CHAPTER 29

Applications of Integration I: Area and Arc Length

Area Between a Curve and the y Axis

We already know how to find the area of a region like that shown in Fig. 29-1, bounded below by the x axis, above by a curve $y = f(x)$, and lying between $x = a$ and $x = b$. The area is the definite integral $\int_a^b f(x)\,dx$.

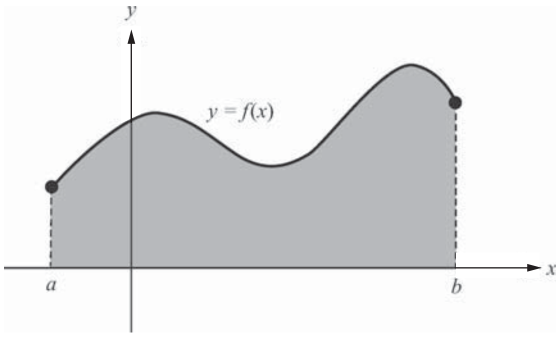

Fig. 29-1

Now consider a region like that shown in Fig. 29-2, bounded on the left by the y axis, on the right by a curve $x = g(y)$, and lying between $y = c$ and $y = d$. Then, by an argument similar to that for the case shown in Fig. 29-1, the area of the region is the definite integral $\int_c^d g(y)\,dy$.

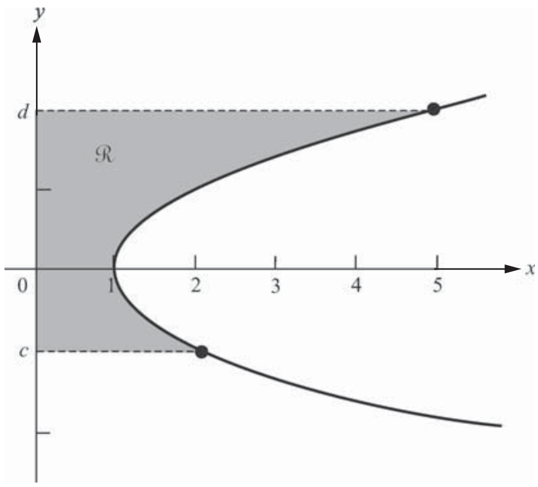

Fig. 29-2

EXAMPLE 29.1: Consider the region bounded on the right by the parabola $x = 4 - y^2$, on the left by the y axis, and above and below by $y = 2$ and $y = -1$. See Fig. 29-3. Then the area of this region is $\int_{-1}^{2}(4 - y^2)\,dy$. By the Fundamental Theorem of Calculus, this is

$$(4y - \tfrac{1}{3}y^3)]_{-1}^{2} = (8 - \tfrac{8}{3}) - (-4 - (-\tfrac{1}{3})) = 12 - \tfrac{9}{3} = 12 - 3 = 9$$

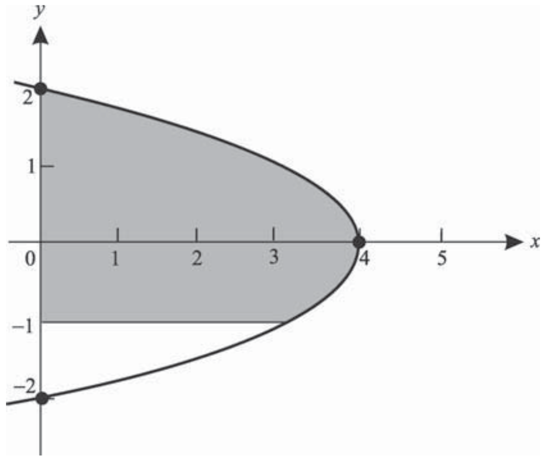

Fig. 29-3

Areas Between Curves

Assume that f and g are continuous functions such that $g(x) \le f(x)$ for $a \le x \le b$. Then the curve $y = f(x)$ lies above the curve $y = g(x)$ between $x = a$ and $x = b$. The area A of the region between the two curves and lying between $x = a$ and $x = b$ is given by the formula

$$A = \int_a^b (f(x) - g(x))\,dx \tag{29.1}$$

To see why this formula holds, first look at the special case where $0 \le g(x) \le f(x)$ for $a \le x \le b$. (See Fig. 29-4.) Clearly, the area is the difference between two areas, the area A_f of the region under the curve $y = f(x)$ and above the x axis, and the area A_g of the region under the curve $y = g(x)$ and above the x axis. Since $A_f = \int_a^b f(x)\,dx$ and $A_g = \int_a^b g(x)\,dx$,

$$\begin{aligned} A = A_f - A_g &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b (f(x) - g(x))\,dx \quad \text{by (23.6)} \end{aligned}$$

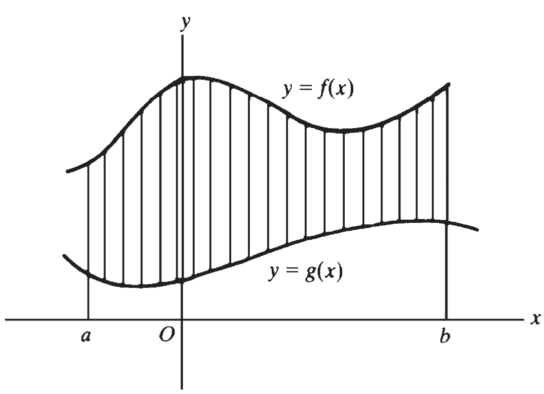

Fig. 29-4

Now look at the general case (see Fig. 29-5), when one or both of the curves $y = f(x)$ and $y = g(x)$ may lie below the x axis. Let $m < 0$ be the absolute minimum of g on $[a, b]$. Raise both curves by $|m|$ units. The new graphs, shown in Fig. 29-6, are on or above the x axis and enclose the same area A as the original graphs. The upper curve is the graph of $y = f(x) + |m|$ and the lower curve is the graph of $y = g(x) + |m|$. Hence, by the special case above,

$$A = \int_a^b ((f(x) + |m| - (g(x) + |m|))\,dx = \int_a^b (f(x) - g(x))\,dx$$

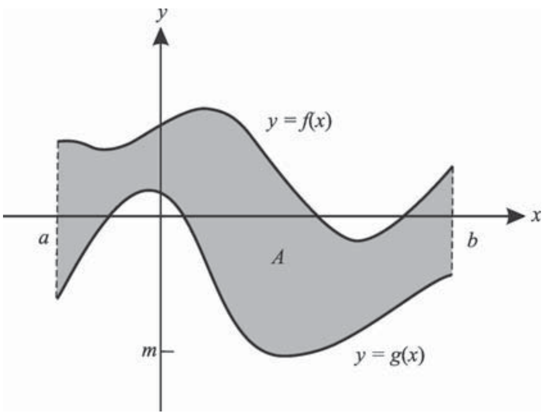

Fig. 29-5

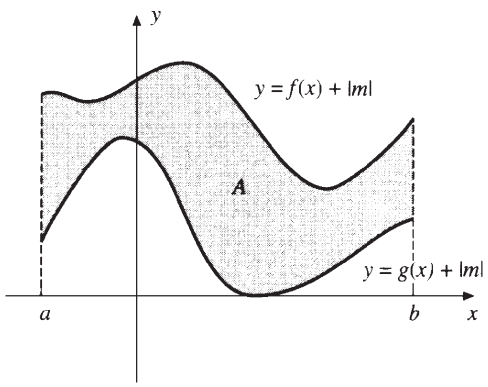

Fig. 29-6

EXAMPLE 29.2: Find the area A of the region $\mathcal{R}$ under the line $y = \frac{1}{2}x + 2$, above the parabola $y = x^2$, and between the y axis and $x = 1$. (See the shaded region in Fig. 29-7.) By (29.1),

$$A = \int_0^1 \left(\left(\frac{1}{2}x + 2\right) - x^2\right)dx = \left(\frac{1}{4}x^2 + 2x - \frac{1}{3}x^3\right)\Big]_0^1 = \left(\frac{1}{4} + 2 - \frac{1}{3}\right) - (0 + 0 - 0) = \frac{3}{12} + \frac{24}{12} - \frac{4}{12} = \frac{23}{12}$$

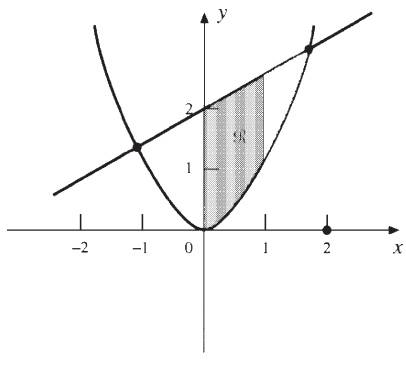

Fig. 29-7

Arc Length

Let f be differentiable on $[a, b]$. Consider the part of the graph of f from $(a, f(a))$ to $(b, f(b))$. Let us find a formula for the length L of this curve. Divide [a, b] into n equal subintervals, each of length Δx. To each point x_k in this subdivision there corresponds a point $P_k(x_k, f(x_k))$ on the curve. (See Fig. 29-8.) For large n, the sum $\overline{P_0P_1} + \overline{P_1P_2} + \ldots + \overline{P_{n-1}P_n} = \sum_{k=1}^{n} \overline{P_{k-1}P_k}$ of the lengths of the line segments $P_{k-1}P_k$ is an approximation to the length of the curve.

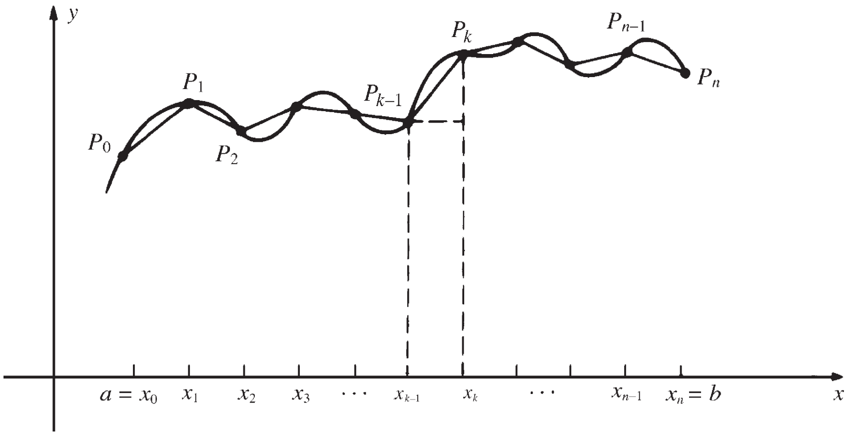

Fig. 29-8

By the distance formula (2.1),

$$\overline{P_{k-1}P_k} = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

Now, $x_k - x_{k-1} = \Delta x$ and, by the law of the mean (Theorem 13.4),

$$f(x_k) - f(x_{k-1}) = (x_k - x_{k-1})f'(x_k^*) = (\Delta x)f'(x_k^*)$$

for some x_k^* in (x_{k-1}, x_k). Thus,

$$\overline{P_{k-1}P_k} = \sqrt{(\Delta x)^2 + (\Delta x)^2(f'(x_k^*))^2} = \sqrt{(1 + (f'(x_k^*))^2)(\Delta x)^2}$$

$$= \sqrt{1 + (f'(x_k^*))^2}\sqrt{(\Delta x)^2} = \sqrt{1 + (f'(x_k^*))^2}\,\Delta x$$

So,
$$\sum_{k=1}^{n} \overline{P_{k-1}P_k} = \sum_{k=1}^{n} \sqrt{1 + (f'(x_k^*))^2}\,\Delta x$$

The right-hand sum is an approximating sum for the definite integral $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$. Therefore, letting $n \to +\infty$, we get the *arc length formula:*

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_a^b \sqrt{1 + (y')^2}\,dx \tag{29.2}$$

EXAMPLE 29.3: Find the arc length L of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.

By (29.2), since $y' = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$,

$$L = \int_0^5 \sqrt{1 + (y')^2}\,dx = \int_0^5 \sqrt{1 + \tfrac{9}{4}x}\,dx$$

$$= \tfrac{4}{9}\int_0^5 (1 + \tfrac{9}{4}x)^{1/2}\left(\tfrac{9}{4}\right)dx = \tfrac{4}{9}\tfrac{2}{3}(1 + \tfrac{9}{4}x)^{3/2}\Big]_0^5$$ (by Quick Formula I and the Fundamental Theorem of Calculus)

$$= \tfrac{8}{27}\left(\left(\tfrac{49}{4}\right)^{3/2} - 1^{3/2}\right) = \tfrac{8}{27}\left(\tfrac{343}{8} - 1\right) = \tfrac{335}{27}$$

SOLVED PROBLEMS

1. Find the area bounded by the parabola $x = 8 + 2y - y^2$, the y axis, and the lines $y = -1$ and $y = 3$.

Note, by completing the square, that $x = -(y^2 - 2y - 8) = -((y-1)^2 - 9) = 9 - (y-1)^2 = (4-y)(2+y)$. Hence, the vertex of the parabola is (9, 1) and the parabola cuts the y axis at $y = 4$ and $y = -2$. We want the area of the shaded region in Fig. 29-9, which is given by

$$\int_{-1}^{3}(8+2y-y^2)\,dy = \left(8y + y^2 - \tfrac{1}{3}y^3\right)\Big]_{-1}^{3} = (24+9-9) - (-8+1-\tfrac{1}{3}) = \tfrac{92}{3}$$

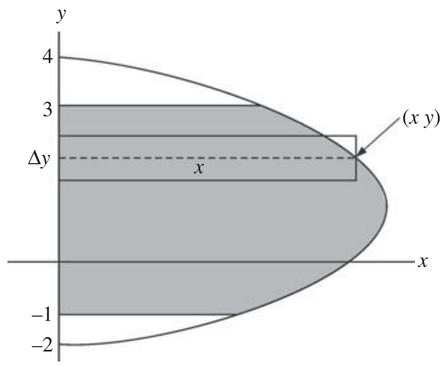

Fig. 29-9

2. Find the area of the region between the curves $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \pi/4$.

The curves intersect at $(\pi/4, \sqrt{2}/2)$, and $0 \le \sin x < \cos x$ for $0 \le x < \pi/4$. (See Fig. 29-10.) Hence, the area is

$$\int_{0}^{\pi/4}(\cos x - \sin x)\,dx = (\sin x + \cos x)\Big]_{0}^{\pi/4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1$$

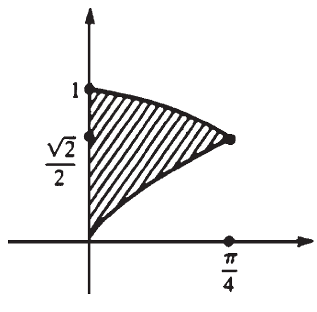

Fig. 29-10

3. Find the area of the region bounded by the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$.

By solving $6x - x^2 = x^2 - 2x$, we see that the parabolas intersect when $x = 0$ and $x = 4$, that is, at (0, 0) and (4, 8). (See Fig. 29-11.) By completing the square, the first parabola has the equation $y = 9 - (x-3)^2$; therefore, it has its vertex at (3, 9) and opens downward. Likewise, the second parabola has the equation $y = (x-1)^2 - 1$; therefore, its vertex is at (1, –1) and it opens upward. Note that the first parabola lies above the second parabola in the given region. By (29.1), the required area is

$$\int_{0}^{4}((6x-x^2)-(x^2-2x))\,dx = \int_{0}^{4}(8x-2x^2)\,dx = \left(4x^2 - \tfrac{2}{3}x^3\right)\Big]_{0}^{4} = \left(64 - \tfrac{128}{3}\right) = \tfrac{64}{3}$$

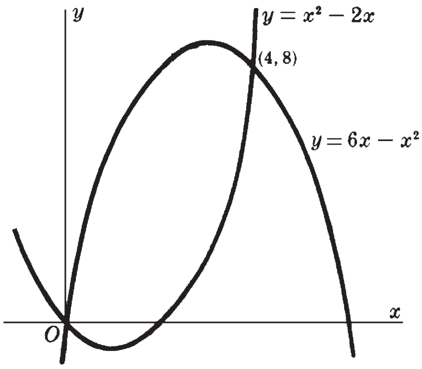

Fig. 29-11

4. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

Solving the equations simultaneously, we get $(2x-4)^2 = 4x$, $x^2 - 4x + 4 = x$, $x^2 - 5x + 4 = 0$, $(x-1)(x-4) = 0$. Hence, the curves intersect when $x = 1$ or $x = 4$, that is, at $(1, -2)$ and $(4, 4)$. (See Fig. 29-12.) Note that neither curve is above the other throughout the region. Hence, it is better to take y as the independent variable and rewrite the curves as $x = \frac{1}{4}y^2$ and $x = \frac{1}{2}(y+4)$. The line is always to the right of the parabola.

The area is obtained by integrating along the y axis:

$$\int_{-2}^{4}\left(\tfrac{1}{2}(y+4) - \tfrac{1}{4}y^2\right)dy = \tfrac{1}{4}\int_{-2}^{4}(2y + 8 - y^2)\,dy$$

$$= \tfrac{1}{4}\left(y^2 + 8y - \tfrac{1}{3}y^3\right)\Big]_{-2}^{4} = \tfrac{1}{4}\left((16 + 32 - \tfrac{64}{3}) - (4 - 16 + \tfrac{8}{3})\right) = 9$$

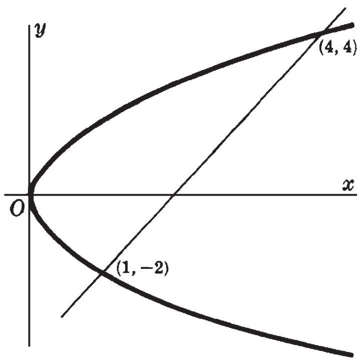

Fig. 29-12

5. Find the area of the region between the curve $y = x^3 - 6x^2 + 8x$ and the x axis.

Since $x^3 - 6x^2 + 8x = x(x^2 - 6x + 8) = x(x-2)(x-4)$, the curve crosses the x axis at $x = 0$, $x = 2$, and $x = 4$. The graph looks like the curve shown in Fig. 29-13. (By applying the quadratic formula to y', we find that the maximum and minimum values occur at $x = 2 \pm \frac{2}{3}\sqrt{3}$.) Since the part of the region with $2 \le x \le 4$ lies below the x axis, we must calculate two separate integrals, one with respect to y between $x = 0$ and $x = 2$, and the other with respect to $-y$ between $x = 2$ and $x = 4$. Thus, the required area is

$$\int_{0}^{2}(x^3 - 6x^2 + 8x)\,dx - \int_{2}^{4}(x^3 - 6x^2 + 8x)\,dx = \left(\tfrac{1}{4}x^4 - 2x^3 + 4x^2\right)\Big]_{0}^{2} - \left(\tfrac{1}{4}x^4 - 2x^3 + 4x^2\right]_{2}^{4} = 4 + 4 = 8$$

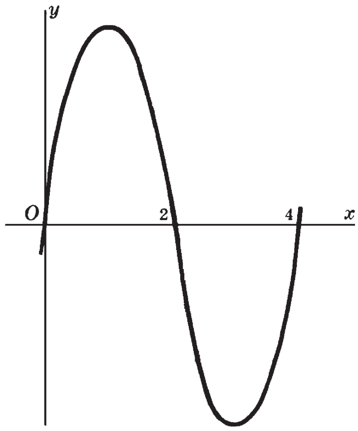

Fig. 29-13

Note that, if we had made the mistake of simply calculating the integral $\int_0^4 (x^3 - 6x^2 + 8x)\,dx$, we would have got the incorrect answer 0.

6. Find the area enclosed by the curve $y^2 = x^2 - x^4$.

The curve is symmetric with respect to the coordinate axes. Hence the required area is four times the portion lying in the first quadrant. (See Fig. 29-14.) In the first quadrant, $y = \sqrt{x^2 - x^4} = x\sqrt{1 - x^2}$ and the curve intersects the x axis at $x = 0$ and $x = 1$. So, the required area is

$$4\int_0^1 x\sqrt{1-x^2}\,dx = -2\int_0^1 (1-x^2)^{1/2}(-2x)\,dx$$

$$= -2\left(\tfrac{2}{3}\right)(1-x^2)^{3/2}\Big]_0^1 \quad \text{(by Quick Formula I)}$$

$$= -\tfrac{4}{3}(0 - 1^{3/2}) = -\tfrac{4}{3}(-1) = \tfrac{4}{3}$$

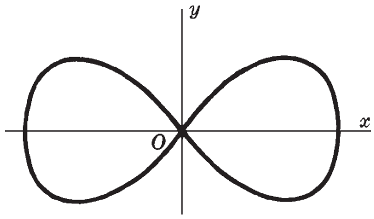

Fig. 29-14

7. Find the arc length of the curve $x = 3y^{3/2} - 1$ from $y = 0$ to $y = 4$.

We can reverse the roles of x and y in the arc length formula (29.2): $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$. Since $\frac{dx}{dy} = \frac{9}{2}y^{1/2}$,

$$L = \int_0^4 \sqrt{1 + \tfrac{81}{4}y}\,dy = \tfrac{4}{81}\int_0^4 (1 + \tfrac{81}{4}y)^{1/2}(\tfrac{81}{4})\,dy = \tfrac{4}{81}(\tfrac{2}{3})(1 + \tfrac{81}{4}y)^{3/2}\Big]_0^4 = \tfrac{8}{243}((82)^{3/2} - 1^{3/2}) = \tfrac{8}{243}(82\sqrt{82} - 1)$$

8. Find the arc length of the curve $24xy = x^4 + 48$ from $x = 2$ to $x = 4$.

$y = \frac{1}{24}x^3 + 2x^{-1}$. Hence, $y' = \frac{1}{8}x^2 - 2/x^2$. Thus,

$$(y')^2 = \tfrac{1}{64}x^4 - \tfrac{1}{2} + \frac{4}{x^4}$$

$$1 + (y')^2 = \tfrac{1}{64}x^4 + \tfrac{1}{2} + \frac{4}{x^4} = \left(\tfrac{1}{8}x^2 + \frac{2}{x^2}\right)^2$$

So, $$L=\int_2^4 \sqrt{1+(y')^2}\,dx=\int_2^4\left(\tfrac{1}{8}x^2+\frac{2}{x^2}\right)dx=\int_2^4\left(\tfrac{1}{8}x^2+2x^{-2}\right)dx$$

$$=\left(\tfrac{1}{24}x^3-2x^{-1}\right)\Big]_2^4=\left(\tfrac{8}{3}-\tfrac{1}{2}\right)-\left(\tfrac{1}{3}-1\right)=\tfrac{17}{6}$$

9. Find the arc length of the catenary $y=\dfrac{a}{2}(e^{x/a}+e^{-x/a})$ from $x=0$ to $x=a$.
$y'=\tfrac{1}{2}(e^{x/a}+e^{-x/a})$ and, therefore,

$$1+(y')^2=1+\tfrac{1}{4}(e^{2x/a}-2+e^{-2x/a})=\tfrac{1}{4}(e^{x/a}+e^{-x/a})^2$$

So, $$L=\tfrac{1}{2}\int_0^a(e^{x/a}+e^{-x/a})dx=\frac{a}{2}(e^{x/a}-e^{-x/a})\Big]_0^a=\frac{a}{2}(e-e^{-1})$$

SUPPLEMENTARY PROBLEMS

10. Find the area of the region lying above the x axis and under the parabola $y=4x-x^2$.

Ans. $\tfrac{32}{3}$

11. Find the area of the region bounded by the parabola $y=x^2-7x+6$, the x axis, and the lines $x=2$ and $x=6$.

Ans. $\tfrac{56}{3}$

12. Find the area of the region bounded by the given curves.

(a) $y=x^2,\ y=0,\ x=2,\ x=5$ *Ans.* 39
(b) $y=x^3,\ y=0,\ x=1,\ x=3$ *Ans.* 20
(c) $y=4x-x^2,\ y=0,\ x=1,\ x=3$ *Ans.* $\tfrac{22}{3}$
(d) $x=1+y^2,\ x=10$ *Ans.* 36
(e) $x=3y^2-9,\ x=0,\ y=0,\ y=1$ *Ans.* 8
(f) $x=y^2+4y,\ x=0$ *Ans.* $\tfrac{32}{3}$
(g) $y=9-x^2,\ y=x+3$ *Ans.* $\tfrac{125}{6}$
(h) $y=2-x^2y=-x$ *Ans.* $\tfrac{9}{2}$
(i) $y=x^2-4,\ y=8-2x^2$ *Ans.* 32
(j) $y=x^4-4x^2,\ y=4x^2$ *Ans.* $\tfrac{512}{15}\sqrt{2}$
(k) $y=e^x,\ y=e^{-x},\ x=0,\ x=2$ *Ans.* $\dfrac{e^2+1}{e^2-2}$
(l) $y=e^{x/a}+e^{-x/a},\ y=0,\ x=\pm a$ *Ans.* $2a\left(\dfrac{e-1}{e}\right)$
(m) $xy=12,\ y=0,\ x=1,\ x=e^2$ *Ans.* 24
(n) $y=\dfrac{1}{1+x^2},\ y=0,\ x=\pm 1$ *Ans.* $\dfrac{\pi}{2}$
(o) $y=\tan x,\ x=0,\ x=\dfrac{\pi}{4}$ *Ans.* $\tfrac{1}{2}\ln 2$
(p) $y=25-x^2,\ 256x=3y^2,\ 16y=9x^2$ *Ans.* $\tfrac{98}{3}$

13. Find the length of the indicated arc of the given curve.

(a) $y^3=8x^2$ from $x=1$ to $x=8$ *Ans.* $\left(104\sqrt{13}-125\right)/27$
(b) $6xy=x^4+3$ from $x=1$ to $x=2$ *Ans.* $\tfrac{17}{12}$
(c) $27y^2=4(x-2)^3$ from (2, 0) to $(11,\ 6\sqrt{3}$ *Ans.* 14

(d) $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $x = 1$ to $x = e$ *Ans.* $\frac{1}{2}e^2 - \frac{1}{4}$

(e) y = ln cos x from $x = \dfrac{\pi}{6}$ to $x = \dfrac{\pi}{4}$ *Ans.* $\ln\left(\dfrac{1+\sqrt{2}}{\sqrt{3}}\right)$

(f) $x^{2/3} + y^{2/3} = 4$ from $x = 1$ to $x = 8$ *Ans.* 9

14. (GC) Estimate the arc length of $y = \sin x$ from $x = 0$ to $x = \pi$ to an accuracy of four decimal places. (Use Simpson's Rule with $n = 10$.)

Ans. 3.8202