CHAPTER 11

Implicit Differentiation

Implicit Functions

An equation $f(x, y) = 0$ defines y *implicitly* as a function of x. The domain of that implicitly defined function consists of those x for which there is a unique y such that $f(x, y) = 0$.

EXAMPLE 11.1:

(a) The equation $xy + x - 2y - 1 = 0$ can be solved for y, yielding $y = \frac{1-x}{x-2}$. This function is defined for $x \neq 2$.

(b) The equation $4x^2 + 9y^2 - 36 = 0$ does not determine a unique function y. If we solve the equation for y, we obtain $y = \pm\frac{2}{3}\sqrt{9-x^2}$. We shall think of the equation as implicitly defining two functions, $y = \frac{2}{3}\sqrt{9-x^2}$ and $y = -\frac{2}{3}\sqrt{9-x^2}$. Each of these functions is defined for $|x| \le 3$. The ellipse determined by the original equation is the union of the graphs of the two functions.

If y is a function implicitly defined by an equation $f(x, y) = 0$, the derivative y' can be found in two different ways:

1. Solve the equation for y and calculate y' directly. Except for very simple equations, this method is usually impossible or impractical.
2. Thinking of y as a function of x, differentiate both sides of the original equation $f(x, y) = 0$ and solve the resulting equation for y'. This differentiation process is known as *implicit differentiation.*

EXAMPLE 11.2:

(a) Find y', given $xy + x - 2y - 1 = 0$. By implicit differentiation, $xy' + y\,D_x(x) - 2y' - D_x(1) = D_x(0)$. Thus, $xy' + y - 2y' = 0$. Solve for y': $y' = \frac{1+y}{2-x}$. In this case, Example 11.1(*a*) shows that we can replace y by $\frac{1-x}{x-2}$ and find y' in terms of x alone. We see that it would have been just as easy to differentiate $y = \frac{1-x}{x-2}$ by the Quotient Rule. However, in most cases, we cannot solve for y or for y' in terms of x alone.

(b) Given $4x^2 + 9y^2 - 36 = 0$, find y' when $x = \sqrt{5}$. By implicit differentiation, $4D_x(x^2) + 9D_x(y^2) - D_x(36) = D_x(0)$. Thus, $4(2x) + 9(2yy') = 0$. (Note that $D_x(y^2) = 2yy'$ by the Power Chain Rule.) Solving for y', we get $y' = -4x/9y$. When $x = \sqrt{5}$, $y = \pm\frac{4}{3}$. For the function y corresponding to the upper arc of the ellipse (see Example 11.1(*b*)), $y = -\frac{4}{3}$ and $y' = -\sqrt{5}/3$. For the function y corresponding to the lower arc of the ellipse, $y = -\frac{4}{3}$ and $y' = -\sqrt{5}/3$.

Derivatives of Higher Order

Derivatives of higher order may be obtained by implicit differentiation or by a combination of direct and implicit differentiation.

EXAMPLE 11.3: In Example 11.2(*a*), $y' = \frac{1+y}{2-x}$. Then

$$y'' = D_x(y') = D_x\left(\frac{1+y}{2-x}\right) = \frac{(2-x)y' - (1+y)(-1)}{(2-x)^2}$$

$$= \frac{(2-x)y' + 1 + y}{(2-x)^2} = \frac{(2-x)\left(\frac{1+y}{2-x}\right) + 1 + y}{(2-x)^2} = \frac{2+2y}{(2-x)^2}$$

EXAMPLE 11.4: Find the value of y'' at the point $(-1, 1)$ of the curve $x^2y + 3y - 4 = 0$.

We differentiate implicitly with respect to x twice. First, $x^2y' + 2xy + 3y' = 0$, and then $x^2y'' + 2xy' + 2xy' + 2y + 3y'' = 0$. We could solve the first equation for y'' and then solve the second equation for y''. However, since we only wish to evaluate y'' at the particular point $(-1, 1)$, we substitute $x = -1$, $y = 1$ in the first equation to find $y' = \frac{1}{2}$ and then substitute $x = -1$, $y = 1$, $y' = \frac{1}{2}$ in the second equation to get $y'' - 1 - 1 + 2 + 3y' = 0$, from which we obtain $y'' = 0$. Notice that this method avoids messy algebraic calculations.

SOLVED PROBLEMS

1. Find y', given $x^2y - xy^2 + x^2 + y^2 = 0$.

$$D_x(x^2y) - D_x(xy^2) + D_x(x^2) + D_x(y^2) = 0$$

$$x^2y' + yD_x(x^2) - xD_x(y^2) - y^2D_x(x) + 2x + 2yy' = 0$$

$$x^2y' + 2xy - x(2yy') - y^2 + 2x + 2yy' = 0$$

$$(x^2 - 2xy + 2y)y' + 2xy - y^2 + 2x = 0$$

$$y' = \frac{y^2 - 2xy - 2x}{x^2 - 2xy + 2y}$$

2. If $x^2 - xy + y^2 = 3$, find y' and y''.

$$D_x(x^2) - D_x(xy) + D_x(y^2) = 0$$

$$2x - xy' - y + 2yy' = 0$$

Hence, $y' = \dfrac{2x - y}{x - 2y}$. Then,

$$y'' = \frac{(x-2y)D_x(2x-y) - (2x-y)D_x(x-2y)}{(x-2y)^2}$$

$$= \frac{(x-2y)(2-y') - (2x-y)(1-2y')}{(x-2y)^2}$$

$$= \frac{2x - xy' - 4y + 2yy' - 2x + 4xy' + y - 2yy'}{(x-2y)^2} = \frac{3xy' - 3y}{(x-2y)^2}$$

$$= \frac{3x\left(\dfrac{2x-y}{x-2y}\right) - 3y}{(x-2y)^2} = \frac{3x(2x-y) - 3y(x-2y)}{(x-2y)^3} = \frac{6(x^2 - xy + y^2)}{(x-2y)^3}$$

$$= \frac{18}{(x-2y)^3}$$

3. Given $x^3y + xy^3 = 2$, find y' and y'' at the point $(1, 1)$.

By implicit differentiation twice,

$$x^3y' + 3x^2y + x(3y^2y') + y^3 = 0$$

and $$x^3y'' + 3x^2y' + 3x^2y' + 6xy + 3xy^2y'' + y'[6xyy' + 3y^2] + 3y^2y' = 0$$

Substituting $x = 1$, $y = 1$ in the first equation yields $y' = -1$. Then substituting $x = 1$, $y = 1$, $y' = -1$ in the second equation yields $y'' = 0$.

SUPPLEMENTARY PROBLEMS

4. Find y'', given: (a) $x + xy + y = 2$; (b) $x^3 - 3xy + y^3 = 1$.

Ans. (a) $y'' = \frac{2(1+y)}{(1+x)^2}$; (b) $y'' = -\frac{4xy}{(y^2-x)^3}$

5. Find y', y'', and y''' at: (a) the point (2, 1) on $x^2 - y^2 - x = 1$; (b) the point (1, 1) on $x^3 + 3x^2y - 6xy^2 + 2y^3 = 0$.

Ans. (a) $\frac{3}{2}, -\frac{5}{4}, \frac{45}{8}$; (b) 1, 0, 0

6. Find the slope of the tangent line at a point (x_0, y_0) of: (a) $b^2x^2 + a^2y^2 = a^2b^2$; (b) $b^2x^2 - a^2y^2 = a^2b^2$; (c) $x^3 + y^3 - 6x^2y = 0$.

Ans. (a) $-\frac{b^2x_0}{a^2y_0}$; (b) $\frac{b^2x_0}{a^2y_0}$; (c) $\frac{4x_0y_0 - x_0^2}{y_0^2 - 2x_0^2}$

7. Prove that the lines tangent to the curves $5y - 2x + y^3 - x^2y = 0$ and $2y + 5x + x^4 - x^3y^2 = 0$ at the origin intersect at right angles.

8. (a) The total surface area of a closed rectangular box whose base is a square with side y and whose height is x is given by $S = 2y^2 + 4xy$. If S is constant, find dy/dx without solving for y.

(b) The total surface area of a right circular cylinder of radius r and height h is given by $S = 2\pi r^2 + 2\pi rh$. If S is constant, find dr/dh.

Ans. (a) $-\frac{y}{x+y}$; (b) $-\frac{r}{2r+h}$

9. For the circle $x^2 + y^2 = r^2$, show that $\left|\frac{y''}{[1+(y')^2]^{3/2}}\right| = \frac{1}{r}$.

10. Given $S = \pi x(x + 2y)$ and $V = \pi x^2 y$, show that $dS/dx = 2\pi(x - y)$ when V is a constant, and $dV/dx = -\pi x(x - y)$ when S is a constant.

11. Derive the formula $D_x(x^m) = mx^{m-1}$ of Theorem 10.1(9) when $m = p/q$, where p and q are nonzero integers. You may assume that $x^{p/q}$ is differentiable. (*Hint*: Let $y = x^{p/q}$. Then $y^q = x^p$. Now use implicit differentiation.)

12. (GC) Use implicit differentation to find an equation of the tangent line to $\sqrt{x} + \sqrt{y} = 4$ at (4, 4), and verify your answer on a graphing calculator.

Ans. $y = -x + 8$