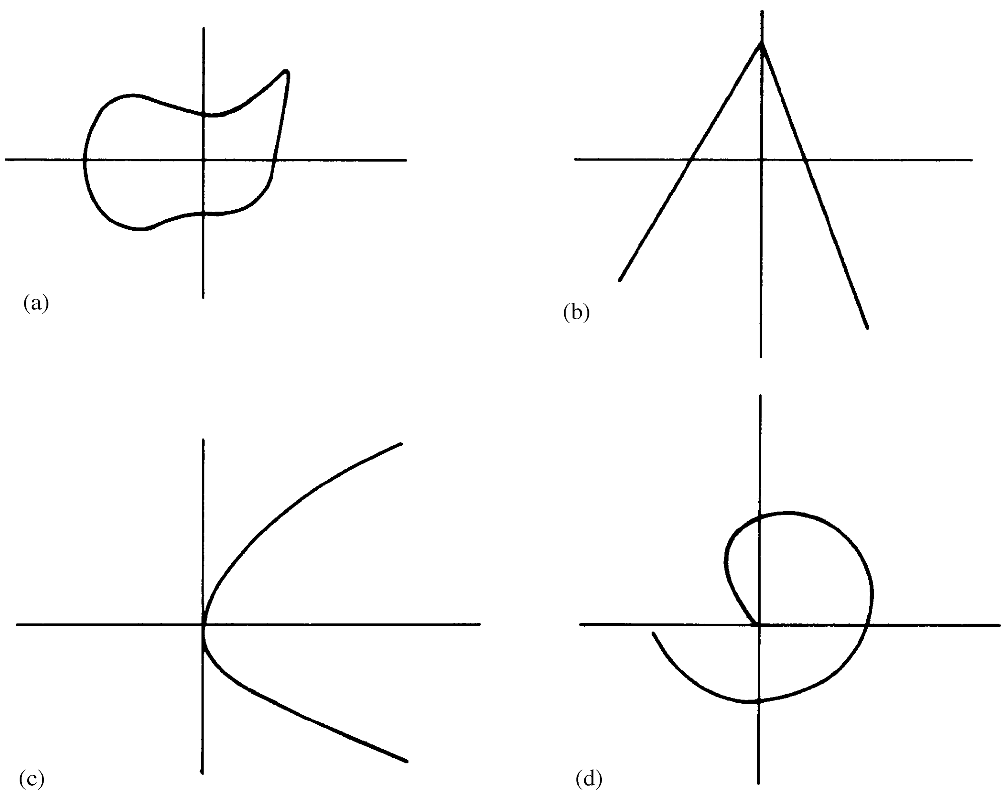

Fig. 6-8