CHAPTER 23

The Definite Integral. Area Under a Curve

Sigma Notation

The Greek capital letter Σ denotes repeated addition.

EXAMPLE 23.1:

(a) $\sum_{j=1}^{5} j = 1+2+3+4+5 = 15.$

(b) $\sum_{i=0}^{3} (2i+1) = 1+3+5+7.$

(c) $\sum_{i=2}^{10} i^2 = 2^2 + 3^2 + \cdots + (10)^2$

(d) $\sum_{j=1}^{4} \cos j\pi = \cos\pi + \cos 2\pi + \cos 3\pi + \cos 4\pi$

In general, if f is a function defined on the integers, and if n and k are integers such that $n \geq k$, then:

$$\sum_{j=k}^{n} f(j) = f(k) + f(k+1) + \cdots + f(n)$$

Area Under a Curve

Assume that f is a function such that $f(x) \geq 0$ for all x in a closed interval $[a, b]$. Its graph is a curve that lies on or above the x axis. (See Fig. 23-1.) We have an intuitive idea of the *area A* of the region $\mathcal{R}$ under the graph, above the x axis, and between the vertical lines $x = a$ and $x = b$. We shall specify a method for evaluating A.

Choose points $x_1, x_2, \ldots, x_{n-1}$ between a and b. Let $x_0 = a$ and $x_n = b$. Thus (see Fig. 23-2),

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

The interval $[a, b]$ is divided into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. Denote the lengths of these subintervals by $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$. Hence, if $1 \leq k \leq n$,

$$\Delta_k x = x_k - x_{k-1}$$

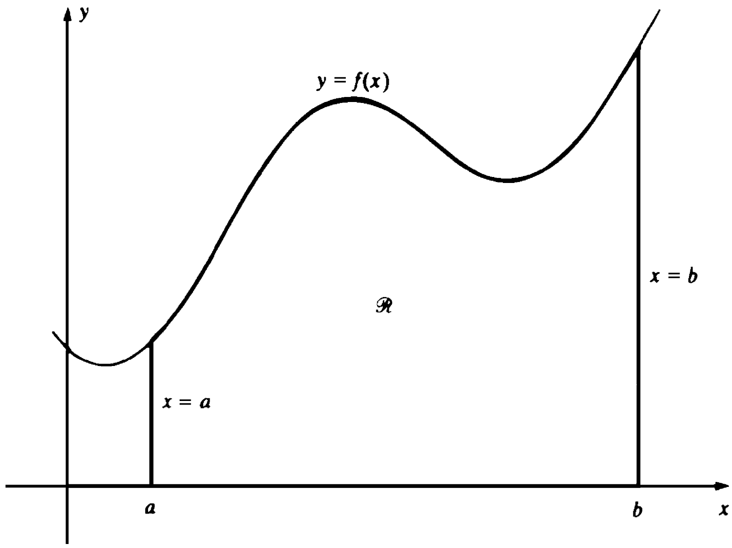

Fig. 23-1

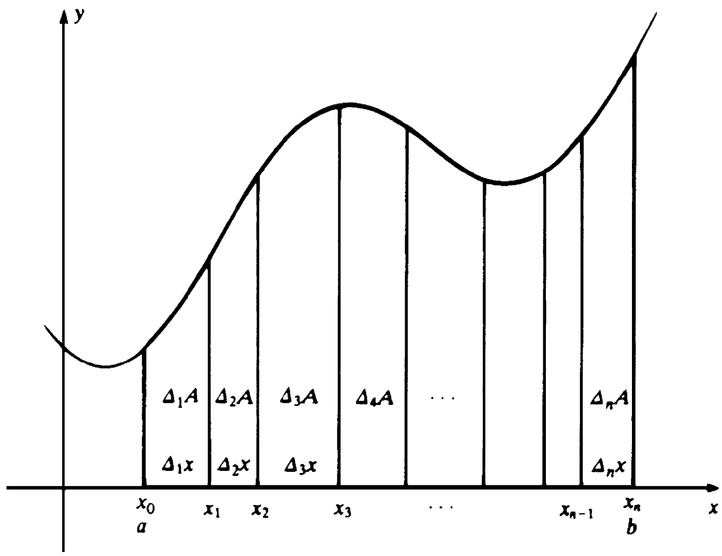

Fig. 23-2

Draw vertical line segments $x = x_k$ from the x axis up to the graph. This divides the region $\mathcal{R}$ into n strips. Letting $\Delta_k A$ denote the area of the kth strip, we obtain

$$A = \sum_{k=1}^{n} \Delta_k A$$

We can approximate the area $\Delta_k A$ in the following manner. Select any point x_k^* in the kth subinterval $[x_{k-1}, x_k]$. Draw the vertical line segment from the point x_k^* on the x axis up to the graph (see the dashed lines in Fig. 23-3); the length of this segment is $f(x_k^*)$. The rectangle with base $\Delta_k x$ and height $f(x_k^*)$ has area $f(x_k^*)\,\Delta_k x$, which is approximately the area $\Delta_k A$ of the kth strip. Hence, the total area A under the curve is approximately the sum

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta_k x = f(x_1^*)\,\Delta_1 x + f(x_2^*)\,\Delta_2 x + \cdots + f(x_n^*)\,\Delta_n x \tag{23.1}$$

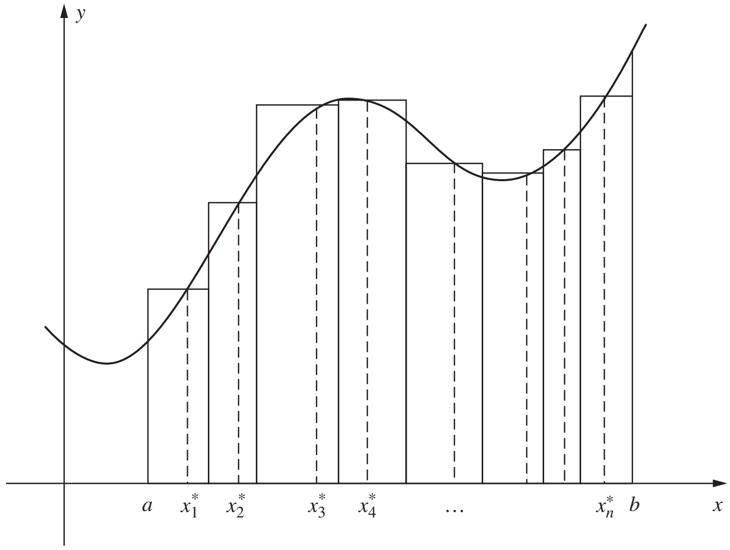

Fig. 23-3

The approximation becomes better and better as we divide the interval $[a, b]$ into more and more subintervals and as we make the lengths of these subintervals smaller and smaller. If successive approximations can be made as close as one wishes to a specific number, then that number will be denoted by

$$\int_a^b f(x)\,dx$$

and will be called the *definite integral* of f from a to b. Such a number does not exist in all cases, but it does exist, for example, when the function f is continuous on $[a, b]$. When $\int_a^b f(x)\,dx$ exists, its value is equal to the area A under the curve.†

In the notation $\int_a^b f(x)\,dx$, b is called the *upper limit* and a is called the *lower limit* of the definite integral.

For any (not necessarily nonnegative) function f on $[a, b]$, sums of the form (23.1) can be defined, without using the notion of area. If there is a number to which these sums can be made as close as we wish, as n gets larger and larger and as the maximum of the lengths $\Delta_k x$ approaches 0, then that number is denoted $\int_a^b f(x)\,dx$ and is called the *definite integral* of f on $[a, b]$. When $\int_a^b f(x)\,dx$ exists, we say that f is *integrable* on $[a, b]$.

We shall assume without proof that $\int_a^b f(x)\,dx$ exists for every function f that is continuous on $[a, b]$. To evaluate $\int_a^b f(x)\,dx$, it suffices to find the limit of a sequence of sums (23.1) for which the number n of subintervals approaches infinity and the maximum lengths of the subintervals approach 0.

EXAMPLE 23.2: Let us show that

$$\int_a^b 1\,dx = b - a \tag{23.2}$$

Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be a subdivision of $[a, b]$. Then a corresponding sum (23.1) is

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta_k x &= \sum_{k=1}^{n} \Delta_k x \qquad \text{(because } f(x) = 1 \text{ for all } x\text{)}\\ &= b - a\end{aligned}$$

Since every approximating sum is $b - a$, $\int_a^b 1\,dx = b - a$.

† The definite integral is also called the *Riemann integral* of f on $[a, b]$, and the sum (23.1) is called a *Riemann sum* for f on $[a, b]$.

An alternative argument would use the fact that the region under the graph of the constant function 1 and above the x axis, between $x = a$ and $x = b$, is a rectangle with base $b - a$ and height 1 (see Fig. 23-4). So, $\int_a^b 1\,dx$, being the area of that rectangle, is $b - a$.

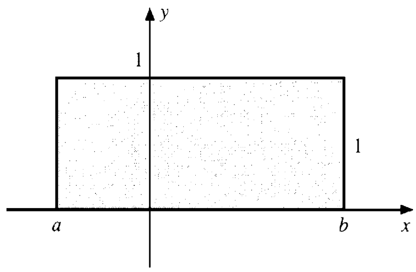

Fig. 23-4

EXAMPLE 23.3: Let us calculate $\int_a^b x\,dx$.

Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be a subdivision of $[a, b]$ into n equal subintervals. Thus, each $\Delta_k x = (b - a)/n$. Denote $(b - a)/n$ by Δx. Then $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, and, in general, $x_k = a + k\,\Delta x$. In the kth subinterval, $[x_{k-1}, x_k]$, choose x_k^* to be the right-hand endpoint x_k. Then the approximating sum (23.1) has the form

$$f(x_k^*)\,\Delta_k x = \sum_{k=1}^{n} x_k^*\,\Delta_k x = \sum_{k=1}^{n} (a + k\,\Delta x)\,\Delta x$$

$$= \sum_{k=1}^{n} (a\,\Delta x + k(\Delta x)^2) = \sum_{k=1}^{n} a\,\Delta x + \sum_{k=1}^{n} k(\Delta x)^2$$

$$= n(a\,\Delta x) + (\Delta x)^2 \sum_{k=1}^{n} k = n\left(a\frac{b-a}{n}\right) + \left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)}{2}\right)$$

$$= a(b-a) + \frac{1}{2}(b-a)^2\,\frac{n+1}{n}$$

Here we have used the fact that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. (See Problem 5.)

Now, as $n \to \infty$, $(n + 1)/n = 1 + 1/n \to 1 + 0 = 1$. Hence, the limit of our approximating sums is

$$a(b-a) + \tfrac{1}{2}(b-a)^2 = (b-a)\left(a + \frac{b-a}{2}\right) = (b-a)\left(\frac{a+b}{2}\right) = \tfrac{1}{2}(b^2 - a^2)$$

Thus, $\int_a^b x\,dx = \tfrac{1}{2}(b^2 - a^2)$.

In the next chapter, we will find a method for calculating $\int_a^b f(x)\,dx$ that will avoid the kind of tedious computation used in this example.

Properties of the Definite Integral

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \tag{23.3}$$

This follows from the fact that an approximating sum $\sum_{k=1}^{n} cf(x_k^*)\,\Delta_k$ for $\int_a^b cf(x)\,dx$ is equal to c times the approximating sum $\sum_{k=1}^{n} f(x_k^*)\,\Delta_k x$ for $\int_a^b f(x)\,dx$, and that the same relation holds for the corresponding limits.

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx \tag{23.4}$$

This is the special case of (23.3) when $c = -1$.

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{23.5}$$

This follows from the fact that an approximating sum $\sum_{k=1}^{n}(f(x_k^*)+g(x_k^*))\,\Delta_k x$ for $\int_a^b (f(x)+g(x))\,dx$ is equal to the sum $\sum_{k=1}^{n} f(x_k^*)\,\Delta_k x + \sum_{k=1}^{n} g(x_k^*)\,\Delta_k x$ of approximating sums for $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$.

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \tag{23.6}$$

Since $f(x) - g(x) = f(x) + (-g(x))$, this follows from (23.5) and (23.4).

If $a < c < b$, then f is integrable on $[a, b]$ if and only if it is integrable on $[a, c]$ and $[c, b]$. Moreover, if f is integrable on $[a, b]$,

$$\int_a^b f(x)\,dx = \int_c^a f(x)\,dx + \int_c^b f(x)\,dx \tag{23.7}$$

This is obvious when $f(x) \geq 0$ and we interpret the integrals as areas. The general result follows from looking at the corresponding approximating sums, although the case where one of the subintervals of $[a, b]$ contains c requires some extra thought.

We have defined $\int_a^b f(x)\,dx$ only when $a < b$. We can extend the definition to all possible cases as follows:

(i) $\int_a^a f(x)\,dx = 0$

(ii) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ when $a < b$

In particular, we always have:

$$\int_c^d f(x)\,dx = -\int_d^c f(x)\,dx \quad \text{for any } c \text{ and } d \tag{23.8}$$

It can readily be verified that the laws (23.2)–(23.6), the equation in (23.7), and the result of Example 23.3 all remain valid for arbitrary upper and lower limits in the integrals.

SOLVED PROBLEMS

1. Assume $f(x) \leq 0$ for all x in $[a, b]$. Let A be the area between the graph of f and the x axis, from $x = a$ to $x = b$. (See Fig. 23-5.) Show that $\int_c^b f(x)\,dx = -A$.

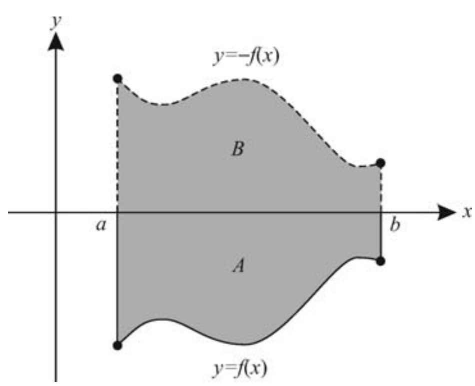

Fig. 23-5

Let B be the area between the graph of $-f$ and the x axis, from $x = a$ to $x = b$. By symmetry, $B = A$. But, $\int_a^b f(x)\,dx = -\int_a^b -f(x)\,dx$ by (23.4).

Since $\int_a^b -f(x)\,dx = B$, $\int_a^b f(x)\,dx = -B = -A$

2. Consider a function f that, between a and b, assumes both positive and negative values. For example, let its graph be as in Fig. 23-6. Then $\int_a^b f(x)\,dx$ is the difference between the sum of the areas above the x axis and below the graph and the sum of the areas below the x axis and above the graph. In the case of the graph shown in Fig. 23-6,

$$\int_a^b f(x)\,dx = (A_1 + A_3 + A_5) - (A_2 + A_4)$$

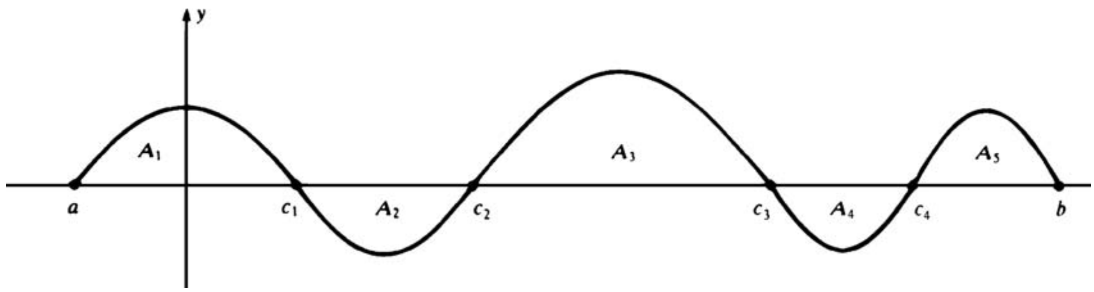

Fig. 23-6

To see this, apply (23.7) and Problem 1:

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \int_{c_3}^{c_4} f(x)\,dx + \int_{c_4}^{b} f(x)\,dx = A_1 - A_2 + A_3 - A_4 + A_5$$

3. Assume that f and g are integrable on $[a, b]$. Prove:

(a) If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

(b) If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

(c) If $m \leq f(x) \leq M$ for all x in $[a, b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

(a) Since every approximating sum $\sum_{k=1}^{n} f(x_k^*)\,\Delta_k x \geq 0$, it follows that

$$\int_a^b f(x)\,dx \geq 0$$

(b) $g(x) - f(x) \geq 0$ on $[a, b]$. So, by (*a*), $\int_a^b (g(x) - f(x))\,dx \geq 0$. By (23.6), $\int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0$. Hence,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

(c) By (b), $\int_a^b m\,dx \leq \int_a^b f(x) \leq \int_a^b M\,dx$. But, by (23.2) and (23.3), $\int_a^b m\,dx = m\int_a^b 1\,dx = m(b-a)$ and $\int_a^b M\,dx = M\int_a^b 1\,dx = M(b-a)$. Hence,

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

4. Evaluate $\int_0^1 x^2\,dx$.

This is the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$. Divide [0, 1] into n equal subintervals. Thus, each $\Delta_k x = 1/n$. In the kth subinterval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, let x_k^* be the right endpoint k/n. Thus, the approximating sum (23.1) is

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta_k x = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{k=1}^{n} k^2.$$

Now, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (see Problem 12).

Hence,

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta_k x = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \frac{1}{6}\left(1+\frac{1}{n}\right)(2+\frac{1}{n})$$

So, the approximating sums approach $\frac{1}{6}(1+0)(2+0) = \frac{1}{3}$ as $n \to \infty$. Therefore, $\int_0^1 x^2\,dx = \frac{1}{3}$. In the next chapter, we will derive a simpler method for obtaining the same result.

5. Prove the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ used in Example 23.3.

Reversing the order of the summands in

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

we get

$$\sum_{k=1}^{n} k = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Adding the two equations yields

$$2\sum_{k=1}^{n} k = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) = n(n+1)$$

since the sum in each column is $n + 1$. Hence, dividing by 2, we get

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

SUPPLEMENTARY PROBLEMS

6. Calculate: (a) $\int_1^4 3\,dx$; (b) $\int_{-2}^5 x\,dx$; (c) $\int_0^1 3x^2\,dx$.

Ans. (a) $3(4-1) = 9$; (b) $\frac{1}{2}(5^2-(-2)^2) = \frac{21}{2}$; (c) $3(\frac{1}{3}) = 1$

7. Find the area under the parabola $y = x^2 - 2x + 2$, above the x axis, and between $x = 0$ and $x = 1$.

Ans. $\frac{1}{3} - 2[\frac{1}{2}(1^2 - 0^2)] + 2(1-0) = \frac{4}{3}$

8. Evaluate $\int_2^6 (3x+4)\,dx$.

Ans. $3((\frac{1}{2})(6^2 - 2^2)) + 4(6-2) = 64$

9. For the function f graphed in Fig. 23-7, express $\int_0^3 f(x)\,dx$ in terms of the areas A_1, A_2, and A_3.

Ans. $A_1 - A_2 + A_3$

10. Show that $3 \le \int_1^4 x^3\,dx \le 192$. [*Hint:* Problem 3(*c*).]

11. Evaluate $\int_0^1 \sqrt{1-x^2}\,dx$. (*Hint:* Find the corresponding area by geometric reasoning.)

Ans. $\pi/4$

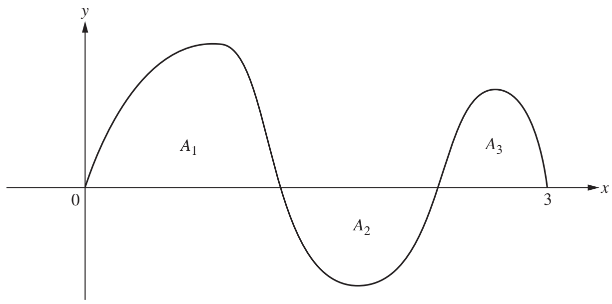

Fig. 23-7

12. Use mathematical induction to prove the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ of Problem 4. (Verify it when $n = 1$, and then show that, if it holds for n, then it holds for $n + 1$.)

13. Evaluate (a) $\sum_{j=0}^{2} \cos\frac{j\pi}{6}$; (b) $\sum_{j=0}^{2}(4j+1)$; (c) $\sum_{j=1}^{100} 4j$; (d) $\sum_{j=1}^{18} 2j^2$.

Ans. (a) $\frac{3+\sqrt{3}}{2}$; (b) 15; (c) 20200; (d) 4218

14. Let the graph of f between $x = 1$ and $x = 6$ be as in Fig. 23-8. Evaluate $\int_1^6 f(x)\,dx$.

Ans. $1 - 3 + \frac{1}{2} = -\frac{3}{2}$

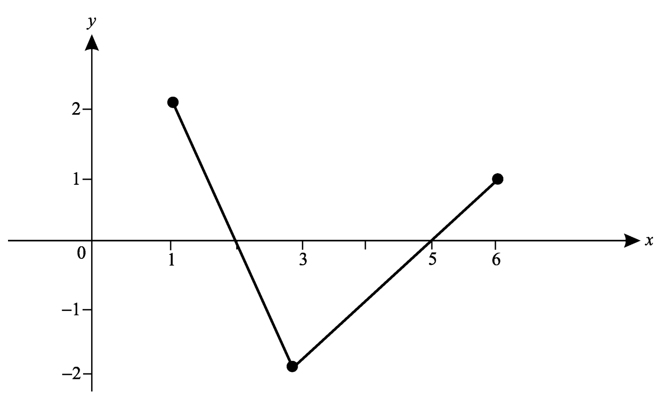

Fig. 23-8

15. If f is continuous on $[a, b]$, $f(x) \ge 0$ on $[a, b]$, and $f(x_0) > 0$ for some x_0 in $[a, b]$, prove that $\int_a^b f(x)\,dx > 0$. [*Hint:* By the continuity of f, $f(x) > \frac{1}{2}f(x_0) > 0$ for all x in some subinterval $[c, d]$. Use (23.7) and Problem 3(a, c).]