CHAPTER 2

Sequences

DEFINITION OF A SEQUENCE

A sequence is a set of numbers $u_1, u_2, u_3, \ldots$ in a definite order of arrangement (i.e., a *correspondence* with the natural numbers) and formed according to a definite rule. Each number in the sequence is called a *term*; u_n is called the nth *term*. The sequence is called *finite* or *infinite* according as there are or are not a finite number of terms. The sequence $u_1, u_2, u_3, \ldots$ is also designated briefly by $\{u_n\}$.

EXAMPLES. 1. The set of numbers $2, 7, 12, 17, \ldots, 32$ is a finite sequence; the nth term is given by $u_n = 2 + 5(n-1) = 5n - 3$, $n = 1, 2, \ldots, 7$.

2. The set of numbers $1, 1/3, 1/5, 1/7, \ldots$ is an infinite sequence with nth term $u_n = 1/(2n-1)$, $n = 1, 2, 3, \ldots$.

Unless otherwise specified, we shall consider infinite sequences only.

LIMIT OF A SEQUENCE

A number l is called the *limit* of an infinite sequence $u_1, u_2, u_3, \ldots$ if for any positive number ϵ we can find a positive number N depending on ϵ such that $|u_n - l| < \epsilon$ for all integers $n > N$. In such case we write $\lim_{n\to\infty} u_n = l$.

EXAMPLE . If $u_n = 3 + 1/n = (3n+1)/n$, the sequence is $4, 7/2, 10/3, \ldots$ and we can show that $\lim_{n\to\infty} u_n = 3$.

If the limit of a sequence exists, the sequence is called *convergent*; otherwise, it is called *divergent*. A sequence can converge to only one limit, i.e., if a limit exists, it is unique. See Problem 2.8.

A more intuitive but unrigorous way of expressing this concept of limit is to say that a sequence $u_1, u_2, u_3, \ldots$ has a limit l if the successive terms get "closer and closer" to l. This is often used to provide a "guess" as to the value of the limit, after which the definition is applied to see if the guess is really correct.

THEOREMS ON LIMITS OF SEQUENCES

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$
2. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$
3. $\lim_{n\to\infty} (a_n \cdot b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = AB$

4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$ if $\lim_{n\to\infty} b_n = B \neq 0$

   If $B = 0$ and $A \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist.

   If $B = 0$ and $A = 0$, $\lim_{n\to\infty} \frac{a_n}{b_n}$ may or may not exist.

5. $\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p = A^p$, for $p =$ any real number if A^p exists.

6. $\lim_{n\to\infty} p^{a_n} = p^{\lim_{n\to\infty} a_n} = p^A$, for $p =$ any real number if p^A exists.

INFINITY

We write $\lim_{n\to\infty} a_n = \infty$ if for each positive number M we can find a positive number N (depending on M) such that $a_n > M$ for all $n > N$. Similarly, we write $\lim_{n\to\infty} a_n = -\infty$ if for each positive number M we can find a positive number N such that $a_n < -M$ for all $n > N$. It should be emphasized that ∞ and $-\infty$ are not numbers and the sequences are not convergent. The terminology employed merely indicates that the sequences diverge in a certain manner. That is, no matter how large a number in absolute value that one chooses there is an n such that the absolute value of a_n is greater than that quantity.

BOUNDED, MONOTONIC SEQUENCES

If $u_n \leqq M$ for $n = 1, 2, 3, \ldots$, where M is a constant (independent of n), we say that the sequence $\{u_n\}$ is *bounded above* and M is called an *upper bound*. If $u_n \geqq m$, the sequence is *bounded below* and m is called a *lower bound*.

If $m \leqq u_n \leqq M$ the sequence is called *bounded*. Often this is indicated by $|u_n| \leqq P$. Every convergent sequence is bounded, but the converse is not necessarily true.

If $u_{n+1} \geqq u_n$ the sequence is called *monotonic increasing*; if $u_{n+1} > u_n$ it is called *strictly increasing*.

Similarly, if $u_{n+1} \leqq u_n$ the sequence is called *monotonic decreasing*, while if $u_{n+1} < u_n$ it is *strictly decreasing*.

EXAMPLES. 1. The sequence $1, 1.1, 1.11, 1.111, \ldots$ is bounded and monotonic increasing. It is also strictly increasing.
2. The sequence $1, -1, 1, -1, 1, \ldots$ is bounded but not monotonic increasing or decreasing.
3. The sequence $-1, -1.5, -2, -2.5, -3, \ldots$ is monotonic decreasing and not bounded. However, it is bounded above.

The following theorem is fundamental and is related to the Bolzano–Weierstrass theorem (Chapter 1, Page 6) which is proved in Problem 2.23.

Theorem. Every bounded monotonic (increasing or decreasing) sequence has a limit.

LEAST UPPER BOUND AND GREATEST LOWER BOUND OF A SEQUENCE

A number $\underline{M}$ is called the *least upper bound* (l.u.b.) of the sequence $\{u_n\}$ if $u_n \leqq \underline{M}$, $n = 1, 2, 3, \ldots$ while at least one term is greater than $\underline{M} - \epsilon$ for any $\epsilon > 0$.

A number $\bar{m}$ is called the *greatest lower bound* (g.l.b.) of the sequence $\{u_n\}$ if $u_n \geqq \bar{m}$, $n = 1, 2, 3, \ldots$ while at least one term is less than $\bar{m} + \epsilon$ for any $\epsilon > 0$.

Compare with the definition of l.u.b. and g.l.b. for sets of numbers in general (see Page 6).

LIMIT SUPERIOR, LIMIT INFERIOR

A number $\bar{l}$ is called the *limit superior*, *greatest limit* or *upper limit* (lim sup or $\overline{\lim}$) of the sequence $\{u_n\}$ if infinitely many terms of the sequence are greater than $\bar{l} - \epsilon$ while only a finite number of terms are greater than $\bar{l} + \epsilon$, where ϵ is any positive number.

A number $\underline{l}$ is called the *limit inferior*, *least limit* or *lower limit* (lim inf or $\underline{\lim}$) of the sequence $\{u_n\}$ if infintely many terms of the sequence are less than $\underline{l} + \epsilon$ while only a finite number of terms are less than $\underline{l} - \epsilon$, where ϵ is any positive number.

These correspond to least and greatest limiting points of general sets of numbers.

If infintely many terms of $\{u_n\}$ exceed any positive number M, we define lim sup $\{u_n\} = \infty$. If infinitely many terms are less than $-M$, where M is any positive number, we define lim inf $\{u_n\} = -\infty$.

If $\lim_{n\to\infty} u_n = \infty$, we define lim sup $\{u_n\} = \liminf \{u_n\} = \infty$.

If $\lim_{n\to\infty} u_n = -\infty$, we define $\limsup \{u_n\} = \liminf \{u_n\} = -\infty$.

Although every bounded sequence is not necessarily convergent, it always has a finite lim sup and lim inf.

A sequence $\{u_n\}$ converges if and only if $\limsup u_n = \liminf u_n$ is finite.

NESTED INTERVALS

Consider a set of intervals $[a_n, b_n]$, $n = 1, 2, 3, \ldots$, where each interval is contained in the preceding one and $\lim_{n\to\infty} (a_n - b_n) = 0$. Such intervals are called *nested intervals*.

We can prove that to every set of nested intervals there corresponds one and only one real number. This can be used to establish the Bolzano–Weierstrass theorem of Chapter 1. (See Problems 2.22 and 2.23.)

CAUCHY'S CONVERGENCE CRITERION

Cauchy's convergence criterion states that a sequence $\{u_n\}$ converges if and only if for each $\epsilon > 0$ we can find a number N such that $|u_p - u_q| < \epsilon$ for all $p, q > N$. This criterion has the advantage that one need not know the limit l in order to demonstrate convergence.

INFINITE SERIES

Let $u_1, u_2, u_3, \ldots$ be a given sequence. Form a new sequence $S_1, S_2, S_3, \ldots$ where

$$S_1 = u_1, S_2 = u_1 + u_2, S_3 = u_1 + u_2 + u_3, \ldots, S_n = u_1 + u_2 + u_3 + \cdots + u_n, \ldots$$

where S_n, called the nth *partial sum*, is the sum of the first n terms of the sequence $\{u_n\}$.

The sequence $S_1, S_2, S_3, \ldots$ is symbolized by

$$u_1 + u_2 + u_3 + \cdots = \sum_{n=1}^{\infty} u_n$$

which is called an *infinite series*. If $\lim_{n\to\infty} S_n = S$ exists, the series is called *convergent* and S is its *sum*, otherwise the series is called *divergent*.

Further discussion of infinite series and other topics related to sequences is given in Chapter 11.

Solved Problems

SEQUENCES

2.1. Write the first five terms of each of the following sequences.

(*a*) $\left\{\dfrac{2n-1}{3n+2}\right\}$

(*b*) $\left\{\dfrac{1-(-1)^n}{n^3}\right\}$

(*c*) $\left\{\dfrac{(-1)^{n-1}}{2\cdot 4\cdot 6\cdots 2n}\right\}$

(*d*) $\left\{\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\cdots+\dfrac{1}{2^n}\right\}$

(*e*) $\left\{\dfrac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}\right\}$

(*a*) $\dfrac{1}{5}, \dfrac{3}{8}, \dfrac{5}{11}, \dfrac{7}{14}, \dfrac{9}{17}$

(*b*) $\dfrac{2}{1^3}, 0, \dfrac{2}{3^3}, 0, \dfrac{2}{5^3}$

(*c*) $\dfrac{1}{2}, \dfrac{-1}{2\cdot 4}, \dfrac{1}{2\cdot 4\cdot 6}, \dfrac{-1}{2\cdot 4\cdot 6\cdot 8}, \dfrac{1}{2\cdot 4\cdot 6\cdot 8\cdot 10}$

(*d*) $\dfrac{1}{2}, \dfrac{1}{2}+\dfrac{1}{4}, \dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}, \dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}, \dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}+\dfrac{1}{32}$

(*e*) $\dfrac{x}{1!}, \dfrac{-x^3}{3!}, \dfrac{x^5}{5!}, \dfrac{-x^7}{7!}, \dfrac{x^9}{9!}$

Note that $n! = 1\cdot 2\cdot 3\cdot 4\cdots n$. Thus $1! = 1$, $3! = 1\cdot 2\cdot 3 = 6$, $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$, etc. We define $0! = 1$.

2.2. Two students were asked to write an nth term for the sequence $1, 16, 81, 256, \ldots$ and to write the 5th term of the sequence. One student gave the nth term as $u_n = n^4$. The other student, who did not recognize this simple law of formation, wrote $u_n = 10n^3 - 35n^2 + 50n - 24$. Which student gave the correct 5th term?

If $u_n = n^4$, then $u_1 = 1^4 = 1$, $u_2 = 2^4 = 16$, $u_3 = 3^4 = 81$, $u_4 = 4^4 = 256$, which agrees with the first four terms of the sequence. Hence the first student gave the 5th term as $u_5 = 5^4 = 625$.

If $u_n = 10n^3 - 35n^2 + 50n - 24$, then $u_1 = 1$, $u_2 = 16$, $u_3 = 81$, $u_4 = 256$, which also agrees with the first four terms given. Hence, the second student gave the 5th term as $u_5 = 601$.

Both students were correct. Merely giving a finite number of terms of a sequence does not define a unique nth term. In fact, an infinite number of nth terms is possible.

LIMIT OF A SEQUENCE

2.3. A sequence has its nth term given by $u_n = \dfrac{3n-1}{4n+5}$. (*a*) Write the 1st, 5th, 10th, 100th, 1000th, 10,000th and 100,000th terms of the sequence in decimal form. Make a *guess* as to the limit of this sequence as $n \to \infty$. (*b*) Using the definition of limit verify that the guess in (*a*) is actually correct.

(*a*)

$n = 1$	$n = 5$	$n = 10$	$n = 100$	$n = 1000$	$n = 10{,}000$	$n = 100{,}000$
.22222 . . .	.56000 . . .	.64444 . . .	.73827 . . .	.74881 . . .	.74988 . . .	.74998 . . .

A good guess is that the limit is $.75000\ldots = \frac{3}{4}$. Note that it is only for *large enough* values of n that a possible limit may become apparent.

(*b*) We must show that for any given $\epsilon > 0$ (no matter how small) there is a number N (depending on ϵ) such that $|u_n - \frac{3}{4}| < \epsilon$ for all $n > N$.

$$\text{Now}\quad \left|\frac{3n-1}{4n+5} - \frac{3}{4}\right| = \left|\frac{-19}{4(4n+5)}\right| < \epsilon \quad \text{when} \quad \frac{19}{4(4n+5)} < \epsilon \quad \text{or}$$

$$\frac{4(4n+5)}{19} > \frac{1}{\epsilon}, \qquad 4n+5 > \frac{19}{4\epsilon}, \qquad n > \frac{1}{4}\left(\frac{19}{4\epsilon} - 5\right)$$

Choosing $N = \frac{1}{4}(19/4\epsilon - 5)$, we see that $|u_n - \frac{3}{4}| < \epsilon$ for all $n > N$, so that $\lim\limits_{n\to\infty} = \frac{3}{4}$ and the proof is complete.

Note that if $\epsilon = .001$ (for example), $N = \frac{1}{4}(19000/4 - 5) = 1186\frac{1}{4}$. This means that all terms of the sequence beyond the 1186th term differ from $\frac{3}{4}$ in absolute value by less than .001.

2.4. Prove that $\lim\limits_{n\to\infty} \dfrac{c}{n^p} = 0$ where $c \neq 0$ and $p > 0$ are constants (independent of n).

We must show that for any $\epsilon > 0$ there is a number N such that $|c/n^p - 0| < \epsilon$ for all $n > N$.

Now $\left|\dfrac{c}{n^p}\right| < \epsilon$ when $\dfrac{|c|}{n^p} < \epsilon$, i.e., $n^p > \dfrac{|c|}{\epsilon}$ or $n > \left(\dfrac{|c|}{\epsilon}\right)^{1/p}$. Choosing $N = \left(\dfrac{|c|}{\epsilon}\right)^{1/p}$ (depending on ϵ), we see that $|c/n^p| < \epsilon$ for all $n > N$, proving that $\lim\limits_{n\to\infty}(c/n^p) = 0$.

2.5. Prove that $\lim\limits_{n\to\infty} \dfrac{1 + 2\cdot 10^n}{5 + 3\cdot 10^n} = \dfrac{2}{3}$.

We must show that for any $\epsilon > 0$ there is a number N such that $\left|\dfrac{1+2\cdot 10^n}{5+3\cdot 10^n} - \dfrac{2}{3}\right| < \epsilon$ for all $n > N$.

Now $\left|\dfrac{1+2\cdot 10^n}{5+3\cdot 10^n} - \dfrac{2}{3}\right| = \left|\dfrac{-7}{3(5+3\cdot 10^n)}\right| < \epsilon$ when $\dfrac{7}{3(5+3\cdot 10^n)} < \epsilon$, i.e. when $\frac{3}{7}(5 + 3\cdot 10^n) > 1/\epsilon$, $3\cdot 10^n > 7/3\epsilon - 5$, $10^n > \frac{1}{8}(7/3\epsilon - 5)$ or $n > \log_{10}\{\frac{1}{3}(7/3\epsilon - 5)\} = N$, proving the existence of N and thus establishing the required result.

Note that the above value of N is real only if $7/3\epsilon - 5 > 0$, i.e., $0 < \epsilon < 7/15$. If $\epsilon \geqq 7/15$, we see that $\left|\dfrac{1+2\cdot 10^n}{5+3\cdot 10^n} - \dfrac{2}{3}\right| < \epsilon$ for *all* $n > 0$.

2.6. Explain exactly what is meant by the statements (*a*) $\lim\limits_{n\to\infty} 3^{2n-1} = \infty$, (*b*) $\lim\limits_{n\to\infty}(1 - 2n) = -\infty$.

(*a*) If for each positive number M we can find a positive number N (depending on M) such that $a_n > M$ for all $n > N$, then we write $\lim\limits_{n\to\infty} a_n = \infty$.

In this case, $3^{2n-1} > M$ when $(2n-1)\log 3 > \log M$, i.e., $n > \dfrac{1}{2}\left(\dfrac{\log M}{\log 3} + 1\right) = N$.

(*b*) If for each positive number M we can find a positive number N (depending on M) such that $a_n < -M$ for all $n > N$, then we write $\lim\limits_{n\to\infty} = -\infty$.

In this case, $1 - 2n < -M$ when $2n - 1 > M$ or $n > \frac{1}{2}(M+1) = N$.

It should be emphasized that the use of the notations ∞ and $-\infty$ for limits does not in any way imply convergence of the given sequences, since ∞ and $-\infty$ are *not* numbers. Instead, these are notations used to describe that the sequences diverge in specific ways.

2.7. Prove that $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$.

Method 1:

We can restrict ourselves to $x \neq 0$, since if $x = 0$, the result is clearly true. Given $\epsilon > 0$, we must show that there exists N such that $|x^n| < \epsilon$ for $n > N$. Now $|x^n| = |x|^n < \epsilon$ when $n \log_{10}|x| < \log_{10}\epsilon$. Dividing by $\log_{10}|x|$, which is negative, yields $n > \dfrac{\log_{10}\epsilon}{\log_{10}|x|} = N$, proving the required result.

Method 2:

Let $|x| = 1/(1+p)$, where $p > 0$. By Bernoulli's inequality (Problem 1.31, Chapter 1), we have $|x^n| = |x|^n = 1/(1+p)^n < 1/(1+np) < \epsilon$ for all $n > N$. Thus $\lim_{n\to\infty} x^n = 0$.

THEOREMS ON LIMITS OF SEQUENCES

2.8. Prove that if $\lim_{n\to\infty} u_n$ exists, it must be unique.

We must show that if $\lim_{n\to\infty} u_n = l_1$ and $\lim_{n\to\infty} u_n = l_2$, then $l_1 = l_2$.

By hypothesis, given any $\epsilon > 0$ we can find N such that

$$|u_n - l_1| < \tfrac{1}{2}\epsilon \quad \text{when} \quad n > N, \qquad |u_n - l_2| < \tfrac{1}{2}\epsilon \quad \text{when} \quad n > N$$

Then

$$|l_1 - l_2| = |l_1 - u_n + u_n - l_2| \leqq |l_1 - u_n| + |u_n - l_2| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon$$

i.e., $|l_1 - l_2|$ is less than any positive ϵ (however small) and so must be zero. Thus, $l_1 = l_2$.

2.9. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, prove that $\lim_{n\to\infty}(a_n + b_n) = A + B$.

We must show that for any $\epsilon > 0$, we can find $N > 0$ such that $|(a_n + b_n) - (A + B)| < \epsilon$ for all $n > N$. From absolute value property 2, Page 3 we have

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leqq |a_n - A| + |b_n - B| \qquad (1)$$

By hypothesis, given $\epsilon > 0$ we can find N_1 and N_2 such that

$$|a_n - A| < \tfrac{1}{2}\epsilon \qquad \text{for all } n > N_1 \qquad (2)$$

$$|b_n - B| < \tfrac{1}{2}\epsilon \qquad \text{for all } n > N_2 \qquad (3)$$

Then from (*1*), (*2*), and (*3*),

$$|(a_n + b_n) - (A + B)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon \qquad \text{for all } n > N$$

where N is chosen as the larger of N_1 and N_2. Thus, the required result follows.

2.10. Prove that a convergent sequence is bounded.

Given $\lim_{n\to\infty} a_n = A$, we must show that there exists a positive number P such that $|a_n| < P$ for all n. Now

$$|a_n| = |a_n - A + A| \leqq |a_n - A| + |A|$$

But by hypothesis we can find N such that $|a_n - A| < \epsilon$ for all $n > N$, i.e.,

$$|a_n| < \epsilon + |A| \qquad \text{for all } n > N$$

It follows that $|a_n| < P$ for all n if we choose P as the largest one of the numbers $a_1, a_2, \ldots, a_N, \epsilon + |A|$.

2.11. If $\lim_{n\to\infty} b_n = B \neq 0$, prove there exists a number N such that $|b_n| > \frac{1}{2}|B|$ for all $n > N$.

Since $B = B - b_n + b_n$, we have: (*1*) $|B| \leqq |B - b_n| + |b_n|$.
Now we can choose N so that $|B - b_n| = |b_n - B| < \frac{1}{2}|B|$ for all $n > N$, since $\lim_{n\to\infty} b_n = B$ by hypothesis.
Hence, from (*1*), $|B| < \frac{1}{2}|B| + |b_n|$ or $|b_n| > \frac{1}{2}|B|$ for all $n > N$.

2.12. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, prove that $\lim_{n\to\infty} a_n b_n = AB$.

We have, using Problem 2.10,

$$\begin{aligned} |a_n b_n - AB| = |a_n(b_n - B) + B(a_n - A)| &\leqq |a_n||b_n - B| + |B||a_n - A| \qquad (1)\\ &\leqq P|b_n - B| + (|B| + 1)|a_n - A| \end{aligned}$$

But since $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, given any $\epsilon > 0$ we can find N_1 and N_2 such that

$$|b_n - B| < \frac{\epsilon}{2P} \text{ for all } n > N_1 \qquad |a_n - A| < \frac{\epsilon}{2(|B| + 1)} \text{ for all } n > N_2$$

Hence, from (*1*), $|a_n b_n - AB| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ for all $n > N$, where N is the larger of N_1 and N_2. Thus, the result is proved.

2.13. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B \neq 0$, prove (*a*) $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$, (*b*) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.

(*a*) We must show that for any given $\epsilon > 0$, we can find N such that

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|B - b_n|}{|B||b_n|} < \epsilon \qquad \text{for all } n > N \qquad (1)$$

By hypothesis, given any $\epsilon > 0$, we can find N_1, such that $|b_n - B| < \frac{1}{2}B^2\epsilon$ for all $n > N_1$.
Also, since $\lim_{n\to\infty} b_n = B \neq 0$, we can find N_2 such that $|b_n| > \frac{1}{2}|B|$ for all $n > N_2$ (see Problem 11).
Then if N is the larger of N_1 and N_2, we can write (*1*) as

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|b_n - B|}{|B||b_n|} < \frac{\frac{1}{2}B^2\epsilon}{|B| \cdot \frac{1}{2}|B|} = \epsilon \qquad \text{for all } n > N$$

and the proof is complete.

(*b*) From part (*a*) and Problem 2.12, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(a_n \cdot \frac{1}{b_n}\right) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = A \cdot \frac{1}{B} = \frac{A}{B}$$

This can also be proved directly (see Problem 41).

2.14. Evaluate each of the following, using theorems on limits.

(*a*) $$\lim_{n\to\infty} \frac{3n^2 - 5n}{5n^2 + 2n - 6} = \lim_{n\to\infty} \frac{3 - 5/n}{5 + 2/n - 6/n^2} = \frac{3 + 0}{5 + 0 + 0} = \frac{3}{5}$$

(*b*) $$\begin{aligned} \lim_{n\to\infty}\left\{\frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1}\right\} &= \lim_{n\to\infty}\left\{\frac{n^3 + n^2 + 2n}{(n+1)(n^2+1)}\right\} = \lim_{n\to\infty}\left\{\frac{1 + 1/n + 2/n^2}{(1 + 1/n)(1 + 1/n^2)}\right\}\\ &= \frac{1 + 0 + 0}{(1 + 0)\cdot(1 + 0)} = 1 \end{aligned}$$

(*c*) $$\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

(d) $\lim_{n\to\infty} \frac{3n^2+4n}{2n-1} = \lim_{n\to\infty} \frac{3+4/n}{2/n-1/n^2}$

Since the limits of the numerator and denominator are 3 and 0, respectively, the limit does not exist.

Since $\frac{3n^2+4n}{2n-1} > \frac{3n^2}{2n} = \frac{3n}{2}$ can be made larger than any positive number M by choosing $n > N$, we can write, if desired, $\lim_{n\to\infty} \frac{3n^2+4n}{2n-1} = \infty$.

(e) $\lim_{n\to\infty} \left(\frac{2n-3}{2n+7}\right)^4 = \left(\lim_{n\to\infty} \frac{2-3/n}{3+7/n}\right)^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

(f) $\lim_{n\to\infty} \frac{2n^5-4n^2}{3n^7+n^3-10} = \lim_{n\to\infty} \frac{2/n^2-4/n^5}{3+1/n^4-10/n^7} = \frac{0}{3} = 0$

(g) $\lim_{n\to\infty} \frac{1+2\cdot 10^n}{5+3\cdot 10^n} = \lim_{n\to\infty} \frac{10^{-n}+2}{5\cdot 10^{-n}+3} = \frac{2}{3}$ (Compare with Problem 2.5.)

BOUNDED MONOTONIC SEQUENCES

2.15. Prove that the sequence with nth $u_n = \frac{2n-7}{3n+2}$ (a) is monotonic increasing, (b) is bounded above, (c) is bounded below, (d) is bounded, (e) has a limit.

(a) $\{u_n\}$ is monotonic increasing if $u_{n+1} \geqq u_n$, $n = 1, 2, 3, \ldots$. Now

$$\frac{2(n+1)-7}{3(n+1)+2} \geqq \frac{2n-7}{3n+2} \quad \text{if and only if} \quad \frac{2n-5}{2n+5} \geqq \frac{2n-7}{3n+2}$$

or $(2n-5)(3n+2) \geqq (2n-7)(3n+5)$, $6n^2-11n-10 \geqq 6n^2-11n-35$, i.e. $-10 \geqq -35$, which is true. Thus, by reversal of steps in the inequalities, we see that $\{u_n\}$ is monotonic increasing. Actually, since $-10 > -35$, the sequence is strictly increasing.

(b) By writing some terms of the sequence, we may *guess* that an upper bound is 2 (for example). To *prove* this we must show that $u_n \leqq 2$. If $(2n-7)/(3n+2) \leqq 2$ then $2n-7 \leqq 6n+4$ or $-4n < 11$, which *is* true. Reversal of steps proves that 2 is an upper bound.

(c) Since this particular sequence is monotonic increasing, the first term -1 is a lower bound, i.e., $u_n \geqq -1$, $n = 1, 2, 3, \ldots$. Any number less than -1 is also a lower bound.

(d) Since the sequence has an upper and lower bound, it is bounded. Thus, for example, we can write $|u_n| \leqq 2$ for all n.

(e) Since every bounded monotonic (increasing or decreasing) sequence has a limit, the given sequence has a limit. In fact, $\lim_{n\to\infty} \frac{2n-7}{3n+2} = \lim_{n\to\infty} \frac{2-7/n}{3+2/n} = \frac{2}{3}$.

2.16. A sequence $\{u_n\}$ is defined by the recursion formula $u_{n+1} = \sqrt{3u_n}$, $u_1 = 1$. (a) Prove that $\lim_{n\to\infty} u_n$ exists. (b) Find the limit in (a).

(a) The terms of the sequence are $u_1 = 1$, $u_2 = \sqrt{3u_1} = 3^{1/2}$, $u_3 = \sqrt{3u_2} = 3^{1/2+1/4}, \ldots$.

The nth term is given by $u_n = 3^{1/2+1/4+\cdots+1/2^{n-1}}$ as can be proved by mathematical induction (Chapter 1).

Clearly, $u_{n+1} \geqq u_n$. Then the sequence is monotone increasing.

By Problem 1.14, Chapter 1, $u_n \leqq 3^1 = 3$, i.e. u_n is bounded above. Hence, u_n is bounded (since a lower bound is zero).

Thus, a limit exists, since the sequence is bounded and monotonic increasing.

(*b*) Let $x =$ required limit. Since $\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} \sqrt{3u_n}$, we have $x = \sqrt{3x}$ and $x = 3$. (The other possibility, $x = 0$, is excluded since $u_n \geqq 1$.)

Another method: $\lim_{n\to\infty} 3^{1/2+1/4+\cdots+1/2^{n-1}} = \lim_{n\to\infty} 3^{1-1/2^n} = 3^{\lim_{n\to\infty}(1-1/2^n)} = 3^1 = 3$

2.17. Verify the validity of the entries in the following table.

Sequence	Bounded	Monotonic Increasing	Monotonic Decreasing	Limit Exists
$2, 1.9, 1.8, 1.7, \ldots, 2-(n-1)/10\ldots$	No	No	Yes	No
$1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$	Yes	No	No	No
$\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, (-1)^{n-1}/(n+1), \ldots$	Yes	No	No	Yes (0)
$.6, .66, .666, \ldots, \frac{2}{3}(1-1/10^n), \ldots$	Yes	Yes	No	Yes ($\frac{2}{3}$)
$-1, +2, -3, +4, -5, \ldots, (-1)^n n, \ldots$	No	No	No	No

2.18. Prove that the sequence with the nth term $u_n = \left(1 + \dfrac{1}{n}\right)^n$ is monotonic, increasing, and bounded, and thus a limit exists. The limit is denoted by the symbol e.

Note: $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$, where $e \cong 2.71828\ldots$ was introduced in the eighteenth century by Leonhart Euler as the base for a system of logarithms in order to simplify certain differentiation and integration formulas.

By the binomial theorem, if n is a positive integer (see Problem 1.95, Chapter 1),

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\cdots(n-n+1)}{n!}x^n$$

Letting $x = 1/n$,

$$\begin{aligned} u_n = \left(1+\frac{1}{n}\right)^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-n+1)}{n!}\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\ &\quad + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \end{aligned}$$

Since each term beyond the first two terms in the last expression is an increasing function of n, it follows that the sequence u_n is a monotonic increasing sequence.

It is also clear that

$$\left(1+\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3$$

by Problem 1.14, Chapter 1.

Thus, u_n is bounded and monotonic increasing, and so has a limit which we denote by e. The value of $e = 2.71828\ldots$.

2.19. Prove that $\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$, where $x \to \infty$ in any manner whatsoever (i.e., not necessarily along the positive integers, as in Problem 2.18).

If $n =$ largest integer $\leqq x$, then $n \leqq x \leqq n+1$ and $\left(1+\dfrac{1}{n+1}\right)^n \leqq \left(1+\dfrac{1}{x}\right)^x \leqq \left(1+\dfrac{1}{n}\right)^{n+1}$.

Since $$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1} \Big/ \left(1+\frac{1}{n+1}\right) = e$$

and
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) = e$$

it follows that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$.

LEAST UPPER BOUND, GREATEST LOWER BOUND, LIMIT SUPERIOR, LIMIT INFERIOR

2.20. Find the (*a*) l.u.b., (*b*) g.l.b., (*c*) lim sup ($\overline{\lim}$), and (*d*) lim inf ($\underline{\lim}$) for the sequence $2, -2, 1, -1, 1, -1, 1, -1, \ldots$.

(*a*) l.u.b. $= 2$, since all terms are less than equal to 2, while at least one term (the 1st) is greater than $2-\epsilon$ for any $\epsilon > 0$.

(*b*) g.l.b. $= -2$, since all terms are greater than or equal to -2, while at least one term (the 2nd) is less than $-2+\epsilon$ for any $\epsilon > 0$.

(*c*) lim sup or $\overline{\lim} = 1$, since infinitely many terms of the sequence are greater than $1-\epsilon$ for any $\epsilon > 0$ (namely, all 1's in the sequence), while only a finite number of terms are greater than $1+\epsilon$ for any $\epsilon > 0$ (namely, the 1st term).

(*d*) lim inf or $\underline{\lim} = -1$, since infinitely many terms of the sequence are less than $-1+\epsilon$ for any $\epsilon > 0$ (namely, all -1's in the sequence), while only a finite number of terms are less than $-1-\epsilon$ for any $\epsilon > 0$ (namely the 2nd term).

2.21. Find the (*a*) l.u.b., (*b*) g.l.b., (*c*) lim sup ($\overline{\lim}$), and (*d*) lim inf ($\underline{\lim}$) for the sequences in Problem 2.17.

The results are shown in the following table.

Sequence	l.u.b.	g.l.b.	lim sup or $\overline{\lim}$	lim inf or $\underline{\lim}$
$2, 1.9, 1.8, 1.7, \ldots, 2-(n-1)/10 \ldots$	2	none	$-\infty$	$-\infty$
$1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$	1	-1	1	-1
$\frac{1}{2}, -\frac{1}{3}, \frac{1}{4} -\frac{1}{5}, \ldots, (-1)^{n-1}/(n+1), \ldots$	$\frac{1}{2}$	$-\frac{1}{3}$	0	0
$.6, .66, .666, \ldots, \frac{2}{3}(1-1/10^n), \ldots$	$\frac{2}{3}$	6	$\frac{2}{3}$	$\frac{2}{3}$
$-1, +2, -3, +4, -5, \ldots, (-1)^n n, \ldots$	none	none	$+\infty$	$-\infty$

NESTED INTERVALS

2.22. Prove that to every set of nested intervals $[a_n, b_n]$, $n = 1, 2, 3, \ldots$, there corresponds one and only one real number.

By definition of nested intervals, $a_{n+1} \geqq a_n$, $b_{n+1} \leqq b_n$, $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty}(a_n - b_n) = 0$.

Then $a_1 \leqq a_n \leqq b_n \leqq b_1$, and the sequences $\{a_n\}$ and $\{b_n\}$ are bounded and respectively monotonic increasing and decreasing sequences and so converge to a and b.

To show that $a = b$ and thus prove the required result, we note that

$$b - a = (b - b_n) + (b_n - a_n) + (a_n - a) \qquad (1)$$
$$|b - a| \leqq |b - b_n| + |b_n - a_n| + |a_n - a| \qquad (2)$$

Now given any $\epsilon > 0$, we can find N such that for all $n > N$

$$|b - b_n| < \epsilon/3, \qquad |b_n - a_n| < \epsilon/3, \qquad |a_n - a| < \epsilon/3 \qquad (3)$$

so that from (*2*), $|b - a| < \epsilon$. Since ϵ is any positive number, we must have $b - a = 0$ or $a = b$.

2.23. Prove the Bolzano–Weierstrass theorem (see Page 6).

Suppose the given bounded infinite set is contained in the finite interval $[a, b]$. Divide this interval into two equal intervals. Then at least one of these, denoted by $[a_1, b_1]$, contains infinitely many points. Dividing $[a_1, b_1]$ into two equal intervals, we obtain another interval, say, $[a_2, b_2]$, containing infinitely many points. Continuing this process, we obtain a set of intervals $[a_n, b_n]$, $n = 1, 2, 3, \ldots$, each interval contained in the preceding one and such that

$$b_1 - a_1 = (b - a)/2, b_2 - a_2 = (b_1 - a_1)/2 = (b - a)/2^2, \ldots, b_n - a_n = (b - a)/2^n$$

from which we see that $\lim_{n\to\infty}(b_n - a_n) = 0$.

This set of nested intervals, by Problem 2.22, corresponds to a real number which represents a limit point and so proves the theorem.

CAUCHY'S CONVERGENCE CRITERION

2.24. Prove Cauchy's convergence criterion as stated on Page 25.

Necessity. Suppose the sequence $\{u_n\}$ converges to l. Then given any $\epsilon > 0$, we can find N such that

$$|u_p - l| < \epsilon/2 \text{ for all } p > N \qquad \text{and} \qquad |u_q - l| < \epsilon/2 \text{ for all } q > N$$

Then for both $p > N$ and $q > N$, we have

$$|u_p - u_q| = |(u_p - l) + (l - u_q)| \leqq |u_p - l| + |l - u_q| < \epsilon/2 + \epsilon/2 = \epsilon$$

Sufficiency. Suppose $|u_p - u_q| < \epsilon$ for all $p, q > N$ and any $\epsilon > 0$. Then all the numbers $u_N, u_{N+1}, \ldots$ lie in a finite interval, i.e., the set is bounded and infinite. Hence, by the Bolzano–Weierstrass theorem there is at least one limit point, say a.

If a is the only limit point, we have the desired proof and $\lim_{n\to\infty} u_n = a$.

Suppose there are two distinct limit points, say a and b, and suppose $b > a$ (see Fig. 2-1). By definition of limit points, we have

$$|u_p - a| < (b - a)/3 \quad \text{for infinnitely many values of } p \qquad (1)$$
$$|u_q - b| < (b - a)/3 \quad \text{for infinitely many values of } q \qquad (2)$$

Fig. 2-1

Then since $b - a = (b - u_q) + (u_q - u_p) + (u_p - a)$, we have

$$|b - a| = b - a \leqq |b - u_q| + |u_p - u_q| + |u_p - a| \qquad (3)$$

Using (*1*) and (*2*) in (*3*), we see that $|u_p - u_q| > (b - a)/3$ for infinitely many values of p and q, thus contradicting the hypothesis that $|u_p - u_q| < \epsilon$ for $p, q > N$ and any $\epsilon > 0$. Hence, there is only one limit point and the theorem is proved.

INFINITE SERIES

2.25. Prove that the infinite series (sometimes called the *geometric series*)

$$a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

(*a*) converges to $a/(1 - r)$ if $|r| < 1$, (*b*) diverges if $|r| \geqq 1$.

Let $\qquad S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

Then $\qquad rS_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$

Subtract, $\qquad (1 - r)S_n = a \qquad\qquad - ar^n$

or
$$S_n = \frac{a(1-r^n)}{1-r}$$

(*a*) If $|r| < 1$, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$ by Problem 7.

(*b*) If $|r| > 1$, $\lim_{n\to\infty} S_n$ does not exist (see Problem 44).

2.26. Prove that if a series converges, its *n*th term must necessarily approach zero.

Since $S_n = u_1 + u_2 + \cdots + u_n$, $S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$ we have $u_n = S_n - S_{n-1}$.
If the series converges to S, then
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

2.27. Prove that the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}$ diverges.

Method 1:
$\lim_{n\to\infty}(-1)^n \neq 0$, in fact it doesn't exist. Then by Problem 2.26 the series cannot converge, i.e., it diverges.

Method 2:
The sequence of partial sums is $1, 1-1, 1-1+1, 1-1+1-1, \ldots$ i.e., $1, 0, 1, 0, 1, 0, 1, \ldots$. Since this sequence has no limit, the series diverges.

MISCELLANEOUS PROBLEMS

2.28. If $\lim_{n\to\infty} u_n = l$, prove that $\lim_{n\to\infty} \frac{u_1 + u_2 + \cdots + u_n}{n} = l$.

Let $u_n = v_n + l$. We must show that $\lim_{n\to\infty} \frac{v_1 + v_2 + \cdots + v_n}{n} = 0$ if $\lim_{n\to\infty} v_n = 0$. Now
$$\frac{v_1 + v_2 + \cdots + v_n}{n} = \frac{v_1 + v_2 + \cdots + v_P}{n} + \frac{v_{P+1} + v_{p+2} + \cdots + v_n}{n}$$
so that
$$\left|\frac{v_1 + v_2 + \cdots + v_n}{n}\right| \leqq \frac{|v_1 + v_2 + \cdots + v_P|}{n} + \frac{|v_{P+1}| + |v_{P+2}| + \cdots + |v_n|}{n} \qquad (1)$$
Since $\lim_{n\to\infty} v_n = 0$, we can choose P so that $|v_n| < \epsilon/2$ for $n > P$. Then
$$\frac{|v_{P+1}| + |v_{P+2}| + \cdots + |v_n|}{n} < \frac{\epsilon/2 + \epsilon/2 + \cdots + \epsilon/2}{n} = \frac{(n-P)\epsilon/2}{n} < \frac{\epsilon}{2} \qquad (2)$$
After choosing P we can choose N so that for $n > N > P$,
$$\frac{|v_1 + v_2 + \cdots + v_P|}{n} < \frac{\epsilon}{2} \qquad (3)$$
Then using (*2*) and (*3*), (*1*) becomes
$$\left|\frac{v_1 + v_2 + \cdots + v_n}{n}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for } n > N$$
thus proving the required result.

2.29. Prove that $\lim_{n\to\infty}(1 + n + n^2)^{1/n} = 1$.

Let $(1 + n + n^2)^{1/n} = 1 + u_n$ where $u_n \geqq 0$. Now by the binomial theorem,

$$1 + n + n^2 = (1 + u_n)^n = 1 + nu_n + \frac{n(n-1)}{2!}u_n^2 + \frac{n(n-1)(n-2)}{3!}u_n^3 + \cdots + u_n^n$$

Then $1 + n + n^2 > 1 + \dfrac{n(n-1)(n-2)}{3!}u_n^3$ or $0 < u_n^3 < \dfrac{6(n^2+n)}{n(n-1)(n-2)}$.

Hence, $\lim_{n\to\infty} u_n^3 = 0$ and $\lim_{n\to\infty} u_n = 0$. Thus $\lim_{n\to\infty}(1+n+n^2)^{1/n} = \lim_{n\to\infty}(1+u_n) = 1$.

2.30. Prove that $\lim_{n\to\infty} \dfrac{a^n}{n!} = 0$ for all constants a.

The result follows if we can prove that $\lim_{n\to\infty} \dfrac{|a|^n}{n!} = 0$ (see Problem 2.38). We can assume $a \neq 0$.

Let $u_n = \dfrac{|a|^n}{n!}$. Then $\dfrac{u_n}{u_{n-1}} = \dfrac{|a|}{n}$. If n is large enough, say, $n > 2|a|$, and if we call $N = [2|a| + 1]$, i.e., the greatest integer $\leqq 2|a| + 1$, then

$$\frac{u_{N+1}}{u_N} < \frac{1}{2}, \frac{u_{N+2}}{u_{N+1}} < \frac{1}{2}, \ldots, \frac{u_n}{u_{n-1}} < \frac{1}{2}$$

Multiplying these inequalities yields $\dfrac{u_n}{u_N} < \left(\tfrac{1}{2}\right)^{n-N}$ or $u_n < \left(\tfrac{1}{2}\right)^{n-N} u_N$.

Since $\lim_{n\to\infty}\left(\dfrac{1}{2}\right)^{n-N} = 0$ (using Problem 2.7), it follows that $\lim_{n\to\infty} u_n = 0$.

Supplementary Problems

SEQUENCES

2.31. Write the first four terms of each of the following sequences:

(*a*) $\left\{\dfrac{\sqrt{n}}{n+1}\right\}$, (*b*) $\left\{\dfrac{(-1)^{n+1}}{n!}\right\}$, (*c*) $\left\{\dfrac{(2x)^{n-1}}{(2n-1)^5}\right\}$, (*d*) $\left\{\dfrac{(-1)^n x^{2n-1}}{1\cdot 3\cdot 5\cdots(2n-1)}\right\}$, (*e*) $\left\{\dfrac{\cos nx}{x^2+n^2}\right\}$.

Ans. (*a*) $\dfrac{\sqrt{1}}{2}, \dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{3}}{4}, \dfrac{\sqrt{4}}{5}$ (*c*) $\dfrac{1}{1^5}, \dfrac{2x}{3^5}, \dfrac{4x^2}{5^5}, \dfrac{8x^3}{7^5}$ (*e*) $\dfrac{\cos x}{x^2+1^2}, \dfrac{\cos 2x}{x^2+2^2}, \dfrac{\cos 3x}{x^2+3^2}, \dfrac{\cos 4x}{x^2+4^2}$

(*b*) $\dfrac{1}{1!}, -\dfrac{1}{2!}, \dfrac{1}{3!}, -\dfrac{1}{4!}$ (*d*) $\dfrac{-x}{1}, \dfrac{x^3}{1\cdot 3}, \dfrac{-x^5}{1\cdot 3\cdot 5}, \dfrac{x^7}{1\cdot 3\cdot 5\cdot 7}$

2.32. Find a possible nth term for the sequences whose first 5 terms are indicated and find the 6th term:

(*a*) $\dfrac{-1}{5}, \dfrac{3}{8}, \dfrac{-5}{11}, \dfrac{7}{14}, \dfrac{-9}{17}, \ldots$ (*b*) $1, 0, 1, 0, 1, \ldots$ (*c*) $\tfrac{2}{3}, 0, \tfrac{3}{4}, 0, \tfrac{4}{5}, \ldots$

Ans. (*a*) $\dfrac{(-1)^n(2n-1)}{(3n+2)}$ (*b*) $\dfrac{1-(-1)^n}{2}$ (*c*) $\dfrac{(n+3)}{(n+5)}\cdot\dfrac{1-(-1)^n}{2}$

2.33. The *Fibonacci sequence* is the sequence $\{u_n\}$ where $u_{n+2} = u_{n+1} + u_n$ and $u_1 = 1$, $u_2 = 1$. (*a*) Find the first 6 terms of the sequence. (*b*) Show that the nth term is given by $u_n = (a^n - b^n)/\sqrt{5}$, where $a = \tfrac{1}{2}(1+\sqrt{5})$, $b = \tfrac{1}{2}(1-\sqrt{5})$.

Ans. (*a*) 1, 1, 2, 3, 5, 8

LIMITS OF SEQUENCES

2.34. Using the definition of limit, prove that:

(*a*) $\lim_{n\to\infty} \frac{4-2n}{3n+2} = \frac{-2}{3}$, (*b*) $\lim_{n\to\infty} 2^{-1/\sqrt{n}} = 1$, (*c*) $\lim_{n\to\infty} \frac{n^4+1}{n^2} = \infty$, (*d*) $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

2.35. Find the least positive integer N such that $|(3n+2)/(n-1) - 3| < \epsilon$ for all $n > N$ if (*a*) $\epsilon = .01$, (*b*) $\epsilon = .001$, (*c*) $\epsilon = .0001$.
Ans. (*a*) 502, (*b*) 5002, (*c*) 50,002

2.36. Using the definition of limit, prove that $\lim_{n\to\infty} (2n-1)/(3n+4)$ cannot be $\frac{1}{2}$.

2.37. Prove that $\lim_{n\to\infty} (-1)^n n$ does not exist.

2.38. Prove that if $\lim_{n\to\infty} |u_n| = 0$ then $\lim_{n\to\infty} u_n = 0$. Is the converse true?

2.39. If $\lim_{n\to\infty} u_n = l$, prove that (*a*) $\lim_{n\to\infty} cu_n = cl$ where c is any constant, (*b*) $\lim_{n\to\infty} u_n^2 = l^2$, (*c*) $\lim_{n\to\infty} u_n^p = l^p$ where p is a positive integer, (*d*) $\lim_{n\to\infty} \sqrt{u_n} = \sqrt{l}, l \geqq 0$.

2.40. Give a direct proof that $\lim_{n\to\infty} a_n/b_n = A/B$ if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B \neq 0$.

2.41. Prove that (*a*) $\lim_{n\to\infty} 3^{1/n} = 1$, (*b*) $\lim_{n\to\infty} \left(\frac{2}{3}\right)^{1/n} = 1$, (*c*) $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$.

2.42. If $r > 1$, prove that $\lim_{n\to\infty} r^n = \infty$, carefully explaining the significance of this statement.

2.43. If $|r| > 1$, prove that $\lim_{n\to\infty} r^n$ does not exist.

2.44. Evaluate each of the following, using theorems on limits:

(*a*) $\lim_{n\to\infty} \frac{4-2n-3n^2}{2n^2+n}$ (*c*) $\lim_{n\to\infty} \frac{\sqrt{3n^2-5n+4}}{2n-7}$ (*e*) $\lim_{n\to\infty} (\sqrt{n^2+n} - n)$

(*b*) $\lim_{n\to\infty} \sqrt[3]{\frac{(3-\sqrt{n})(\sqrt{n}+2)}{8n-4}}$ (*d*) $\lim_{n\to\infty} \frac{4\cdot 10^n - 3\cdot 10^{2n}}{3\cdot 10^{n-1} + 2\cdot 10^{2n-1}}$ (*f*) $\lim_{n\to\infty} (2^n + 3^n)^{1/n}$

Ans. (*a*) $-3/2$, (*b*) $-1/2$, (*c*) $\sqrt{3}/2$, (*d*) -15, (*e*) $1/2$, (*f*) 3

BOUNDED MONOTONIC SEQUENCES

2.45. Prove that the sequence with nth term $u_n = \sqrt{n}/(n+1)$ (*a*) is monotonic decreasing, (*b*) is bounded below, (*c*) is bounded above, (*d*) has a limit.

2.46. If $u_n = \frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \cdots + \frac{1}{n+n}$, prove that $\lim_{n\to\infty} u_n$ exists and lies between 0 and 1.

2.47. If $u_{n+1} = \sqrt{u_n + 1}$, $u_1 = 1$, prove that $\lim_{n\to\infty} u_n = \frac{1}{2}(1+\sqrt{5})$.

2.48. If $u_{n+1} = \frac{1}{2}(u_n + p/u_n)$ where $p > 0$ and $u_1 > 0$, prove that $\lim_{n\to\infty} u_n = \sqrt{p}$. Show how this can be used to determine $\sqrt{2}$.

2.49. If u_n is monotonic increasing (or monotonic decreasing), prove that S_n/n, where $S_n = u_1 + u_2 + \cdots + u_n$, is also monotonic increasing (or monotonic decreasing).

LEAST UPPER BOUND, GREATEST LOWER BOUND, LIMIT SUPERIOR, LIMIT INFERIOR

2.50. Find the l.u.b., g.l.b., lim sup ($\overline{\lim}$), lim inf ($\underline{\lim}$) for each sequence:

(*a*) $-1, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, \ldots, (-1)^n/(2n-1), \ldots$ (*c*) $1, -3, 5, -7, \ldots, (-1)^{n-1}(2n-1), \ldots$

(*b*) $\frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots, (-1)^{n+1}(n+1)/(n+2), \ldots$ (*d*) $1, 4, 1, 16, 1, 36, \ldots, n^{1+(-1)^n}, \ldots$

Ans. (*a*) $\frac{1}{3}, -1, 0, 0$ (*b*) $1, -1, 1, -1$ (*c*) none, none, $+\infty$, $-\infty$ (*d*) none, $1, +\infty, 1$

2.51. Prove that a bounded sequence $\{u_n\}$ is convergent if and only if $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

INFINITE SERIES

2.52. Find the sum of the series $\sum_{n=1}^{\infty} (\frac{2}{3})^n$. *Ans.* 2

2.53. Evaluate $\sum_{n=1}^{\infty} (-1)^{n-1}/5^n$. *Ans.* $\frac{1}{6}$

2.54. Prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. $\left[\text{Hint: } \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\right]$

2.55. Prove that multiplication of each term of an infinite series by a constant (not zero) does not affect the convergence or divergence.

2.56. Prove that the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ diverges. $\Big[$Hint: Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Then prove that $|S_{2n} - S_n| > \frac{1}{2}$, giving a contradiction with Cauchy's convergence criterion.$\Big]$

MISCELLANEOUS PROBLEMS

2.57. If $a_n \leqq u_n \leqq b_n$ for all $n > N$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, prove that $\lim_{n\to\infty} u_n = l$.

2.58. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, and θ is independent of n, prove that $\lim_{n\to\infty} (a_n \cos n\theta + b_n \sin n\theta) = 0$. Is the result true when θ depends on n?

2.59. Let $u_n = \frac{1}{2}\{1 + (-1)^n\}$, $n = 1, 2, 3, \ldots$. If $S_n = u_1 + u_2 + \cdots + u_n$, prove that $\lim_{n\to\infty} S_n/n = \frac{1}{2}$.

2.60. Prove that (*a*) $\lim_{n\to\infty} n^{1/n} = 1$, (*b*) $\lim_{n\to\infty} (a+n)^{p/n} = 1$ where a and p are constants.

2.61. If $\lim_{n\to\infty} |u_{n+1}/u_n| = |a| < 1$, prove that $\lim_{n\to\infty} u_n = 0$.

2.62. If $|a| < 1$, prove that $\lim_{n\to\infty} n^p a^n = 0$ where the constant $p > 0$.

2.63. Prove that $\lim \frac{2^n n!}{n^n} = 0$.

2.64. Prove that $\lim_{n\to\infty} n \sin 1/n = 1$. Hint: Let the central angle, θ, of a circle be measured in radians. Geometrically illustrate that $\sin\theta \leq \theta \leq \tan\theta$, $0 \leq \theta \leq \pi$.

Let $\theta = 1/n$. Observe that since n is restricted to positive integers, the angle is restricted to the first quadrant.

2.65. If $\{u_n\}$ is the Fibonacci sequence (Problem 2.33), prove that $\lim_{n\to\infty} u_{n+1}/u_n = \frac{1}{2}(1+\sqrt{5})$.

2.66. Prove that the sequence $u_n = (1 + 1/n)^{n+1}$, $n = 1, 2, 3, \ldots$ is a monotonic decreasing sequence whose limit is e. [Hint: Show that $u_n/u_{n-1} \leqq 1$.]

2.67. If $a_n \geqq b_n$ for all $n > N$ and $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, prove that $A \geqq B$.

2.68. If $|u_n| \leqq |v_n|$ and $\lim_{n\to\infty} v_n = 0$, prove that $\lim_{n\to\infty} u_n = 0$.

2.69. Prove that $\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = 0$.

2.70. Prove that $[a_n, b_n]$, where $a_n = (1 + 1/n)^n$ and $b_n = (1 + 1/n)^{n+1}$, is a set of nested intervals defining the number e.

2.71. Prove that every bounded monotonic (increasing or decreasing) sequence has a limit.

2.72. Let $\{u_n\}$ be a sequence such that $u_{n+2} = au_{n+1} + bu_n$ where a and b are constants. This is called a second order difference equation for u_n. (*a*) Assuming a solution of the form $u_n = r^n$ where r is a constant, prove that r must satisfy the equation $r^2 - ar - b = 0$. (*b*) Use (*a*) to show that a solution of the difference equation (called a general solution) is $u_n = Ar_1^n + Br_2^n$, where A and B are arbitrary constants and r_1 and r_2 are the two solutions of $r^2 - ar - b = 0$ assumed different. (*c*) In case $r_1 = r_2$ in (*b*), show that a (general) solution is $u_n = (A + Bn)r_1^n$.

2.73. Solve the following difference equations subject to the given conditions: (*a*) $u_{n+2} = u_{n+1} + u_n$, $u_1 = 1$, $u_2 = 1$ (compare Prob. 34); (*b*) $u_{n+2} = 2u_{n+1} + 3u_n$, $u_1 = 3$, $u_2 = 5$; (*c*) $u_{n+2} = 4u_{n+1} - 4u_n$, $u_1 = 2$, $u_2 = 8$.
Ans. (*a*) Same as in Prob. 34, (*b*) $u_n = 2(3)^{n-1} + (-1)^{n-1}$ (*c*) $u_n = n \cdot 2^n$

CHAPTER 3

Functions, Limits, and Continuity

FUNCTIONS

A function is composed of a domain set, a range set, and a rule of correspondence that assigns exactly one element of the range to each element of the domain.

This definition of a function places no restrictions on the nature of the elements of the two sets. However, in our early exploration of the calculus, these elements will be real numbers. The rule of correspondence can take various forms, but in advanced calculus it most often is an equation or a set of equations.

If the elements of the domain and range are represented by x and y, respectively, and f symbolizes the function, then the rule of correspondence takes the form $y = f(x)$.

The distinction between f and $f(x)$ should be kept in mind. f denotes the function as defined in the first paragraph. y and $f(x)$ are different symbols for the range (or image) values corresponding to domain values x. However a "common practice" that provides an expediency in presentation is to read $f(x)$ as, "the image of x with respect to the function f" and then use it when referring to the function. (For example, it is simpler to write $\sin x$ than "the sine function, the image value of which is $\sin x$.") This deviation from precise notation will appear in the text because of its value in exhibiting the ideas.

The domain variable x is called the independent variable. The variable y representing the corresponding set of values in the range, is the dependent variable.

Note: There is nothing exclusive about the use of x, y, and f to represent domain, range, and function. Many other letters will be employed.

There are many ways to relate the elements of two sets. [Not all of them correspond a unique range value to a given domain value.] For example, given the equation $y^2 = x$, there are two choices of y for each positive value of x. As another example, the pairs (a, b), (a, c), (a, d), and (a, e) can be formed and again the correspondence to a domain value is not unique. Because of such possibilities, some texts, especially older ones, distinguish between multiple-valued and single-valued functions. This viewpoint is not consistent with our definition or modern presentations. In order that there be no ambiguity, the calculus and its applications require a single image associated with each domain value. A multiple-valued rule of correspondence gives rise to a collection of functions (i.e., single-valued). Thus, the rule $y^2 = x$ is replaced by the pair of rules $y = x^{1/2}$ and $y = -x^{1/2}$ and the functions they generate through the establishment of domains. (See the following section on graphs for pictorial illustrations.)

EXAMPLES.

1. If to each number in $-1 \leqq x \leqq 1$ we associate a number y given by x^2, then the interval $-1 \leqq x \leqq 1$ is the domain. The rule $y = x^2$ generates the range $-1 \leqq y \leqq 1$. The totality is a function f.

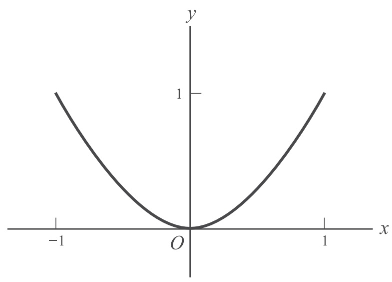
   The functional image of x is given by $y = f(x) = x^2$. For example, $f(-\frac{1}{3}) = (-\frac{1}{3})^2 = \frac{1}{9}$ is the image of $-\frac{1}{3}$ with respect to the function f.
2. The sequences of Chapter 2 may be interpreted as functions. For infinite sequences consider the domain as the set of positive integers. The rule is the definition of u_n, and the range is generated by this rule. To illustrate, let $u_n = \frac{1}{n}$ with $n = 1, 2, \ldots$. Then the range contains the elements $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. If the function is denoted by f, then we may write $f(n) = \frac{1}{n}$.

   As you read this chapter, reviewing Chapter 2 will be very useful, and in particular comparing the corresponding sections.
3. With each time t after the year 1800 we can associate a value P for the population of the United States. The correspondence between P and t defines a function, say F, and we can write $P = F(t)$.
4. For the present, both the domain and the range of a function have been restricted to sets of real numbers. Eventually this limitation will be removed. To get the flavor for greater generality, think of a map of the world on a globe with circles of latitude and longitude as coordinate curves. Assume there is a rule that corresponds this domain to a range that is a region of a plane endowed with a rectangular Cartesian coordinate system. (Thus, a flat map usable for navigation and other purposes is created.) The points of the domain are expressed as pairs of numbers (θ, ϕ) and those of the range by pairs (x, y). These sets and a rule of correspondence constitute a function whose independent and dependent variables are not single real numbers; rather, they are pairs of real numbers.

GRAPH OF A FUNCTION

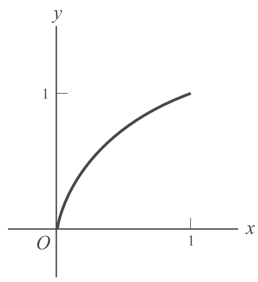
A function f establishes a set of ordered pairs (x, y) of real numbers. The plot of these pairs $(x, f(x))$ in a coordinate system is the graph of f. The result can be thought of as a pictorial representation of the function.

For example, the graphs of the functions described by $y = x^2$, $-1 \leqq x \leqq 1$, and $y^2 = x$, $0 \leqq x \leqq 1$, $y \geqq 0$ appear in Fig. 3-1.



$y = x^2$

(*a*)



$y^2 = x, y \geq 0$

(*b*)

Fig. 3-1

BOUNDED FUNCTIONS

If there is a constant M such that $f(x) \leqq M$ for all x in an interval (or other set of numbers), we say that f is *bounded above* in the interval (or the set) and call M an *upper bound* of the function.

If a constant m exists such that $f(x) \geqq m$ for all x in an interval, we say that $f(x)$ is *bounded below* in the interval and call m a *lower bound*.

If $m \leqq f(x) \leqq M$ in an interval, we call $f(x)$ *bounded*. Frequencly, when we wish to indicate that a function is bounded, we shall write $|f(x)| < P$.

EXAMPLES. 1. $f(x) = 3 + x$ is bounded in $-1 \leqq x \leqq 1$. An upper bound is 4 (or any number greater than 4). A lower bound is 2 (or any number less than 2).

2. $f(x) = 1/x$ is not bounded in $0 < x < 4$ since by choosing x sufficiently close to zero, $f(x)$ can be made as large as we wish, so that there is no upper bound. However, a lower bound is given by $\frac{1}{4}$ (or any number less than $\frac{1}{4}$).

If $f(x)$ has an upper bound it has a *least upper bound* (l.u.b.); if it has a lower bound it has a *greatest lower bound* (g.l.b.). (See Chapter 1 for these definitions.)

MONOTONIC FUNCTIONS

A function is called *monotonic increasing* in an interval if for any two points x_1 and x_2 in the interval such that $x_1 < x_2$, $f(x_1) \leqq f(x_2)$. If $f(x_1) < f(x_2)$ the function is called *strictly increasing*.

Similarly if $f(x_1) \geqq f(x_2)$ whenever $x_1 < x_2$, then $f(x)$ is *monotonic decreasing*; while if $f(x_1) > f(x_2)$, it is *strictly decreasing*.

INVERSE FUNCTIONS. PRINCIPAL VALUES

Suppose y is the range variable of a function f with domain variable x. Furthermore, let the correspondence between the domain and range values be one-to-one. Then a new function f^{-1}, called the *inverse function* of f, can be created by interchanging the domain and range of f. This information is contained in the form $x = f^{-1}(y)$.

As you work with the inverse function, it often is convenient to rename the domain variable as x and use y to symbolize the images, then the notation is $y = f^{-1}(x)$. In particular, this allows graphical expression of the inverse function with its domain on the horizontal axis.

Note: f^{-1} does *not* mean f to the negative one power. When used with functions the notation f^{-1} always designates the inverse function to f.

If the domain and range elements of f are not in one-to-one correspondence (this would mean that distinct domain elements have the same image), then a collection of one-to-one functions may be created. Each of them is called a *branch*. It is often convenient to choose one of these branches, called the *principal branch*, and denote it as the inverse function, f^{-1}. The range values of f that compose the principal branch, and hence the domain of f^{-1}, are called the *principal values*. (As will be seen in the section of elementary functions, it is common practice to specify these principal values for that class of functions.)

EXAMPLE. Suppose f is generated by $y = \sin x$ and the domain is $-\infty \leqq x \leqq \infty$. Then there are an infinite number of domain values that have the same image. (A finite portion of the graph is illustrated below in Fig. 3-2(*a*.) In Fig. 3-2(*b*) the graph is rotated about a line at 45° so that the x-axis rotates into the y-axis. Then the variables are interchanged so that the x-axis is once again the horizontal one. We see that the image of an x value is not unique. Therefore, a set of principal values must be chosen to establish an inverse function. A choice of a branch is accomplished by restricting the domain of the starting function, $\sin x$. For example, choose $-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$. Then there is a one-to-one correspondence between the elements of this domain and the images in $-1 \leqq x \leqq 1$. Thus, f^{-1} may be defined with this interval as its domain. This idea is illustrated in Fig. 3-2(*c*) and Fig. 3-2(*d*). With the domain of f^{-1} represented on the horizontal axis and by the variable x, we write $y = \sin^{-1} x$, $-1 \leqq x \leqq 1$.

If $x = -\frac{1}{2}$, then the corresponding range value is $y = -\frac{\pi}{6}$.

Note: In algebra, b^{-1} means $\frac{1}{b}$ and the fact that bb^{-1} produces the identity element 1 is simply a rule of algebra generalized from arithmetic. Use of a similar exponential notation for inverse functions is justified in that corresponding algebraic characteristics are displayed by $f^{-1}[f(x)] = x$ and $f[f^{-1}(x)] = x$.

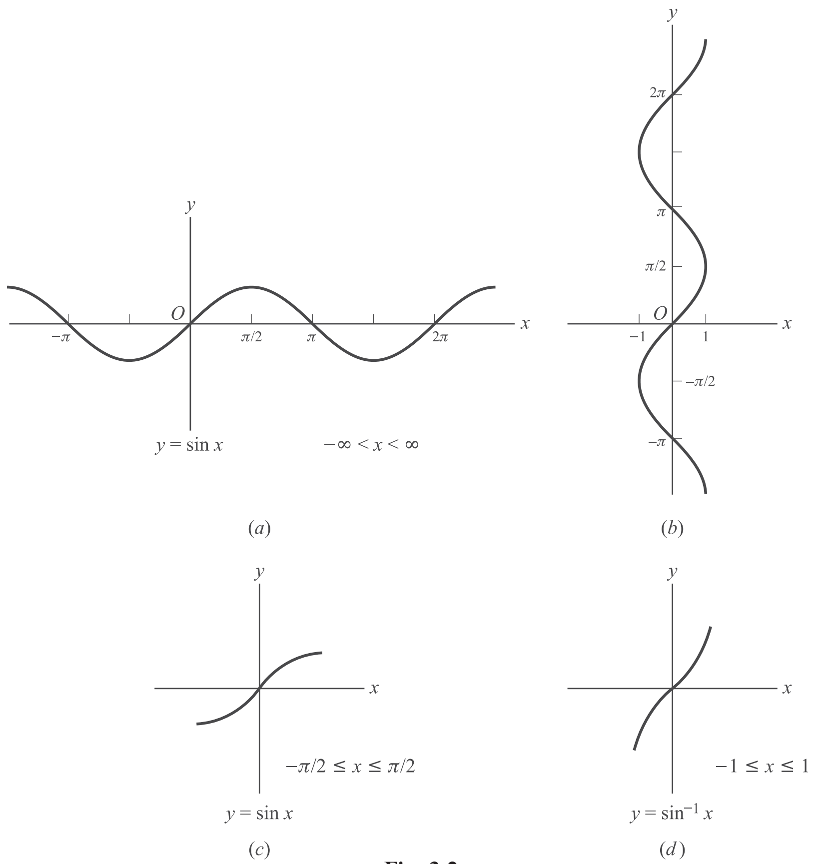


Fig. 3-2

MAXIMA AND MINIMA

The seventeenth-century development of the calculus was strongly motivated by questions concerning extreme values of functions. Of most importance to the calculus and its applications were the notions of *local extrema*, called *relative maximums* and *relative minimums*.

If the graph of a function were compared to a path over hills and through valleys, the local extrema would be the high and low points along the way. This intuitive view is given mathematical precision by the following definition.

Definition: If there exists an open interval (a, b) containing c such that $f(x) < f(c)$ for all x other than c in the interval, then $f(c)$ is a *relative maximum* of f. If $f(x) > f(c)$ for all x in (a, b) other than c, then $f(c)$ is a *relative minimum* of f. (See Fig. 3-3.)

Functions may have any number of relative extrema. On the other hand, they may have none, as in the case of the strictly increasing and decreasing functions previously defined.

Definition: If c is in the domain of f and for all x in the domain of the function $f(x) \leqq f(c)$, then $f(c)$ is an *absolute maximum* of the function f. If for all x in the domain $f(x) \geqq f(c)$ then $f(c)$ is an *absolute minimum* of f. (See Fig. 3-3.)

Note: If defined on closed intervals the strictly increasing and decreasing functions possess *absolute extrema*.

Absolute extrema are not necessarily unique. For example, if the graph of a function is a horizontal line, then every point is an *absolute maximum* and an *absolute minimum*.

Note: A *point of inflection* also is represented in Fig. 3-3. There is an overlap with relative extrema in representation of such points through derivatives that will be addressed in the problem set of Chapter 4.

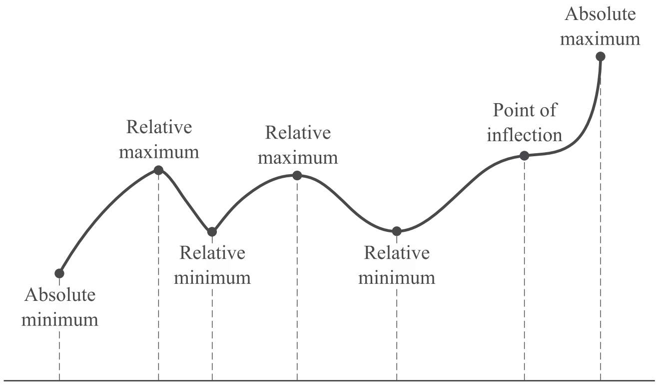


Fig. 3-3

TYPES OF FUNCTIONS

It is worth realizing that there is a fundamental pool of functions at the foundation of calculus and advanced calculus. These are called *elementary functions*. Either they are generated from a real variable x by the fundamental operations of algebra, including powers and roots, or they have relatively simple geometric interpretations. As the title "elementary functions" suggests, there is a more general category of functions (which, in fact, are dependent on the elementary ones). Some of these will be explored later in the book. The *elementary functions* are described below.

1. **Polynomial functions** have the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \qquad (1)$$

where $a_0, \ldots, a_n$ are constants and n is a positive integer called the *degree* of the polynomial if $a_0 \neq 0$.

The *fundamental theorem of algebra* states that in the field of complex numbers every polynomial equation has at least one root. As a consequence of this theorem, it can be proved that every nth degree polynomial has n roots in the complex field. When complex numbers are admitted, the polynomial theoretically may be expressed as the product of n linear factors; with our restriction to real numbers, it is possible that $2k$ of the roots may be complex. In this case, the k factors generating them will be quadratic. (The corresponding roots are in complex conjugate pairs.) The polynomial $x^3 - 5x^2 + 11x - 15 = (x-3)(x^2 - 2x + 5)$ illustrates this thought.

2. **Algebraic functions** are functions $y = f(x)$ satisfying an equation of the form

$$p_0(x)y^n + p_1(x)y^{n-1} + \cdots + p_{n-1}(x)y + p_n(x) = 0 \qquad (2)$$

where $p_0(x), \ldots, p_n(x)$ are polynomials in x.

If the function can be expressed as the quotient of two polynomials, i.e., $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, it is called a *rational algebraic function*; otherwise it is an *irrational algebraic function*.

3. **Transcendental functions** are functions which are not algebraic, i.e., they do not satisfy equations of the form (*2*).

Note the analogy with real numbers, polynomials corresponding to integers, rational functions to rational numbers, and so on.

TRANSCENDENTAL FUNCTIONS

The following are sometimes called *elementary transcendental functions*.

1. **Exponential function:** $f(x) = a^x$, $a \neq 0, 1$. For properties, see Page 3.
2. **Logarithmic function:** $f(x) = \log_a x$, $a \neq 0, 1$. This and the exponential function are inverse functions. If $a = e = 2.71828\ldots$, called the *natural base of logarithms*, we write $f(x) = \log_e x = \ln x$, called the *natural logarithm* of x. For properties, see Page 4.
3. **Trigonometric functions** (Also called circular functions because of their geometric interpretation with respect to the unit circle):

$$\sin x, \cos x, \tan x = \frac{\sin x}{\cos x}, \csc x = \frac{1}{\sin x}, \sec x = \frac{1}{\cos x}, \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

The variable x is generally expressed in radians (π radians $= 180°$). For real values of x, $\sin x$ and $\cos x$ lie between -1 and 1 inclusive.

The following are some properties of these functions:

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \sin(-x) = -\sin x$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \qquad \cos(-x) = \cos x$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \qquad \tan(-x) = -\tan x$$

4. **Inverse trigonometric functions.** The following is a list of the inverse trigonometric functions and their principal values:

(*a*) $y = \sin^{-1} x, \; (-\pi/2 \leqq y \leqq \pi/2)$

(*b*) $y = \cos^{-1} x, \; (0 \leqq y \leqq \pi)$

(*c*) $y = \tan^{-1} x, \; (-\pi/2 < y < \pi/2)$

(*d*) $y = \csc^{-1} x = \sin^{-1} 1/x, \; (-\pi/2 \leqq y \leqq \pi/2)$

(*e*) $y = \sec^{-1} x = \cos^{-1} 1/x, \; (0 \leqq y \leqq \pi)$

(*f*) $y = \cot^{-1} x = \pi/2 - \tan^{-1} x, \; (0 < y < \pi)$

5. **Hyperbolic functions** are defined in terms of exponential functions as follows. These functions may be interpreted geometrically, much as the trigonometric functions but with respect to the unit hyperbola.

(*a*) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(*b*) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(*c*) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(*d*) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

(*e*) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(*f*) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

The following are some properties of these functions:

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \coth^2 x - 1 = \operatorname{csch}^2 x$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \qquad \sinh(-x) = -\sinh x$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \qquad \cosh(-x) = \cosh x$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \qquad \tanh(-x) = -\tanh x$$

6. **Inverse hyperbolic functions.** If $x = \sinh y$ then $y = \sinh^{-1} x$ is the *inverse hyperbolic sine* of x. The following list gives the principal values of the inverse hyperbolic functions in terms of natural logarithms and the domains for which they are real.

(*a*) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}),\ \text{all } x$

(*b*) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}),\ x \geqq 1$

(*c*) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right),\ |x| < 1$

(*d*) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2+1}}{|x|}\right),\ x \neq 0$

(*e*) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right),\ 0 < x \leqq 1$

(*f*) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right),\ |x| > 1$

LIMITS OF FUNCTIONS

Let $f(x)$ be defined and single-valued for all values of x near $x = x_0$ with the possible exception of $x = x_0$ itslef (i.e., in a deleted δ neighborhood of x_0). We say that the number l is the *limit of* $f(x)$ *as* x *approaches* x_0 and write $\lim_{x \to x_0} f(x) = l$ if for any positive number ϵ (however small) we can find some positive number δ (usually depending on ϵ) such that $|f(x) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$. In such case we also say that $f(x)$ approaches l as x approaches x_0 and write $f(x) \to l$ as $x \to x_0$.

In words, this means that we can make $f(x)$ arbitrarily close to l by choosing x sufficiently close to x_0.

EXAMPLE. Let $f(x) = \begin{vmatrix} x^2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{vmatrix}$. Then as x gets closer to 2 (i.e., x approaches 2), $f(x)$ gets closer to 4. We thus *suspect* that $\lim_{x \to 2} f(x) = 4$. To *prove* this we must see whether the above definition of limit (with $l = 4$) is satisfied. For this proof see Problem 3.10.

Note that $\lim_{x \to 2} f(x) \neq f(2)$, i.e., the limit of $f(x)$ as $x \to 2$ is not the same as the value of $f(x)$ *at* $x = 2$ since $f(2) = 0$ by definition. The limit would in fact be 4 even if $f(x)$ were not defined at $x = 2$.

When the limit of a function exists it is unique, i.e., it is the only one (see Problem 3.17).

RIGHT- AND LEFT-HAND LIMITS

In the definition of limit no restriction was made as to how x should approach x_0. It is sometimes found convenient to restrict this approach. Considering x and x_0 as points on the real axis where x_0 is fixed and x is moving, then x can approach x_0 from the right or from the left. We indicate these respective approaches by writing $x \to x_0+$ and $x \to x_0-$.

If $\lim_{x \to x_0+} f(x) = l_1$ and $\lim_{x \to x_0-} f(x) = l_2$, we call l_1 and l_2, respectively, the *right- and left-hand limits* of f at x_0 and denote them by $f(x_0+)$ or $f(x_0 + 0)$ and $f(x_0-)$ or $f(x_0 - 0)$. The ϵ, δ definitions of limit of $f(x)$ as $x \to x_0+$ or $x \to x_0-$ are the same as those for $x \to x_0$ except for the fact that values of x are restricted to $x > x_0$ or $x < x_0$, respectively.

We have $\lim_{x \to x_0} f(x) = l$ if and only if $\lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x) = l$.

THEOREMS ON LIMITS

If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, then

1. $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = A + B$

2. $\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x) = A - B$
3. $\lim_{x \to x_0} (f(x)g(x)) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right) = AB$
4. $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \dfrac{A}{B} \qquad \text{if } B \neq 0$

Similar results hold for right- and left-hand limits.

INFINITY

It sometimes happens that as $x \to x_0$, $f(x)$ increases or decreases without bound. In such case it is customary to write $\lim_{x \to x_0} f(x) = +\infty$ or $\lim_{x \to x_0} f(x) = -\infty$, respectively. The symbols $+\infty$ (also written ∞) and $-\infty$ are read *plus infinity* (or *infinity*) and *minus infinity*, respectively, but it must be emphasized that they are not numbers.

In precise language, we say that $\lim_{x \to x_0} f(x) = \infty$ if for each positive number M we can find a positive number δ (depending on M in general) such that $f(x) > M$ whenever $0 < |x - x_0| < \delta$. Similarly, we say that $\lim_{x \to x_0} f(x) = -\infty$ if for each positive number M we can find a positive number δ such that $f(x) < -M$ whenever $0 < |x - x_0| < \delta$. Analogous remarks apply in case $x \to x_0+$ or $x \to x_0-$.

Frequently we wish to examine the behavior of a function as x increases or decreases without bound. In such cases it is customary to write $x \to +\infty$ (or ∞) or $x \to -\infty$, respectively.

We say that $\lim_{x \to +\infty} f(x) = l$, or $f(x) \to l$ as $x \to +\infty$, if for any positive number ϵ we can find a positive number N (depending on ϵ in general) such that $|f(x) - l| < \epsilon$ whenever $x > N$. A similar definition can be formulated for $\lim_{x \to -\infty} f(x)$.

SPECIAL LIMITS

1. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \dfrac{1 - \cos x}{x} = 0$
2. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e, \qquad \lim_{x \to 0+} (1 + x)^{1/x} = e$
3. $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1, \qquad \lim_{x \to 1} \dfrac{x - 1}{\ln x} = 1$

CONTINUITY

Let f be defined for all values of x near $x = x_0$ as well as at $x = x_0$ (i.e., in a δ neighborhood of x_0). The function f is called *continuous* at $x = x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$. Note that this implies three conditions which must be met in order that $f(x)$ be continuous at $x = x_0$.

1. $\lim_{x \to x_0} f(x) = l$ must exist.
2. $f(x_0)$ must exist, i.e., $f(x)$ is defined *at* x_0.
3. $l = f(x_0)$.

In summary, $\lim_{x \to x_0} f(x)$ is the value suggested for f at $x = x_0$ by the behavior of f in arbitrarily small neighborhoods of x_0. If in fact this limit is the actual value, $f(x_0)$, of the function at x_0, then f is continuous there.

Equivalently, if f is continuous at x_0, we can write this in the suggestive form $\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x)$.

EXAMPLES. 1. If $f(x) = \begin{cases} x^2, & x \neq 2 \\ 0, & x = 2 \end{cases}$ then from the example on Page 45 $\lim_{x\to 2} f(x) = 4$. But $f(2) = 0$. Hence $\lim_{x\to 2} f(x) \neq f(2)$ and the function is not continuous at $x = 2$.

2. If $f(x) = x^2$ for all x, then $\lim_{x\to 2} f(x) = f(2) = 4$ and $f(x)$ is continuous at $x = 2$.

Points where f fails to be continuous are called *discontinuities* of f and f is said to be *discontinuous* at these points.

In constructing a graph of a continuous function the pencil need never leave the paper, while for a discontinuous function this is not true since there is generally a jump taking place. This is of course merely a characteristic property and not a definition of continuity or discontinuity.

Alternative to the above definition of continuity, we can define f as continuous at $x = x_0$ if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Note that this is simply the definition of limit with $l = f(x_0)$ and removal of the restriction that $x \neq x_0$.

RIGHT- AND LEFT-HAND CONTINUITY

If f is defined only for $x \geqq x_0$, the above definition does not apply. In such case we call f *continuous* (*on the right*) at $x = x_0$ if $\lim_{x\to x_0+} f(x) = f(x_0)$, i.e., if $f(x_0+) = f(x_0)$. Similarly, f is *continuous* (*on the left*) at $x = x_0$ if $\lim_{x\to x_0-} f(x) = f(x_0)$, i.e., $f(x_0-) = f(x_0)$. Definitions in terms of ϵ and δ can be given.

CONTINUITY IN AN INTERVAL

A function f is said to be *continuous in an interval* if it is continuous at all points of the interval. In particular, if f is defined in the closed interval $a \leqq x \leqq b$ or $[a, b]$, then f is continuous in the interval if and only if $\lim_{x\to x_0} f(x) = f(x_0)$ for $a < x_0 < b$, $\lim_{x\to a+} f(x) = f(a)$ and $\lim_{x\to b-} f(x) = f(b)$.

THEOREMS ON CONTINUITY

Theorem 1. If f and g are continuous at $x = x_0$, so also are the functions whose image values satisfy the relations $f(x) + g(x)$, $f(x) - g(x)$, $f(x)g(x)$ and $\dfrac{f(x)}{g(x)}$, the last only if $g(x_0) \neq 0$. Similar results hold for continuity in an interval.

Theorem 2. Functions described as follows are continuous in every finite interval: (*a*) all polynomials; (*b*) $\sin x$ and $\cos x$; (*c*) a^x, $a > 0$

Theorem 3. Let the function f be continuous at the domain value $x = x_0$. Also suppose that a function g, represented by $z = g(y)$, is continuous at y_0, where $y = f(x)$ (i.e., the range value of f corresponding to x_0 is a domain value of g). Then a new function, called a *composite function*, $f(g)$, represented by $z = g[f(x)]$, may be created which is continuous at its domain point $x = x_0$. [One says that *a continuous function of a continuous function is continuous*.]

Theorem 4. If $f(x)$ is continuous in a closed interval, it is bounded in the interval.

Theorem 5. If $f(x)$ is continuous at $x = x_0$ and $f(x_0) > 0$ [or $f(x_0) < 0$], there exists an interval about $x = x_0$ in which $f(x) > 0$ [or $f(x) < 0$].

Theorem 6. If a function $f(x)$ is continuous in an interval and either strictly increasing or strictly decreasing, the inverse function $f^{-1}(x)$ is single-valued, continuous, and either strictly increasing or strictly decreasing.

Theorem 7. If $f(x)$ is continuous in $[a, b]$ and if $f(a) = A$ and $f(b) = B$, then corresponding to any number C between A and B there exists at least one number c in $[a, b]$ such that $f(c) = C$. This is sometimes called the *intermediate value theorem.*

Theorem 8. If $f(x)$ is continuous in $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, there is at least one number c for which $f(c) = 0$ where $a < c < b$. This is related to Theorem 7.

Theorem 9. If $f(x)$ is continuous in a closed interval, then $f(x)$ has a maximum value M for at least one value of x in the interval and a minimum value m for at least one value of x in the interval. Furthermore, $f(x)$ assumes all values between m and M for one or more values of x in the interval.

Theorem 10. If $f(x)$ is continuous in a closed interval and if M and m are respectively the least upper bound (l.u.b.) and greatest lower bound (g.l.b.) of $f(x)$, there exists at least one value of x in the interval for which $f(x) = M$ or $f(x) = m$. This is related to Theorem 9.

PIECEWISE CONTINUITY

A function is called *piecewise continuous* in an interval $a \leqq x \leqq b$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right- and left-hand limits. Such a function has only a finite number of discontinuities. An example of a function which is piecewise continuous in $a \leqq x \leqq b$ is shown graphically in Fig. 3-4 below. This function has discontinuities at x_1, x_2, x_3, and x_4.

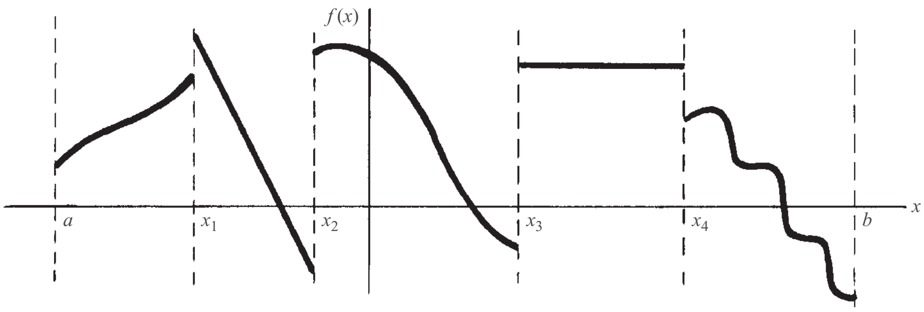


Fig. 3-4

UNIFORM CONTINUITY

Let f be continuous in an interval. Then by definition at each point x_0 of the interval and for any $\epsilon > 0$, we can find $\delta > 0$ (which will in general depend on both ϵ and the particular point x_0) such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. If we can find δ for each ϵ which holds for all points of the interval (i.e., if δ depends *only* on ϵ and *not* on x_0), we say that f is *uniformly continuous* in the interval.

Alternatively, f is uniformly continuous in an interval if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ where x_1 and x_2 are any two points in the interval.

Theorem. If f is continuous in a *closed* interval, it is uniformly continuous in the interval.

Solved Problems

FUNCTIONS

3.1. Let $f(x) = (x-2)(8-x)$ for $2 \leqq x \leqq 8$. (*a*) Find $f(6)$ and $f(-1)$. (*b*) What is the domain of definition of $f(x)$? (*c*) Find $f(1-2t)$ and give the domain of definition. (*d*) Find $f[f(3)]$, $f[f(5)]$. (*e*) Graph $f(x)$.

(*a*) $f(6) = (6-2)(8-6) = 4 \cdot 2 = 8$
$f(-1)$ is not defined since $f(x)$ is defined only for $2 \leqq x \leqq 8$.

(*b*) The set of all x such that $2 \leqq x \leqq 8$.

(*c*) $f(1-2t) = \{(1-2t)-2\}\{8-(1-2t)\} = -(1+2t)(7+2t)$ where t is such that $2 \leqq 1-2t \leqq 8$, i.e., $-7/2 \leqq t \leqq -1/2$.

(*d*) $f(3) = (3-2)(8-3) = 5$,
$f[f(3)] = f(5) = (5-2)(8-5) = 9$.
$f(5) = 9$ so that $f[f(5)] = f(9)$ is not defined.

(*e*) The following table shows $f(x)$ for various values of x.

x	2	3	4	5	6	7	8	2.5	7.5
$f(x)$	0	5	8	9	8	5	0	2.75	2.75

Plot points (2, 0), (3, 5), (4, 8), (5, 9), (6, 8), (7, 5), (8, 0), (2.5, 2.75), (7.5, 2.75).

These points are only a few of the infinitely many points on the required graph shown in the adjoining Fig. 3-5. This set of points defines a curve which is part of a *parabola*.

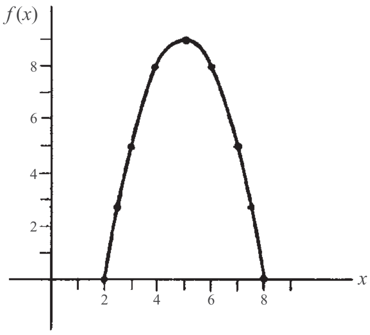


Fig. 3-5

3.2. Let $g(x) = (x-2)(8-x)$ for $2 < x < 8$. (*a*) Discuss the difference between the graph of $g(x)$ and that of $f(x)$ in Problem 3.1. (*b*) What is the l.u.b. and g.l.b. of $g(x)$? (*c*) Does $g(x)$ attain its l.u.b. and g.l.b. for any value of x in the domain of definition? (*d*) Answer parts (*b*) and (*c*) for the function $f(x)$ of Problem 3.1.

(*a*) The graph of $g(x)$ is the same as that in Problem 3.1 except that the two points $(2, 0)$ and $(8, 0)$ are missing, since $g(x)$ is not defined at $x = 2$ and $x = 8$.

(*b*) The l.u.b. of $g(x)$ is 9. The g.l.b. of $g(x)$ is 0.

(*c*) The l.u.b. of $g(x)$ is attained for the value of $x = 5$. The g.l.b. of $g(x)$ is not attained, since there is no value of x in the domain of definition such that $g(x) = 0$.

(*d*) As in (*b*), the l.u.b. of $f(x)$ is 9 and the g.l.b. of $f(x)$ is 0. The l.u.b. of $f(x)$ is attained for the value $x = 5$ and the g.l.b. of $f(x)$ is attained at $x = 2$ and $x = 8$.

Note that a function, such as $f(x)$, which is *continuous* in a closed interval attains its l.u.b. and g.l.b. at some point of the interval. However, a function, such as $g(x)$, which is not continuous in a closed interval need not attain its l.u.b. and g.l.b. See Problem 3.34.

3.3. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$. (*a*) Find $f(\frac{2}{3})$, $f(-5)$, $f(1.41423)$, $f(\sqrt{2})$, (*b*) Construct a graph of $f(x)$ and explain why it is misleading by itself.

(*a*)
$f(\frac{2}{3}) = 1$ since $\frac{2}{3}$ is a rational number
$f(-5) = 1$ since -5 is a rational number
$f(1.41423) = 1$ since 1.41423 is a rational number
$f(\sqrt{2}) = 0$ since $\sqrt{2}$ is an irrational number

(*b*) The graph is shown in the adjoining Fig. 3-6. Because both the sets of rational numbers and irrational numbers are dense, the visual impression is that there are two images corresponding to each domain value. In actuality, each domain value has only one corresponding range value.

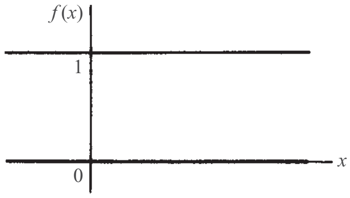


Fig. 3-6

3.4. Referring to Problem 3.1: (*a*) Draw the graph with axes interchanged, thus illustrating the two possible choices available for definition of f^{-1}. (*b*) Solve for x in terms of y to determine the equations describing the two branches, and then interchange the variables.

(*a*) The graph of $y = f(x)$ is shown in Fig. 3-5 of Problem 3.1(a). By interchanging the axes (and the variables), we obtain the graphical form of Fig. 3-7. This figure illustrates that there are two values of y corresponding to each value of x, and hence two branches. Either may be employed to define f^{-1}.

(*b*) We have $y = (x-2)(8-x)$ or $x^2 - 10x + 16 + y = 0$. The solution of this quadratic equation is

$$x = 5 \pm \sqrt{9-y}.$$

After interchanging variables

$$y = 5 \pm \sqrt{9-x}.$$

In the graph, AP represents $y = 5 + \sqrt{9-x}$, and BP designates $y = 5 - \sqrt{9-x}$. Either branch may represent f^{-1}.

Note: The point at which the two branches meet is called a *branch point*.

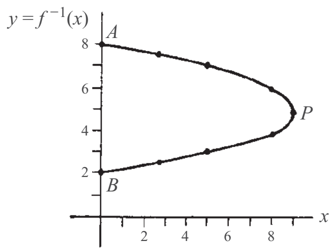


Fig. 3-7

3.5. (*a*) Prove that $g(x) = 5 + \sqrt{9-x}$ is strictly decreasing in $0 \leqq x \leqq 9$. (*b*) Is it monotonic decreasing in this interval? (*c*) Does $g(x)$ have a single-valued inverse?

(*a*) $g(x)$ is strictly decreasing if $g(x_1) > g(x_2)$ whenever $x_1 < x_2$. If $x_1 < x_2$ then $9 - x_1 > 9 - x_2$, $\sqrt{9-x_1} > \sqrt{9-x_2}$, $5 + \sqrt{9-x_1} > 5 + \sqrt{9-x_2}$ showing that $g(x)$ is strictly decreasing.

(*b*) Yes, any strictly decreasing function is also monotonic decreasing, since if $g(x_1) > g(x_2)$ it is also true that $g(x_1) \geqq g(x_2)$. However, if $g(x)$ is monotonic decreasing, it is not necessarily strictly decreasing.

(*c*) If $y = 5 + \sqrt{9-x}$, then $y - 5 = \sqrt{9-x}$ or squaring, $x = -16 + 10y - y^2 = (y-2)(8-y)$ and x is a single-valued function of y, i.e., the inverse function is single-valued.

In general, any strictly decreasing (or increasing) function has a single-valued inverse (see Theorem 6, Page 47).

The results of this problem can be interpreted graphically using the figure of Problem 3.4.

3.6. Construct graphs for the functions (*a*) $f(x) = \begin{cases} x \sin 1/x, & x > 0 \\ 0, & x = 0 \end{cases}$, (*b*) $f(x) = [x] =$ greatest integer $\leqq x$.

(*a*) The required graph is shown in Fig. 3-8. Since $|x \sin 1/x| \leqq |x|$, the graph is included between $y = x$ and $y = -x$. Note that $f(x) = 0$ when $\sin 1/x = 0$ or $1/x =, m\pi$, $m = 1, 2, 3, 4, \ldots$, i.e., where $x = 1/\pi, 1/2\pi, 1/3\pi, \ldots$. The curve oscillates infinitely often between $x = 1/\pi$ and $x = 0$.

(*b*) The required graph is shown in Fig. 3-9. If $1 \leqq x < 2$, then $[x] = 1$. Thus $[1.8] = 1$, $[\sqrt{2}] = 1$, $[1.99999] = 1$. However, $[2] = 2$. Similarly for $2 \leqq x < 3$, $[x] = 2$, etc. Thus there are *jumps* at the integers. The function is sometimes called the *staircase function* or *step function*.

3.7. (*a*) Construct the graph of $f(x) = \tan x$. (*b*) Construct the graph of some of the infinite number of branches available for a definition of $\tan^{-1} x$. (*c*) Show graphically why the relationship of x

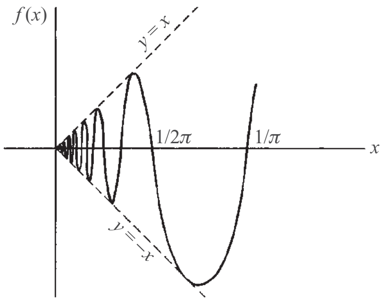


Fig. 3-8

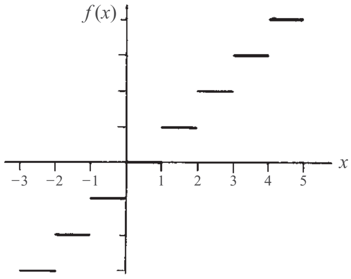


Fig. 3-9

to y is multivalued. (*d*) Indicate possible principal values for $\tan^{-1} x$. (*e*) Using your choice, evaluate $\tan^{-1}(-1)$.

(*a*) The graph of $f(x) = \tan x$ appears in Fig. 3-10 below.

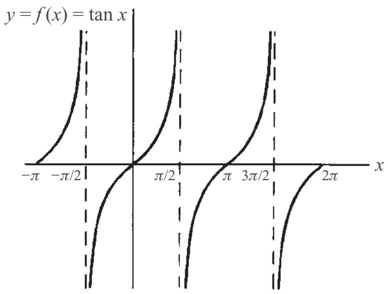


Fig. 3-10

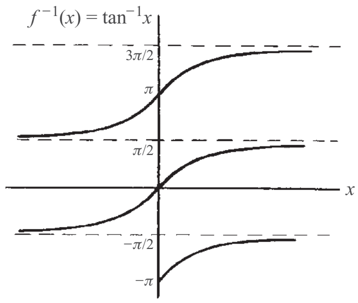


Fig. 3-11

(*b*) The required graph is obtained by interchanging the x and y axes in the graph of (*a*). The result, with axes oriented as usual, appears in Fig. 3-11 above.

(*c*) In Fig. 3-11 of (*b*), any vertical line meets the graph in infinitely many points. Thus, the relation of y to x is multivalued and infinitely many branches are available for the purpose of defining $\tan^{-1} x$.

(*d*) To define $\tan^{-1} x$ as a single-valued function, it is clear from the graph that we can only do so by restricting its value to any of the following: $-\pi/2 < \tan^{-1} x < \pi/2, \pi/2 < \tan^{-1} x < 3\pi/2$, etc. We shall agree to take the first as defining the principal value.

Note that no matter which branch is used to define $\tan^{-1} x$, the resulting function is strictly increasing.

(*e*) $\tan^{-1}(-1) = -\pi/4$ is the only value lying between $-\pi/2$ and $\pi/2$, i.e., it is the principal value according to our choice in (*d*).

3.8. Show that $f(x) = \dfrac{\sqrt{x}+1}{x+1}$, $x \neq -1$, describes an irrational algebraic function.

If $y = \dfrac{\sqrt{x}+1}{x+1}$ then $(x+1)y - 1 = \sqrt{x}$ or squaring, $(x+1)^2y^2 - 2(x+1)y + 1 - x = 0$, a polynomial equation in y whose coefficients are polynomials in x. Thus $f(x)$ is an algebraic function. However, it is not the quotient of two polynomials, so that it is an irrational algebraic function.

3.9. If $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$, prove that we can choose as the principal value of the inverse function, $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geqq 1$.

If $y = \frac{1}{2}(e^x + e^{-x})$, $e^{2x} - 2ye^x + 1 = 0$. Then using the quadratic formula, $e^x = \dfrac{2y \pm \sqrt{4y^2 - 4}}{2} = y \pm \sqrt{y^2 - 1}$. Thus $x = \ln(y \pm \sqrt{y^2 - 1})$.

Since $y - \sqrt{y^2 - 1} = (y - \sqrt{y^2 - 1})\left(\dfrac{y + \sqrt{y^2 - 1}}{y + \sqrt{y^2 - 1}}\right) = \dfrac{1}{y + \sqrt{y^2 - 1}}$, we can also write

$$x = \pm \ln(y + \sqrt{y^2 - 1}) \quad \text{or} \quad \cosh^{-1} y = \pm \ln(y + \sqrt{y^2 - 1})$$

Choosing the $+$ sign as defining the principal value and replacing y by x, we have $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$. The choice $x \geqq 1$ is made so that the inverse function is real.

LIMITS

3.10. If (*a*) $f(x) = x^2$, (*b*) $f(x) = \begin{cases} x^2, & x \neq 2 \\ 0, & x = 2 \end{cases}$, prove that $\lim_{x \to 2} f(x) = 4$.

(*a*) We must show that given any $\epsilon > 0$ we can find $\delta > 0$ (depending on ϵ in general) such that $|x^2 - 4| < \epsilon$ when $0 < |x - 2| < \delta$.

Choose $\delta \leqq 1$ so that $0 < |x - 2| < 1$ or $1 < x < 3$, $x \neq 2$. Then $|x^2 - 4| = |(x-2)(x+2)| = |x - 2||x + 2| < \delta|x + 2| < 5\delta$.

Take δ as 1 or $\epsilon/5$, whichever is smaller. Then we have $|x^2 - 4| < \epsilon$ whenever $0 < |x - 2| < \delta$ and the required result is proved.

It is of interest to consider some numerical values. If for example we wish to make $|x^2 - 4| < .05$, we can choose $\delta = \epsilon/5 = .05/5 = .01$. To see that this is actually the case, note that if $0 < |x - 2| < .01$ then $1.99 < x < 2.01$ $(x \neq 2)$ and so $3.9601 < x^2 < 4.0401$, $-.0399 < x^2 - 4 < .0401$ and certainly $|x^2 - 4| < .05$ $(x^2 \neq 4)$. The fact that these inequalities also happen to hold at $x = 2$ is merely coincidental.

If we wish to make $|x^2 - 4| < 6$, we can choose $\delta = 1$ and this will be satisfied.

(*b*) There is no difference between the proof for this case and the proof in (*a*), since in both cases we exclude $x = 2$.

3.11. Prove that $\lim_{x \to 1} \dfrac{2x^4 - 6x^3 + x^2 + 3}{x - 1} = -8$.

We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that $\left| \dfrac{2x^4 - 6x^3 + x^2 + 3}{x - 1} - (-8) \right| < \epsilon$ when $0 < |x - 1| < \delta$. Since $x \neq 1$, we can write $\dfrac{2x^4 - 6x^3 + x^2 + 3}{x - 1} = \dfrac{(2x^3 - 4x^2 - 3x - 3)(x - 1)}{x - 1} = 2x^3 - 4x^2 - 3x - 3$ on cancelling the common factor $x - 1 \neq 0$.

Then we must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that $|2x^3 - 4x^2 - 3x + 5| < \epsilon$ when $0 < |x - 1| < \delta$. Choosing $\delta \leqq 1$, we have $0 < x < 2$, $x \neq 1$.

Now $|2x^3 - 4x^2 - 3x + 5| = |x - 1||2x^2 - 2x - 5| < \delta|2x^2 - 2x - 5| < \delta(|2x^2| + |2x| + 5) < (8 + 4 + 5)\delta = 17\delta$. Taking δ as the smaller of 1 and $\epsilon/17$, the required result follows.

3.12. Let $f(x) = \begin{cases} \dfrac{|x - 3|}{x - 3}, & x \neq 3 \\ 0, & x = 3 \end{cases}$, (*a*) Graph the function. (*b*) Find $\lim_{x \to 3+} f(x)$. (*c*) Find $\lim_{x \to 3-} f(x)$. (*d*) Find $\lim_{x \to 3} f(x)$.

(*a*) For $x > 3$, $\dfrac{|x - 3|}{x - 3} = \dfrac{x - 3}{x - 3} = 1$.

For $x < 3$, $\dfrac{|x - 3|}{x - 3} = \dfrac{-(x - 3)}{x - 3} = -1$.

Then the graph, shown in the adjoining Fig. 3-12, consists of the lines $y = 1$, $x > 3$; $y = -1$, $x < 3$ and the point $(3, 0)$.

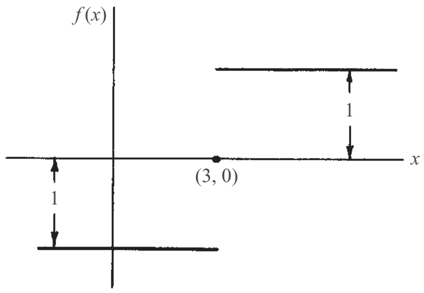


Fig. 3-12

(*b*) As $x \to 3$ from the right, $f(x) \to 1$, i.e., $\lim_{x \to 3+} f(x) = 1$, as seems clear from the graph. To prove this we must show that given any $\epsilon > 0$, we can find $\delta > 0$ such that $|f(x) - 1| < \epsilon$ whenever $0 < x - 1 < \delta$.

Now since $x > 1$, $f(x) = 1$ and so the proof consists in the triviality that $|1 - 1| < \epsilon$ whenever $0 < x - 1 < \delta$.

(*c*) As $x \to 3$ from the left, $f(x) \to -1$, i.e., $\lim_{x \to 3-} f(x) = -1$. A proof can be formulated as in (*b*).

(*d*) Since $\lim_{x \to 3+} f(x) \neq \lim_{x \to 3-} f(x)$, $\lim_{x \to 3} f(x)$ does not exist.

3.13. Prove that $\lim_{x \to 0} x \sin 1/x = 0$.

We must show that given any $\epsilon > 0$, we can find $\delta > 0$ such that $|x \sin 1/x - 0| < \epsilon$ when $0 < |x - 0| < \delta$.

If $0 < |x| < \delta$, then $|x \sin 1/x| = |x||\sin 1/x| \leqq |x| < \delta$ since $|\sin 1/x| \leqq 1$ for all $x \neq 0$.

Making the choice $\delta = \epsilon$, we see that $|x \sin 1/x| < \epsilon$ when $0 < |x| < \delta$, completing the proof.

3.14. Evaluate $\lim_{x \to 0+} \dfrac{2}{1 + e^{-1/x}}$.

As $x \to 0+$ we *suspect* that $1/x$ increases indefinitely, $e^{1/x}$ increases indefinitely, $e^{-1/x}$ approaches 0, $1 + e^{-1/x}$ approaches 1; thus the required limit is 2.

To *prove* this conjecture we must show that, given $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left|\frac{2}{1 + e^{-1/x}} - 2\right| < \epsilon \quad \text{when} \quad 0 < x < \delta$$

Now

$$\left|\frac{2}{1 + e^{-1/x}} - 2\right| = \left|\frac{2 - 2 - 2e^{-1/x}}{1 + e^{-1/x}}\right| = \frac{2}{e^{1/x} + 1}$$

Since the function on the right is smaller than 1 for all $x > 0$, any $\delta > 0$ will work when $e \geq 1$. If $0 < \epsilon < 1$, then $\dfrac{2}{e^{1/x} + 1} < \epsilon$ when $\dfrac{e^{1/x} + 1}{2} > \dfrac{1}{\epsilon}$, $e^{1/x} > \dfrac{2}{\epsilon} - 1$, $\dfrac{1}{x} > \ln\left(\dfrac{2}{\epsilon} - 1\right)$; or $0 < x < \dfrac{1}{\ln(2/\epsilon - 1)} = \delta$.

3.15. Explain exactly what is meant by the statement $\lim_{x \to 1} \dfrac{1}{(x - 1)^4} = \infty$ and prove the validity of this statement.

The statement means that for each positive number M, we can find a positive number δ (depending on M in general) such that

$$\frac{1}{(x - 1)^4} > 4 \qquad \text{when} \qquad 0 < |x - 1| < \delta$$

To prove this note that $\dfrac{1}{(x - 1)^4} > M$ when $0 < (x - 1)^4 < \dfrac{1}{M}$ or $0 < |x - 1| < \dfrac{1}{\sqrt[4]{M}}$.

Choosing $\delta = 1/\sqrt[4]{M}$, the required results follows.

3.16. Present a geometric proof that $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$.

Construct a circle with center at O and radius $OA = OD = 1$, as in Fig. 3-13 below. Choose point B on OA extended and point C on OD so that lines BD and AC are perpendicular to OD.

It is geometrically evident that

Area of triangle OAC < Area of sector OAD < Area of triangle OBD

i.e., $$\tfrac{1}{2}\sin\theta\cos\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta$$

Dividing by $\frac{1}{2}\sin\theta$,

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

or $$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}$$

As $\theta \to 0$, $\cos\theta \to 1$ and it follows that $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$.

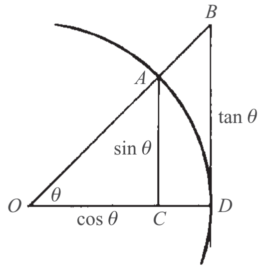


Fig. 3-13

THEOREMS ON LIMITS

3.17. If $\lim\limits_{x\to x_0} f(x)$ exists, prove that it must be unique.

We must show that if $\lim\limits_{x\to x_0} f(x) = l_1$ and $\lim\limits_{x\to x_0} f(x) = l_2$, then $l_1 = l_2$.

By hypothesis, given any $\epsilon > 0$ we can find $\delta > 0$ such that

$$|f(x) - l_1| < \epsilon/2 \quad \text{when} \quad 0 < |x - x_0| < \delta$$
$$|f(x) - l_2| < \epsilon/2 \quad \text{when} \quad 0 < |x - x_0| < \delta$$

Then by the absolute value property 2 on Page 3,

$$|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \leqq |l_1 - f(x)| + |f(x) - l_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

i.e., $|l_1 - l_2|$ is less than any positive number ϵ (however small) and so must be zero. Thus $l_1 = l_2$.

3.18. If $\lim\limits_{x\to x_0} g(x) = B \neq 0$, prove that there exists $\delta > 0$ such that

$$|g(x)| > \tfrac{1}{2}|B| \quad \text{for} \quad 0 < |x - x_0| < \delta$$

Since $\lim\limits_{x\to x_0} g(x) = B$, we can find $\delta > 0$ such that $|g(x) - B| < \frac{1}{2}|B|$ for $0 < |x - x_0| < \delta$.

Writing $B = B - g(x) + g(x)$, we have

$$|B| \leqq |B - g(x)| + |g(x)| < \tfrac{1}{2}|B| + |g(x)|$$

i.e., $|B| < \frac{1}{2}|B| + |g(x)|$, from which $|g(x)| > \frac{1}{2}|B|$.

3.19. Given $\lim\limits_{x\to x_0} f(x) = A$ and $\lim\limits_{x\to x_0} g(x) = B$, prove (*a*) $\lim\limits_{x\to x_0}[f(x) + g(x)] = A + B$, (*b*) $\lim\limits_{x\to x_0} f(x)g(x) = AB$, (*c*) $\lim\limits_{x\to x_0}\dfrac{1}{g(x)} = \dfrac{1}{B}$ if $B \neq 0$, (*d*) $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)} = \dfrac{A}{B}$ if $B \neq 0$.

(*a*) We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$|[f(x) + g(x)] - (A + B)| < \epsilon \quad \text{when} \quad 0 < |x - x_0| < \delta$$

Using absolute value property 2, Page 3, we have

$$|[f(x) + g(x)] - (A + B)| = |[f(x) - A] + [g(x) - B]| \leqq |f(x) - A| + |g(x) - B| \tag{1}$$

By hypothesis, given $\epsilon > 0$ we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - A| < \epsilon/2 \quad \text{when} \quad 0 < |x - x_0| < \delta_1 \tag{2}$$
$$|g(x) - B| < \epsilon/2 \quad \text{when} \quad 0 < |x - x_0| < \delta_2 \tag{3}$$

Then from (*1*), (*2*), and (*3*),

$$|[f(x) + g(x)] - (A + B)| < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{when} \quad 0 < |x - x_0| < \delta$$

where δ is chosen as the smaller of δ_1 and δ_2.

(*b*) We have

$$|f(x)g(x) - AB| = |f(x)[g(x) - B] + B[f(x) - A]| \qquad (4)$$
$$\leqq |f(x)||g(x) - B| + |B||f(x) - A|$$
$$\leqq |f(x)||g(x) - B| + (|B| + 1)|f(x) - A|$$

Since $\lim_{x \to x_0} f(x) = A$, we can find δ_1 such $|f(x) - A| < 1$ for $0 < |x - x_0| < \delta_1$, i.e., $A - 1 < f(x) < A + 1$, so that $f(x)$ is bounded, i.e., $|f(x)| < P$ where P is a positive constant.

Since $\lim_{x \to x_0} g(x) = B$, given $\epsilon > 0$ we can find $\delta_2 > 0$ such that $|g(x) - B| < \epsilon/2P$ for $0 < |x - x_0| < \delta_2$.

Since $\lim_{x \to x_0} f(x) = A$, given $\epsilon > 0$ we can find $\delta_3 > 0$ such that $|f(x) - A| < \dfrac{\epsilon}{2(|B| + 1)}$ for $0 < |x - x_0| < \delta_2$.

Using these in (*4*), we have

$$|f(x)g(x) - AB| < P \cdot \frac{\epsilon}{2P} + (|B| + 1) \cdot \frac{\epsilon}{2(|B| + 1)} = \epsilon$$

for $0 < |x - x_0| < \delta$ where δ is the smaller of $\delta_1, \delta_2, \delta_3$ and the proof is complete.

(*c*) We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|g(x) - B|}{|B||g(x)|} < \epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta \qquad (5)$$

By hypothesis, given $\epsilon > 0$ we can find $\delta_1 > 0$ such that

$$|g(x) - B| < \tfrac{1}{2}B^2\epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta_1$$

By Problem 3.18, since $\lim_{x \to x_0} g(x) = B \neq 0$, we can find $\delta_2 > 0$ such that

$$|g(x)| > \tfrac{1}{2}|B| \qquad \text{when} \qquad 0 < |x - x_0| < \delta_2$$

Then if δ is the smaller of δ_1 and δ_2, we can write

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|g(x) - B|}{|B||g(x)|} < \frac{\frac{1}{2}B^2\epsilon}{|B| \cdot \frac{1}{2}|B|} = \epsilon \qquad \text{whenever} \qquad 0 < |x - x_0| < \delta$$

and the required result is proved.

(*d*) From parts (*b*) and (*c*),

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} f(x) \cdot \frac{1}{g(x)} = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} \frac{1}{g(x)} = A \cdot \frac{1}{B} = \frac{A}{B}$$

This can also be proved directly (see Problem 3.69).

The above results can also be proved in the cases $x \to x_0+$, $x \to x_0-$, $x \to \infty$, $x \to -\infty$.

Note: In the proof of (*a*) we have used the results $|f(x) - A| < \epsilon/2$ and $|g(x) - B| < \epsilon/2$, so that the final result would come out to be $|f(x) + g(x) - (A + B)| < \epsilon$. Of course the proof would be *just as valid* if we had used 2ϵ (or any other positive multiple of ϵ) in place of ϵ. A similar remark holds for the proofs of (*b*), (*c*), and (*d*).

3.20. Evaluate each of the following, using theorems on limits.

(*a*)
$$\lim_{x \to 2}(x^2 - 6x + 4) = \lim_{x \to 2} x^2 + \lim_{x \to 2}(-6x) + \lim_{x \to 2} 4$$
$$= (\lim_{x \to 2} x)(\lim_{x \to 2} x) + (\lim_{x \to 2} -6)(\lim_{x \to 2} x) + \lim_{x \to 2} 4$$
$$= (2)(2) + (-6)(2) + 4 = -4$$

In practice the intermediate steps are omitted.

(*b*) $\lim_{x\to -1} \dfrac{(x+3)(2x-1)}{x^2+3x-2} = \dfrac{\lim_{x\to -1}(x+3)\lim_{x\to -1}(2x-1)}{\lim_{x\to -1}(x^2+3x-2)} = \dfrac{2\cdot(-3)}{-4} = \dfrac{3}{2}$

(*c*) $$\lim_{x\to\infty} \frac{2x^4-3x^2+1}{6x^4+x^3-3x} = \lim_{x\to\infty} \frac{2-\dfrac{3}{x^2}+\dfrac{1}{x^4}}{6+\dfrac{1}{x}-\dfrac{3}{x^3}}$$
$$= \frac{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \dfrac{-3}{x^2} + \lim_{x\to\infty} \dfrac{1}{x^4}}{\lim_{x\to\infty} 6 + \lim_{x\to\infty} \dfrac{1}{x} + \lim_{x\to\infty} \dfrac{-3}{x^3}} = \frac{2}{6} = \frac{1}{3}$$

by Problem 3.19.

(*d*) $$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$
$$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{2+2} = \frac{1}{4}$$

(*e*) $$\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0+} \frac{\sin x}{x} \cdot \sqrt{x} = \lim_{x\to 0+} \frac{\sin x}{x} \cdot \lim_{x\to 0+} \sqrt{x} = 1 \cdot 0 = 0.$$

Note that in (*c*), (*d*), and (*e*) if we use the theorems on limits indiscriminately we obtain the so called *indeterminate forms* ∞/∞ and $0/0$. To avoid such predicaments, note that in each case the form of the limit is suitably modified. For other methods of evaluating limits, see Chapter 4.

CONTINUITY

(Assume that values at which continuity is to be demonstrated, are interior domain values unless otherwise stated.)

3.21. Prove that $f(x) = x^2$ is continuous at $x = 2$.

Method 1: By Problem 3.10, $\lim_{x\to 2} f(x) = f(2) = 4$ and so $f(x)$ is continuous at $x = 2$.

Method 2: We must show that given any $\epsilon > 0$, we can find $\delta > 0$ (depending on ϵ) such that $|f(x) - f(2)| = |x^2 - 4| < \epsilon$ when $|x - 2| < \delta$. The proof patterns that are given in Problem 3.10.

3.22. (*a*) Prove that $f(x) = \begin{cases} x\sin 1/x, & x \neq 0 \\ 5, & x = 0 \end{cases}$ is not continuous at $x = 0$. (*b*) Can one redefine $f(0)$ so that $f(x)$ is continuous at $x = 0$?

(*a*) From Problem 3.13, $\lim_{x\to 0} f(x) = 0$. But this limit is not equal to $f(0) = 5$, so that $f(x)$ is discontinuous at $x = 0$.

(*b*) By redefining $f(x)$ so that $f(0) = 0$, the function becomes continuous. Because the function can be made continuous at a point simply by redefining the function at the point, we call the point a *removable discontinuity*.

3.23. Is the function $f(x) = \dfrac{2x^4 - 6x^3 + x^2 + 3}{x - 1}$ continuous at $x = 1$?

$f(1)$ does not exist, so that $f(x)$ is not continuous at $x = 1$. By redefining $f(x)$ so that $f(1) = \lim_{x\to 1} f(x) = -8$ (see Problem 3.11), it becomes continuous at $x = 1$, i.e., $x = 1$ is a removable discontinuity.

3.24. Prove that if $f(x)$ and $g(x)$ are continuous at $x = x_0$, so also are (*a*) $f(x) + g(x)$, (*b*) $f(x)g(x)$, (*c*) $\dfrac{f(x)}{g(x)}$ if $f(x_0) \neq 0$.

These results follow at once from the proofs given in Problem 3.19 by taking $A = f(x_0)$ and $B = g(x_0)$ and rewriting $0 < |x - x_0| < \delta$ as $|x - x_0| < \delta$, i.e., *including* $x = x_0$.

3.25. Prove that $f(x) = x$ is continuous at any point $x = x_0$.

We must show that, given any $\epsilon > 0$, we can find $\delta > 0$ such that $|f(x) - f(x_0)| = |x - x_0| < \epsilon$ when $|x - x_0| < \delta$. By choosing $\delta = \epsilon$, the result follows at once.

3.26. Prove that $f(x) = 2x^3 + x$ is continuous at any point $x = x_0$.

Since x is continuous at any point $x = x_0$ (Problem 3.25) so also is $x \cdot x = x^2$, $x^2 \cdot x = x^3$, $2x^3$, and finally $2x^3 + x$, using the theorem (Problem 3.24) that sums and products of continuous functions are continuous.

3.27. Prove that if $f(x) = \sqrt{x - 5}$ for $5 \leqq x \leqq 9$, then $f(x)$ is continuous in this interval.

If x_0 is any point such that $5 < x_0 < 9$, then $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \sqrt{x-5} = \sqrt{x_0 - 5} = f(x_0)$. Also, $\lim_{x\to 5+} \sqrt{x-5} = 0 = f(5)$ and $\lim_{x\to 9-} \sqrt{x-5} = 2 = f(9)$. Thus the result follows.

Here we have used the result that $\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{\lim_{x\to x_0} f(x)} = \sqrt{f(x_0)}$ if $f(x)$ is continuous at x_0. An ϵ, δ proof, directly from the definition, can also be employed.

3.28. For what values of x in the domain of definition is each of the following functions continuous?

(*a*) $f(x) = \dfrac{x}{x^2 - 1}$ *Ans.* all x except $x = \pm 1$ (where the denominator is zero)

(*b*) $f(x) = \dfrac{1 + \cos x}{3 + \sin x}$ *Ans.* all x

(*c*) $f(x) = \dfrac{1}{\sqrt[4]{10 + 4}}$ *Ans.* All $x > -10$

(*d*) $f(x) = 10^{-1/(x-3)^2}$ *Ans.* all $x \neq 3$ (see Problem 3.55)

(*e*) $f(x) = \begin{cases} 10^{-1/(x-3)^2}, & x \neq 3 \\ 0, & x = 3 \end{cases}$ *Ans.* all x, since $\lim_{x\to 3} f(x) = f(3)$

(*f*) $f(x) = \dfrac{x - |x|}{x}$

If $x > 0$, $f(x) = \dfrac{x - x}{x} = 0$. If $x < 0$, $f(x) = \dfrac{x + x}{x} = 2$. At $x = 0$, $f(x)$ is undefined. Then $f(x)$ is continuous for all x except $x = 0$.

(*g*) $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x < 0 \\ 2, & x = 0 \end{cases}$

As in (*f*), $f(x)$ is continuous for $x < 0$. Then since

$$\lim_{x\to 0-} \frac{x - |x|}{x} = \lim_{x\to 0-} \frac{x + x}{x} = \lim_{x\to 0-} 2 = 2 = f(0)$$

if follows that $f(x)$ is continuous (from the left) at $x = 0$.

Thus, $f(x)$ is continuous for all $x \leqq 0$, i.e., everywhere in its domain of definition.

(*h*) $f(x) = x \csc x = \dfrac{x}{\sin x}$. *Ans.* all x except $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

(*i*) $f(x) = x \csc x$, $f(0) = 1$. Since $\lim_{x\to 0} x \csc x = \lim_{x\to 0} \dfrac{x}{\sin x} = 1 = f(0)$, we see that $f(x)$ is continuous for all x except $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ [compare (*h*)].

UNIFORM CONTINUITY

3.29. Prove that $f(x) = x^2$ is uniformly continuous in $0 < x < 1$.

Method 1: Using definition.

We must show that given any $\epsilon > 0$ we can find $\delta > 0$ such that $|x^2 - x_0^2| < \epsilon$ when $|x - x_0| < \delta$, where δ depends *only* on ϵ and *not* on x_0 where $0 < x_0 < 1$.

If x and x_0 are any points in $0 < x < 1$, then

$$|x^2 - x_0^2| = |x + x_0||x - x_0| < |1 + 1||x - x_0| = 2|x - x_0|$$

Thus if $|x - x_0| < \delta$ it follows that $|x^2 - x_0^2| < 2\delta$. Choosing $\delta = \epsilon/2$, we see that $|x^2 - x_0^2| < \epsilon$ when $|x - x_0| < \delta$, where δ depends only on ϵ and not on x_0. Hence, $f(x) = x^2$ is uniformly continuous in $0 < x < 1$.

The above can be used to prove that $f(x) = x^2$ is uniformly continuous in $0 \leqq x \leqq 1$.

Method 2: The function $f(x) = x^2$ is continuous in the closed interval $0 \leqq x \leqq 1$. Hence, by the theorem on Page 48 is uniformly continuous in $0 \leqq x \leqq 1$ and thus in $0 < x < 1$.

3.30. Prove that $f(x) = 1/x$ is not uniformly continuous in $0 < x < 1$.

Method 1: Suppose $f(x)$ is uniformly continuous in the given interval. Then for any $\epsilon > 0$ we should be able to find δ, say, between 0 and 1, such that $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$ for all x and x_0 in the interval.

Let $x = \delta$ and $x_0 = \dfrac{\delta}{1+\epsilon}$. Then $|x - x_0| = \left|\delta - \dfrac{\delta}{1+\epsilon}\right| = \dfrac{\epsilon}{1+\epsilon}\delta < \delta$.

However, $\left|\dfrac{1}{x} - \dfrac{1}{x_0}\right| = \left|\dfrac{1}{\delta} - \dfrac{1+\epsilon}{\delta}\right| = \dfrac{\epsilon}{\delta} > \epsilon$ (since $0 < \delta < 1$).

Thus, we have a contradiction and it follows that $f(x) = 1/x$ cannot be uniformly continuous in $0 < x < 1$.

Method 2: Let x_0 and $x_0 + \delta$ be any two points in $(0, 1)$. Then

$$|f(x_0) - f(x_0 + \delta)| = \left|\frac{1}{x_0} - \frac{1}{x_0 + \delta}\right| = \frac{\delta}{x_0(x_0 + \delta)}$$

can be made larger than any positive number by choosing x_0 sufficiently close to 0. Hence, the function cannot be uniformly continuous.

MISCELLANEOUS PROBLEMS

3.31. If $y = f(x)$ is continuous at $x = x_0$, and $z = g(y)$ is continuous at $y = y_0$ where $y_0 = f(x_0)$, prove that $z = g\{f(x)\}$ is continuous at $x = x_0$.

Let $h(x) = g\{f(x)\}$. Since by hypothesis $f(x)$ and $g(y)$ are continuous at x_0 and y_0, respectively, we have

$$\lim_{x\to x_0} f(x) = f(\lim_{x\to x_0} x) = f(x_0)$$
$$\lim_{y\to y_0} g(y) = g(\lim_{y\to y_0} y) = g(y_0) = g\{f(x_0)\}$$

Then

$$\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g\{f(x)\} = g\{\lim_{x\to x_0} f(x)\} = g\{f(x_0)\} = h(x_0)$$

which proves that $h(x) = g\{f(x)\}$ is continuous at $x = x_0$.

3.32. Prove Theorem 8, Page 48.

Suppose that $f(a) < 0$ and $f(b) > 0$. Since $f(x)$ is continuous there must be an interval $(a, a+h)$, $h > 0$, for which $f(x) < 0$. The set of points $(a, a+h)$ has an upper bound and so has a least upper bound which we call c. Then $f(c) \leqq 0$. Now we cannot have $f(c) < 0$, because if $f(c)$ were negative we would be able to find an interval about c (including values greater than c) for which $f(x) < 0$; but since c is the least upper bound, this is impossible, and so we must have $f(c) = 0$ as required.

If $f(a) > 0$ and $f(b) < 0$, a similar argument can be used.

3.33. (*a*) Given $f(x) = 2x^3 - 3x^2 + 7x - 10$, evaluate $f(1)$ and $f(2)$. (*b*) Prove that $f(x) = 0$ for some real number x such that $1 < x < 2$. (*c*) Show how to calculate the value of x in (*b*).

(*a*) $f(1) = 2(1)^3 - 3(1)^2 + 7(1) - 10 = -4$, $f(2) = 2(2)^3 - 3(2)^2 + 7(2) - 10 = 8$.

(*b*) If $f(x)$ is continuous in $a \leqq x \leqq b$ and if $f(a)$ and $f(b)$ have opposite signs, then there is a value of x between a and b such that $f(x) = 0$ (Problem 3.32).

To apply this theorem we need only realize that the given polynomial is continuous in $1 \leqq x \leqq 2$, since we have already shown in (*a*) that $f(1) < 0$ and $f(2) > 0$. Thus there *exists* a number c between 1 and 2 such that $f(c) = 0$.

(*c*) $f(1.5) = 2(1.5)^3 - 3(1.5)^2 + 7(1.5) - 10 = 0.5$. Then applying the theorem of (*b*) again, we see that the required root lies between 1 and 1.5 and is "most likely" closer to 1.5 than to 1, since $f(1.5) = 0.5$ has a value closer to 0 than $f(1) = -4$ (this is not always a valid conclusion but is worth pursuing in practice).

Thus we consider $x = 1.4$. Since $f(1.4) = 2(1.4)^3 - 3(1.4)^2 + 7(1.4) - 10 = -0.592$, we conclude that there is a root between 1.4 and 1.5 which is most likely closer to 1.5 than to 1.4.

Continuing in this manner, we find that the root is 1.46 to 2 decimal places.

3.34. Prove Theorem 10, Page 48.

Given any $\epsilon > 0$, we can find x such that $M - f(x) < \epsilon$ by definition of the l.u.b. M.

Then $\dfrac{1}{M - f(x)} > \dfrac{1}{\epsilon}$, so that $\dfrac{1}{M - f(x)}$ is not bounded and hence cannot be continuous in view of Theorem 4, Page 47. However, if we suppose that $f(x) \neq M$, then since $M - f(x)$ is continuous, by hypothesis, we must have $\dfrac{1}{M - f(x)}$ also continuous. In view of this contradiction, we must have $f(x) = M$ for at least one value of x in the interval.

Similarly, we can show that there exists an x in the interval such that $f(x) = m$ (Problem 3.93).

Supplementary Problems

FUNCTIONS

3.35. Give the largest domain of definition for which each of the following rules of correspondence support the construction of a function.

(*a*) $\sqrt{(3 - x)(2x + 4)}$, (*b*) $(x - 2)/(x^2 - 4)$, (*c*) $\sqrt{\sin 3x}$, (*d*) $\log_{10}(x^3 - 3x^2 - 4x + 12)$.

Ans. (*a*) $-2 \leqq x \leqq 3$, (*b*) all $x \neq \pm 2$, (*c*) $2m\pi/3 \leqq x \leqq (2m+1)\pi/3$, $m = 0, \pm 1, \pm 2, \ldots$, (*d*) $x > 3$, $-2 < x < 2$.

3.36. If $f(x) = \dfrac{3x + 1}{x - 2}$, $x \neq 2$, find: (*a*) $\dfrac{5f(-1) - 2f(0) + 3f(5)}{6}$; (*b*) $\{f(-\frac{1}{2})\}^2$; (*c*) $f(2x - 3)$; (*d*) $f(x) + f(4/x)$, $x \neq 0$; (*e*) $\dfrac{f(h) - f(0)}{h}$, $h \neq 0$; (*f*) $f(\{f(x)\}$.

Ans. (*a*) $\frac{61}{18}$ (*b*) $\frac{1}{25}$ (*c*) $\dfrac{6x - 8}{2x - 5}$, $x \neq 0, \frac{5}{2}, 2$ (*d*) $\frac{5}{2}$, $x \neq 0, 2$ (*e*) $\dfrac{7}{2h - 4}$, $h \neq 0, 2$
(*f*) $\dfrac{10x + 1}{x + 5}$, $x \neq -5, 2$

3.37. If $f(x) = 2x^2$, $0 < x \leqq 2$, find (*a*) the l.u.b. and (*b*) the g.l.b. of $f(x)$. Determine whether $f(x)$ attains its l.u.b. and g.l.b.
Ans. (*a*) 8, (*b*) 0

3.38. Construct a graph for each of the following functions.

(*a*) $f(x) = |x|, -3 \leqq x \leqq 3$

(*b*) $f(x) = 2 - \dfrac{|x|}{x}, -2 \leqq x \leqq 2$

(*c*) $f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$

(*d*) $f(x) = \begin{cases} -x, & -2 \leqq x \leqq 0 \\ x, & 0 \leqq x \leqq 2 \end{cases}$

(*e*) $f(x) = x^2 \sin 1/x, x \neq 0$

(*f*) $\dfrac{x - [x]}{x}$ where $[x]$ = greatest integer $\leqq x$

(*g*) $f(x) = \cosh x$

(*h*) $f(x) = \dfrac{\sin x}{x}$

(*i*) $f(x) = \dfrac{x}{(x-1)(x-2)(x-3)}$

(*j*) $f(x) = \dfrac{\sin^2 x}{x^2}$

3.39. Construct graphs for (*a*) $x^2/a^2 + y^2/b^2 = 1$, (*b*) $x^2/a^2 - y^2/b^2 = 1$, (*c*) $y^2 = 2px$, and (*d*) $y = 2ax - x^2$, where a, b, p are given constants. In which cases when solved for y is there exactly one value of y assigned to each value of x, thus making possible definitions of functions f, and enabling us to write $y = f(x)$? In which cases must branches be defined?

3.40. (*a*) From the graph of $y = \cos x$ construct the graph obtained by interchanging the variables, and from which $\cos^{-1} x$ will result by choosing an appropriate branch. Indicate possible choices of a principal value of $\cos^{-1} x$. Using this choice, find $\cos^{-1}(1/2) - \cos^{-1}(-1/2)$. Does the value of this depend on the choice? Explain.

3.41. Work parts (*a*) and (*b*) of Problem 40 for (*a*) $y = \sec^{-1} x$, (*b*) $y = \cot^{-1} x$.

3.42. Given the graph for $y = f(x)$, show how to obtain the graph for $y = f(ax + b)$, where a and b are given constants. Illustrate the procedure by obtaining the graphs of
(*a*) $y = \cos 3x$, (*b*) $y = \sin(5x + \pi/3)$, (*c*) $y = \tan(\pi/6 - 2x)$.

3.43. Construct graphs for (*a*) $y = e^{-|x|}$, (*b*) $y = \ln|x|$, (*c*) $y = e^{-|x|}\sin x$.

3.44. Using the conventional principal values on Pages 44 and 45, evaluate:

(*a*) $\sin^{-1}(-\sqrt{3}/2)$
(*b*) $\tan^{-1}(1) - \tan^{-1}(-1)$
(*c*) $\cot^{-1}(1/\sqrt{3}) - \cot^{-1}(-1/\sqrt{3})$
(*d*) $\cosh^{-1}\sqrt{2}$
(*e*) $e^{-\coth^{-1}(25/7)}$
(*f*) $\sin^{-1} x + \cos^{-1} x, -1 \leqq x \leqq 1$
(*g*) $\sin^{-1}(\cos 2x), 0 \leqq x \leqq \pi/2$
(*h*) $\sin^{-1}(\cos 2x), \pi/2 \leqq x \leqq 3\pi/2$
(*i*) $\tanh(\operatorname{csch}^{-1} 3x), x \neq 0$
(*j*) $\cos(2\tan^{-1} x^2)$

Ans. (*a*) $-\pi/3$ (*c*) $-\pi/3$ (*e*) $\frac{3}{4}$ (*g*) $\pi/2 - 2x$ (*i*) $\dfrac{|x|}{x\sqrt{9x^2+1}}$ (*j*) $\dfrac{1-x^4}{1+x^4}$
(*b*) $\pi/2$ (*d*) $\ln(1+\sqrt{2})$ (*f*) $\pi/2$ (*h*) $2x - 3\pi/2$

3.45. Evaluate (*a*) $\cos\{\pi \sinh(\ln 2)\}$, (*b*) $\cosh^{-1}\{\coth(\ln 3)\}$.
Ans. (*a*) $-\sqrt{2}/2$, (*b*) $\ln 2$

3.46. (*a*) Prove that $\tan^{-1} x + \cot^{-1} x = \pi/2$ if the conventional principal values on Page 44 are taken. (*b*) Is $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$ also? Explain.

3.47. If $f(x) = \tan^{-1} x$, prove that $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$, discussing the case $xy = 1$.

3.48. Prove that $\tan^{-1} a - \tan^{-1} b = \cot^{-1} b - \cot^{-1} a$.

3.49. Prove the identities:
(*a*) $1 - \tanh^2 x = \operatorname{sech}^2 x$, (*b*) $\sin 3x = 3\sin x - 4\sin^3 x$, (*c*) $\cos 3x = 4\cos^3 x - 3\cos x$, (*d*) $\tanh \frac{1}{2}x = (\sinh x)/(1 + \cosh x)$, (*e*) $\ln|\csc x - \cot x| = \ln|\tan \frac{1}{2}x|$.

3.50. Find the relative and absolute maxima and minima of: (*a*) $f(x) = (\sin x)/x, f(0) = 1$; (*b*) $f(x) = (\sin^2 x)/x^2, f(0) = 1$. Discuss the cases when $f(0)$ is undefined or $f(0)$ is defined but $\neq 1$.

LIMITS

3.51. Evaluate the following limits, first by using the definition and then using theorems on limits.

(*a*) $\lim\limits_{x\to 3}(x^2 - 3x + 2)$, (*b*) $\lim\limits_{x\to -1}\dfrac{1}{2x-5}$, (*c*) $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$, (*d*) $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{4-x}$, (*e*) $\lim\limits_{h\to 0}\dfrac{(2+h)^4 - 16}{h}$,

(*f*) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}}{x+1}$.

Ans. (*a*) 2, (*b*) $-\frac{1}{7}$, (*c*) 4, (*d*) $-\frac{1}{4}$, (*e*) 32, (*f*) $\frac{1}{2}$

3.52. Let $f(x) = \begin{cases} 3x-1, & x < 0 \\ 0, & x = 0 \\ 2x+5, & x > 0 \end{cases}$. (*a*) Construct a graph of $f(x)$.

Evaluate (*b*) $\lim\limits_{x\to 2} f(x)$, (*c*) $\lim\limits_{x\to -3} f(x)$, (*d*) $\lim\limits_{x\to 0+} f(x)$, (*e*) $\lim\limits_{x\to 0-} f(x)$, (*f*) $\lim\limits_{x\to 0} f(x)$, justifying your answer in each case.
Ans. (*b*) 9, (*c*) -10, (*d*) 5, (*e*) -1, (*f*) does not exist

3.53. Evaluate (*a*) $\lim\limits_{h\to 0+}\dfrac{f(h) - f(0+)}{h}$ and (*b*) $\lim\limits_{h\to 0-}\dfrac{f(h) - f(0-)}{h}$, where $f(x)$ is the function of Prob. 3.52.
Ans. (*a*) 2, (*b*) 3

3.54. (*a*) If $f(x) = x^2 \cos 1/x$, evaluate $\lim\limits_{x\to 0} f(x)$, justifying your answer. (*b*) Does your answer to (*a*) still remain the same if we consider $f(x) = x^2 \cos 1/x$, $x \neq 0$, $f(0) = 2$? Explain.

3.55. Prove that $\lim\limits_{x\to 3} 10^{-1/(x-3)^2} = 0$ using the definition.

3.56. Let $f(x) = \dfrac{1 + 10^{-1/x}}{2 - 10^{-1/x}}$, $x \neq 0$, $f(0) = \frac{1}{2}$. Evaluate (*a*) $\lim\limits_{x\to 0+} f(x)$, (*b*) $\lim\limits_{x\to 0-} f(x)$, (*c*) $\lim\limits_{x\to 0} f(x)$, justifying answers in all cases.
Ans. (*a*) $\frac{1}{2}$, (*b*) -1, (*c*) does not exist.

3.57. Find (*a*) $\lim\limits_{x\to 0+}\dfrac{|x|}{x}$, (*b*) $\lim\limits_{x\to 0-}\dfrac{|x|}{x}$. Illustrate your answers graphically.
Ans. (*a*) 1, (*b*) -1

3.58. If $f(x)$ is the function defined in Problem 3.56, does $\lim\limits_{x\to 0} f(|x|)$ exist? Explain.

3.59. Explain *exactly* what is meant when one writes:

(*a*) $\lim\limits_{x\to 3}\dfrac{2-x}{(x-3)^2} = -\infty$, (*b*) $\lim\limits_{x\to 0+}(1 - e^{1/x}) = -\infty$, (*c*) $\lim\limits_{x\to\infty}\dfrac{2x+5}{3x-2} = \dfrac{2}{3}$.

3.60. Prove that (*a*) $\lim_{x\to\infty} 10^{-x} = 0$, (*b*) $\lim_{x\to-\infty} \dfrac{\cos x}{x+\pi} = 0$.

3.61. Explain why (*a*) $\lim_{x\to\infty} \sin x$ does not exist, (*b*) $\lim_{x\to\infty} e^{-x}\sin x$ does not exist.

3.62. If $f(x) = \dfrac{3x+|x|}{7x-5|x|}$, evaluate (*a*) $\lim_{x\to\infty} f(x)$, (*b*) $\lim_{x\to-\infty} f(x)$, (*c*) $\lim_{x\to 0+} f(x)$, (*d*) $\lim_{x\to 0-} f(x)$, (*e*) $\lim_{x\to 0} f(x)$.
Ans. (*a*) 2, (*b*) 1/6, (*c*) 2, (*d*) 1/6, (*e*) does not exist.

3.63. If $[x]$ = largest integer $\leqq x$, evaluate (*a*) $\lim_{x\to 2+} \{x-[x]\}$, (*b*) $\lim_{x\to 2-} \{x-[x]\}$.
Ans. (*a*) 0, (*b*) 1

3.64. If $\lim_{x\to x_0} f(x) = A$, prove that (*a*) $\lim_{x\to x_0} \{f(x)\}^2 = A^2$, (*b*) $\lim_{x\to x_0} \sqrt[3]{f(x)} = \sqrt[3]{A}$.
What generalizations of these do you suspect are true? Can you prove them?

3.65. If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$, prove that
(*a*) $\lim_{x\to x_0} \{f(x)-g(x)\} = A-B$, (*b*) $\lim_{x\to x_0} \{af(x)+bg(x)\} = aA+bB$ where a, b = any constants.

3.66. If the limits of $f(x)$, $g(x)$, and $h(x)$ are A, B, and C respectively, prove that:
(*a*) $\lim_{x\to x_0} \{f(x)+g(x)+h(x)\} = A+B+C$, (*b*) $\lim_{x\to x_0} f(x)g(x)h(x) = ABC$. Generalize these results.

3.67. Evaluate each of the following using the theorems on limits.

(*a*) $\lim_{x\to 1/2} \left\{\dfrac{2x^2-1}{(3x+2)(5x-3)} - \dfrac{2-3x}{x^2-5x+3}\right\}$ *Ans.* (*a*) $-8/21$

(*b*) $\lim_{x\to\infty} \dfrac{(3x-1)(2x+3)}{(5x-3)(4x+5)}$ (*b*) 3/10

(*c*) $\lim_{x\to-\infty} \left(\dfrac{3x}{x-1} - \dfrac{2x}{x+1}\right)$ (*c*) 1

(*d*) $\lim_{x\to 1} \dfrac{1}{x-1}\left(\dfrac{1}{x+3} - \dfrac{2x}{3x+5}\right)$ (*d*) 1/32

3.68. Evaluate $\lim_{h\to 0} \dfrac{\sqrt[3]{8+h}-2}{h}$. (Hint: Let $8+h = x^3$). *Ans.* 1/12

3.69. If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B \neq 0$, prove directly that $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$.

3.70. Given $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, evaluate:

(*a*) $\lim_{x\to 0} \dfrac{\sin 3x}{x}$ (*c*) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ (*e*) $\lim_{x\to 0} \dfrac{6x-\sin 2x}{2x+3\sin 4x}$ (*g*) $\lim_{x\to 0} \dfrac{1-2\cos x+\cos 2x}{x^2}$

(*b*) $\lim_{x\to 0} \dfrac{1-\cos x}{x}$ (*d*) $\lim_{x\to 3} (x-3)\csc \pi x$ (*f*) $\lim_{x\to 0} \dfrac{\cos ax - \cos bx}{x^2}$ (*h*) $\lim_{x\to 1} \dfrac{3\sin \pi x - \sin 3\pi x}{x^3}$

Ans. (*a*) 3, (*b*) 0, (*c*) 1/2, (*d*) $-1/\pi$, (*e*) 2/7, (*f*) $\frac{1}{2}(b^2-a^2)$, (*g*) -1, (*h*) $4\pi^3$

3.71. If $\lim_{x\to 0} \dfrac{e^x-1}{x} = 1$, prove that:
(*a*) $\lim_{x\to 0} \dfrac{e^{-ax}-e^{-bx}}{x} = b-a$; (*b*) $\lim_{x\to 0} \dfrac{a^x-b^x}{x} = \ln \dfrac{a}{b}$, $a, b > 0$; (*c*) $\lim_{x\to 0} \dfrac{\tanh ax}{x} = a$.

3.72. Prove that $\lim_{x \to x_0} f(x) = l$ if and only if $\lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x) = l$.

CONTINUITY

In the following problems assume the largest possible domain unless otherwise stated.

3.73. Prove that $f(x) = x^2 - 3x + 2$ is continuous at $x = 4$.

3.74. Prove that $f(x) = 1/x$ is continuous (*a*) at $x = 2$, (*b*) in $1 \leqq x \leqq 3$.

3.75. Investigate the continuity of each of the following functions at the indicated points:

(*a*) $f(x) = \dfrac{\sin x}{x}$; $x \neq 0$, $f(0) = 0$; $x = 0$

(*b*) $f(x) = x - |x|$; $x = 0$

(*c*) $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$; $x \neq 2$, $f(2) = 3$; $x = 2$

(*d*) $f(x) = \begin{cases} \sin \pi x, & 0 < x < 1 \\ \ln x & 1 < x < 2 \end{cases}$; $x = 1$.

Ans. (*a*) discontinuous, (*b*) continuous, (*c*) continuous, (*d*) discontinuous

3.76. If $[x]$ = greatest integer $\leqq x$, investigate the continuity of $f(x) = x - [x]$ in the interval (*a*) $1 < x < 2$, (*b*) $1 \leqq x \leqq 2$.

3.77. Prove that $f(x) = x^3$ is continuous in every finite interval.

3.78. If $f(x)/g(x)$ and $g(x)$ are continuous at $x = x_0$, prove that $f(x)$ must be continuous at $x = x_0$.

3.79. Prove that $f(x) = (\tan^{-1} x)/x$, $f(0) = 1$ is continuous at $x = 0$.

3.80. Prove that a polynomial is continuous in every finite interval.

3.81. If $f(x)$ and $g(x)$ are polynomials, prove that $f(x)/g(x)$ is continuous at each point $x = x_0$ for which $g(x_0) \neq 0$.

3.82. Give the points of discontinuity of each of the following functions.

(*a*) $f(x) = \dfrac{x}{(x-2)(x-4)}$

(*b*) $f(x) = x^2 \sin 1/x$, $x \neq 0$, $f(0) = 0$

(*c*) $f(x) = \sqrt{(x-3)(6-x)}$, $3 \leqq x \leqq 6$

(*d*) $f(x) = \dfrac{1}{1 + 2\sin x}$.

Ans. (*a*) $x = 2, 4$, (*b*) none, (*c*) none, (*d*) $x = 7\pi/6 \pm 2m\pi, 11\pi/6 \pm 2m\pi, m = 0, 1, 2, \ldots$

UNIFORM CONTINUITY

3.83. Prove that $f(x) = x^3$ is uniformly continuous in (*a*) $0 < x < 2$, (*b*) $0 \leqq x \leqq 2$, (*c*) any finite interval.

3.84. Prove that $f(x) = x^2$ is not uniformly continuous in $0 < x < \infty$.

3.85. If a is a constant, prove that $f(x) = 1/x^2$ is (*a*) continuous in $a < x < \infty$ if $a \geqq 0$, (*b*) uniformly continuous in $a < x < \infty$ if $a > 0$, (*c*) not uniformly continuous in $0 < x < 1$.

3.86. If $f(x)$ and $g(x)$ are uniformly continuous in the same interval, prove that (*a*) $f(x) \pm g(x)$ and (*b*) $f(x)g(x)$ are uniformly continuous in the interval. State and prove an analogous theorem for $f(x)/g(x)$.

MISCELLANEOUS PROBLEMS

3.87. Give an "ϵ, δ" proof of the theorem of Problem 3.31.

3.88. (*a*) Prove that the equation $\tan x = x$ has a real positive root in each of the intervals $\pi/2 < x < 3\pi/2$, $3\pi/2 < x < 5\pi/2$, $5\pi/2 < x < 7\pi/2, \ldots$.

(*b*) Illustrate the result in (*a*) graphically by constructing the graphs of $y = \tan x$ and $y = x$ and locating their points of intersection.
(*c*) Determine the value of the smallest positive root of $\tan x = x$.
Ans. (*c*) 4.49 approximately

3.89. Prove that the only real solution of $\sin x = x$ is $x = 0$.

3.90. (*a*) Prove that $\cos x \cosh x + 1 = 0$ has infinitely many real roots.
(*b*) Prove that for large values of x the roots approximate those of $\cos x = 0$.

3.91. Prove that $\lim_{x \to 0} \dfrac{x^2 \sin(1/x)}{\sin x} = 0$.

3.92. Suppose $f(x)$ is continuous at $x = x_0$ and assume $f(x_0) > 0$. Prove that there exists an interval $(x_0 - h, x_0 + h)$, where $h > 0$, in which $f(x) > 0$. (See Theorem 5, page 47.) [Hint: Show that we can make $|f(x) - f(x_0)| < \frac{1}{2} f(x_0)$. Then show that $f(x) \geqq f(x_0) - |f(x) - f(x_0)| > \frac{1}{2} f(x_0) > 0$.]

3.93. (*a*) Prove Theorem 10, Page 48, for the greatest lower bound m (see Problem 3.34). (*b*) Prove Theorem 9, Page 48, and explain its relationship to Theorem 10.

CHAPTER 4

Derivatives

THE CONCEPT AND DEFINITION OF A DERIVATIVE

Concepts that shape the course of mathematics are few and far between. The derivative, the fundamental element of the differential calculus, is such a concept. That branch of mathematics called analysis, of which advanced calculus is a part, is the end result. There were two problems that led to the discovery of the derivative. The older one of defining and representing the tangent line to a curve at one of its points had concerned early Greek philosophers. The other problem of representing the instantaneous velocity of an object whose motion was not constant was much more a problem of the seventeenth century. At the end of that century, these problems and their relationship were resolved. As is usually the case, many mathematicians contributed, but it was Isaac Newton and Gottfried Wilhelm Leibniz who independently put together organized bodies of thought upon which others could build.

The tangent problem provides a visual interpretation of the derivative and can be brought to mind no matter what the complexity of a particular application. It leads to the definition of the derivative as the limit of a difference quotient in the following way. (See Fig. 4-1.)

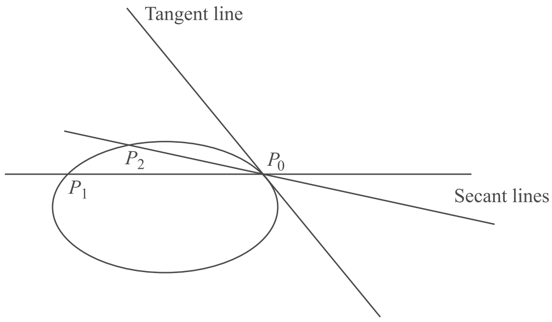


Fig. 4-1

Let $P_o(x_0)$ be a point on the graph of $y = f(x)$. Let $P(x)$ be a nearby point on this same graph of the function f. Then the line through these two points is called a *secant line*. Its slope, m_s, is the difference quotient

$$m_s = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\Delta y}{\Delta x}$$

where Δx and Δy are called the increments in x and y, respectively. Also this slope may be written

$$m_s = \frac{f(x_0 + h) - f(x_0)}{h}$$

where $h = x - x_0 = \Delta x$. See Fig. 4-2.

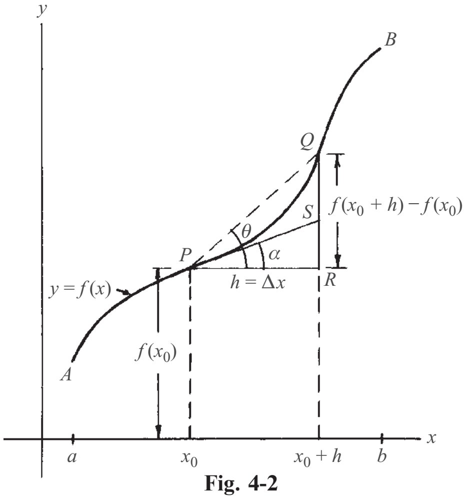


Fig. 4-2

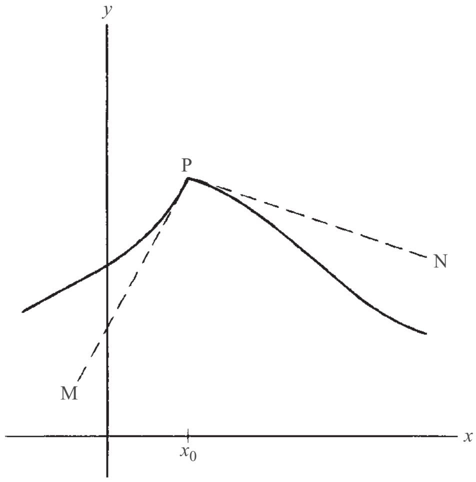


Fig. 4-3

We can imagine a sequence of lines formed as $h \to 0$. It is the limiting line of this sequence that is the natural one to be the tangent line to the graph at P_0.

To make this mode of reasoning precise, the limit (when it exists), is formed as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

As indicated, this limit is given the name $f'(x_0)$. It is called the *derivative* of the function f at its domain value x_0. If this limit can be formed at each point of a subdomain of the domain of f, then f is said to be *differentiable* on that subdomain and a new function f' has been constructed.

This limit concept was not understood until the middle of the nineteenth century. A simple example illustrates the conceptual problem that faced mathematicians from 1700 until that time. Let the graph of f be the parabola $y = x^2$, then a little algebraic manipulation yields

$$m_s = \frac{2x_0 h + h^2}{h} = 2x_0 + h$$

Newton, Leibniz, and their contemporaries simply let $h = 0$ and said that $2x_0$ was the slope of the tangent line at P_0. However, this raises the ghost of a $\frac{0}{0}$ form in the middle term. True understanding of the calculus is in the comprehension of how the introduction of something new (the derivative, i.e., the limit of a difference quotient) resolves this dilemma.

Note 1: The creation of new functions from difference quotients is not limited to f'. If, starting with f', the limit of the difference quotient exists, then f'' may be constructed and so on and so on.

Note 2: Since the continuity of a function is such a strong property, one might think that differentiability followed. This is not necessarily true, as is illustrated in Fig. 4-3.

The following theorem puts the matter in proper perspective:

Theorem: If f is differentiable at a domain value, then it is continuous at that value.

As indicated above, the converse of this theorem is not true.

RIGHT- AND LEFT-HAND DERIVATIVES

The status of the derivative at end points of the domain of f, and in other special circumstances, is clarified by the following definitions.

The *right-hand derivative* of $f(x)$ at $x = x_0$ is defined as

$$f'_+(x_0) = \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} \tag{3}$$

if this limit exists. Note that in this case $h(= \Delta x)$ is restricted only to positive values as it approaches zero.

Similarly, the *left-hand derivative* of $f(x)$ at $x = x_0$ is defined as

$$f'_-(x_0) = \lim_{h \to 0-} \frac{f(x_0 + h) - f(x_0)}{h} \tag{4}$$

if this limit exists. In this case h is restricted to negative values as it approaches zero.

A function f has a derivative at $x = x_0$ if and only if $f'_+(x_0) = f'_-(x_0)$.

DIFFERENTIABILITY IN AN INTERVAL

If a function has a derivative at all points of an interval, it is said to be *differentiable in the interval*. In particular if f is defined in the closed interval $a \leqq x \leqq b$, i.e. $[a, b]$, then f is differentiable in the interval if and only if $f'(x_0)$ exists for each x_0 such that $a < x_0 < b$ and if $f'_+(a)$ and $f'_-(b)$ both exist.

If a function has a continuous derivative, it is sometimes called *continuously differentiable*.

PIECEWISE DIFFERENTIABILITY

A function is called *piecewise differentiable* or *piecewise smooth* in an interval $a \leqq x \leqq b$ if $f'(x)$ is piecewise continuous. An example of a piecewise continuous function is shown graphically on Page 48.

An equation for the tangent line to the curve $y = f(x)$ at the point where $x = x_0$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{7}$$

The fact that a function can be continuous at a point and yet not be differentiable there is shown graphically in Fig. 4-3. In this case there are two tangent lines at P represented by PM and PN. The slopes of these tangent lines are $f'_-(x_0)$ and $f'_+(x_0)$ respectively.

DIFFERENTIALS

Let $\Delta x = dx$ be an increment given to x. Then

$$\Delta y = f(x + \Delta x) - f(x) \tag{8}$$

is called the *increment* in $y = f(x)$. If $f(x)$ is continuous and has a continuous first derivative in an interval, then

$$\Delta y = f'(x)\Delta x + \epsilon\Delta x = f'(x)dx + dx \tag{9}$$

where $\epsilon \to 0$ as $\Delta x \to 0$. The expression

$$\mathrm{d}y = f'(x)\mathrm{d}x \tag{10}$$

is called the *differential of y or f(x)* or the *principal part of* Δy. Note that $\Delta y \neq dy$ in general. However if $\Delta x = dx$ is small, then dy is a close approximation of Δy (see Problem 11). The quantity dx, called the *differential of* x, and dy need not be small.

Because of the definitions *(8)* and *(10)*, we often write

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{11}$$

It is emphasized that dx and dy are *not* the limits of Δx and Δy as $\Delta x \to 0$, since these limits are zero whereas dx and dy are not necessarily zero. Instead, given dx we determine dy from *(10)*, i.e., dy is a dependent variable determined from the independent variable dx for a given x.

Geometrically, dy is represented in Fig. 4-1, for the particular value $x = x_0$, by the line segment SR, whereas Δy is represented by QR.

The geometric interpretation of the derivative as the slope of the tangent line to a curve at one of its points is fundamental to its application. Also of importance is its use as representative of instantaneous velocity in the construction of physical models. In particular, this physical viewpoint may be used to introduce the notion of differentials.

Newton's Second and First Laws of Motion imply that the path of an object is determined by the forces acting on it, and that if those forces suddenly disappear, the object takes on the tangential direction of the path at the point of release. Thus, the nature of the path in a small neighborhood of the point of release becomes of interest. With this thought in mind, consider the following idea.

Suppose the graph of a function f is represented by $y = f(x)$. Let $x = x_0$ be a domain value at which f' exists (i.e., the function is differentiable at that value). Construct a new linear function

$$dy = f'(x_0)\, dx$$

with dx as the (independent) domain variable and dy the range variable generated by this rule. This linear function has the graphical interpretation illustrated in Fig. 4-4.

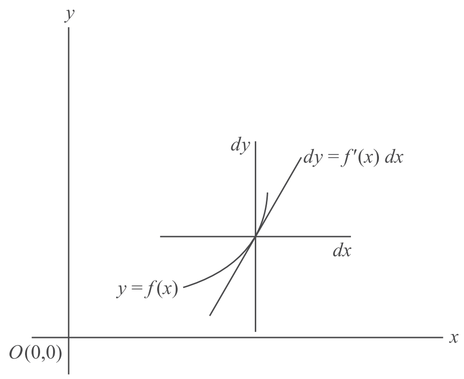


Fig. 4-4

That is, a coordinate system may be constructed with its origin at P_0 and the dx and dy axes parallel to the x and y axes, respectively. In this system our linear equation is the equation of the tangent line to the graph at P_0. It is representative of the path in a small neighborhood of the point; and if the path is that of an object, the linear equation represents its new path when all forces are released.

dx and dy are called differentials of x and y, respectively. Because the above linear equation is valid at every point in the domain of f at which the function has a derivative, the subscript may be dropped and we can write

$$dy = f'(x)\, dx$$

The following important observations should be made. $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, thus $\frac{dy}{dx}$ is not the same thing as $\frac{\Delta y}{\Delta x}$.

On the other hand, dy and Δy are related. In particular, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $-\varepsilon < \frac{\Delta y}{\Delta x} - \frac{dy}{dx} < \varepsilon$ whenever $|\Delta x| < \delta$. Now dx is an independent variable and the axes of x and dx are parallel; therefore, dx may be chosen equal to Δx. With this choice

$$-\varepsilon \Delta x < \Delta y - dy < \varepsilon \Delta x$$

or

$$dy - \varepsilon \Delta x < \Delta y < dy + \varepsilon \Delta x$$

From this relation we see that dy is an approximation to Δy in small neighborhoods of x. dy is called the *principal part* of Δy.

The representation of f' by $\frac{dy}{dx}$ has an algebraic suggestiveness that is very appealing and will appear in much of what follows. In fact, this notation was introduced by Leibniz (without the justification provided by knowledge of the limit idea) and was the primary reason his approach to the calculus, rather than Newton's was followed.

THE DIFFERENTIATION OF COMPOSITE FUNCTIONS

Many functions are a composition of simpler ones. For example, if f and g have the rules of correspondence $u = x^3$ and $y = \sin u$, respectively, then $y = \sin x^3$ is the rule for a composite function $F = g(f)$. The domain of F is that subset of the domain of F whose corresponding range values are in the domain of g. The rule of composite function differentiation is called the *chain rule* and is represented by $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ $[F'(x) = g'(u)f'(x)]$.

In the example

$$\frac{dy}{dx} \equiv \frac{d(\sin x^3)}{dx} = \cos x^3(3x^2 dx)$$

The importance of the chain rule cannot be too greatly stressed. Its proper application is essential in the differentiation of functions, and it plays a fundamental role in changing the variable of integration, as well as in changing variables in mathematical models involving differential equations.

IMPLICIT DIFFERENTIATION

The rule of correspondence for a function may not be explicit. For example, the rule $y = f(x)$ is *implicit* to the equation $x^2 + 4xy^5 + 7xy + 8 = 0$. Furthermore, there is no reason to believe that this equation can be solved for y in terms of x. However, assuming a common domain (described by the independent variable x) the left-hand member of the equation can be construed as a composition of functions and differentiated accordingly. (The rules of differentiation are listed below for your review.)

In this example, differentiation with respect to x yields

$$2x + 4\left(y^5 + 5xy^4\frac{dy}{dx}\right) + 7\left(y + x\frac{dy}{dx}\right) = 0$$

Observe that this equation can be solved for $\frac{dy}{dx}$ as a function of x and y (but not of x alone).

RULES FOR DIFFERENTIATION

If f, g, and h are differentiable functions, the following differentiation rules are valid.

1. $\dfrac{d}{dx}\{f(x)+g(x)\} = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x) = f'(x) + g'(x)$ (Addition Rule)

2. $\dfrac{d}{dx}\{f(x)-g(x)\} = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x) = f'(x) - g'(x)$

3. $\dfrac{d}{dx}\{Cf(x)\} = C\dfrac{d}{dx}f(x) = Cf'(x)$ where C is any constant

4. $\dfrac{d}{dx}\{f(x)g(x)\} = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

5. $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$ (Quotient Rule)

6. If $y = f(u)$ where $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\frac{du}{dx} = f'\{g(x)\}g'(x) \tag{12}$$

Similarly if $y = f(u)$ where $u = g(v)$ and $v = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} \tag{13}$$

The results (*12*) and (*13*) are often called *chain rules* for differentiation of composite functions.

7. If $y = f(x)$, and $x = f^{-1}(y)$; then dy/dx and dx/dy are related by

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{14}$$

8. If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \tag{15}$$

Similar rules can be formulated for differentials. For example,

$$d\{f(x)+g(x)\} = df(x) + dg(x) = f'(x)dx + g'(x)dx = \{f'(x)+g'(x)\}dx$$

$$d\{f(x)g(x)\} = f(x)dg(x) + g(x)df(x) = \{f(x)g'(x) + g(x)f'(x)\}dx$$

DERIVATIVES OF ELEMENTARY FUNCTIONS

In the following we assume that u is a differentiable function of x; if $u = x$, $du/dx = 1$. The inverse functions are defined according to the principal values given in Chapter 3.

1. $\frac{d}{dx}(C) = 0$
2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
3. $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$
4. $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$
5. $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$
6. $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$
7. $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$
8. $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$
9. $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \quad a > 0, a \neq 1$
10. $\frac{d}{dx}\log_e u = \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
11. $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$
12. $\frac{d}{dx}e^u = e^u \frac{du}{dx}$
13. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
14. $\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
15. $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$
16. $\frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$
17. $\frac{d}{dx}\sec^{-1} u = \pm\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx} \begin{cases} + \text{ if } u > 1 \\ - \text{ if } u < -1 \end{cases}$
18. $\frac{d}{dx}\csc^{-1} u = \mp\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx} \begin{cases} - \text{ if } u > 1 \\ + \text{ if } u < -1 \end{cases}$
19. $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$
20. $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$
21. $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$
22. $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}$
23. $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$
24. $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$
25. $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
26. $\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
27. $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$
28. $\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$
29. $\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
30. $\frac{d}{dx}\operatorname{csch}^{-1} u = -\frac{1}{u\sqrt{u^2+1}}\frac{du}{dx}$

HIGHER ORDER DERIVATIVES

If $f(x)$ is differentiable in an interval, its derivative is given by $f'(x)$, y' or dy/dx, where $y = f(x)$. If $f'(x)$ is also differentiable in the interval, its derivative is denoted by $f''(x)$, y'' or $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. Similarly, the nth derivative of $f(x)$, if it exists, is denoted by $f^{(n)}(x)$, $y^{(n)}$ or $\frac{d^ny}{dx^n}$, where n is called the order of the derivative. Thus derivatives of the first, second, third, . . . orders are given by $f'(x), f''(x), f'''(x), \ldots$.

Computation of higher order derivatives follows by repeated application of the differentiation rules given above.

MEAN VALUE THEOREMS

These theorems are fundamental to the rigorous establishment of numerous theorems and formulas. (See Fig. 4-5.)

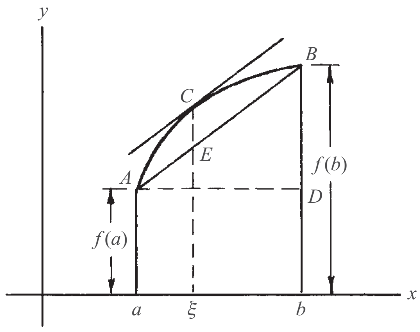


Fig. 4-5

1. **Rolle's theorem**. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and if $f(a) = f(b) = 0$, then there exists a point ξ in (a, b) such that $f'(\xi) = 0$.

   Rolle's theorem is employed in the proof of the mean value theorem. It then becomes a special case of that theorem.

2. **The mean value theorem**. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \qquad a < \xi < b \tag{16}$$

   Rolle's theorem is the special case of this where $f(a) = f(b) = 0$.

   The result (*16*) can be written in various alternative forms; for example, if x and x_0 are in (a, b), then

$$f(x) = f(x_0) + f'(\xi)(x - x_0) \qquad \xi \text{ between } x_0 \text{ and } x \tag{17}$$

   We can also write (*16*) with $b = a + h$, in which case $\xi = a + \theta h$, where $0 < \theta < 1$.

   The mean value theorem is also called the *law of the mean*.

3. **Cauchy's generalized mean value theorem**. If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad a < \xi < b \tag{18}$$

   where we assume $g(a) \neq g(b)$ and $f'(x)$, $g'(x)$ are not simultaneously zero. Note that the special case $g(x) = x$ yields (*16*).

L'HOSPITAL'S RULES

If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, where A and B are either both zero or both infinite, $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is often called an *indeterminate* of the form $0/0$ or ∞/∞, respectively, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. The following theorems, called *L'Hospital's rules*, facilitate evaluation of such limits.

1. If $f(x)$ and $g(x)$ are differentiable in the interval (a, b) except possibly at a point x_0 in this interval, and if $g'(x) \neq 0$ for $x \neq x_0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \tag{19}$$

whenever the limit on the right can be found. In case $f'(x)$ and $g'(x)$ satisfy the same conditions as $f(x)$ and $g(x)$ given above, the process can be repeated.

2. If $\lim_{x \to x_0} f(x) = \infty$ and $\lim_{x \to x_0} g(x) = \infty$, the result (*19*) is also valid.

These can be extended to cases where $x \to \infty$ or $-\infty$, and to cases where $x_0 = a$ or $x_0 = b$ in which only one sided limits, such as $x \to a+$ or $x \to b-$, are involved.

Limits represented by the so-called *indeterminate forms* $0 \cdot \infty$, ∞^0, 0^0, 1^∞, and $\infty - \infty$ can be evaluated on replacing them by equivalent limits for which the above rules are applicable (see Problem 4.29).

APPLICATIONS

1. Relative Extrema and Points of Inflection

See Chapter 3 where relative extrema and points of inflection were described and a diagram is presented. In this chapter such points are characterized by the variation of the tangent line, and then by the derivative, which represents the slope of that line.

Assume that f has a derivative at each point of an open interval and that P_1 is a point of the graph of f associated with this interval. Let a varying tangent line to the graph move from left to right through P_1. If the point is a relative minimum, then the tangent line rotates counterclockwise. The slope is negative to the left of P_1 and positive to the right. At P_1 the slope is zero. At a relative maximum a similar analysis can be made except that the rotation is clockwise and the slope varies from positive to negative. Because f'' designates the change of f', we can state the following theorem. (See Fig. 4-6.)

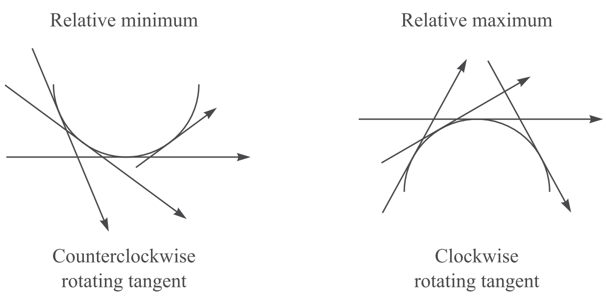


Fig. 4-6

Theorem. Assume that x_1 is a number in an open set of the domain of f at which f' is continuous and f'' is defined. If $f'(x_1) = 0$ and $f''(x_1) \neq 0$, then $f(x_1)$ is a relative extreme of f. Specifically:

(*a*) If $f''(x_1) > 0$, then $f(x_1)$ is a relative minimum,

(*b*) If $f''(x_1) < 0$, then $f(x_1)$ is a relative maximum.

(The domain value x_1 is called a *critical value*.)

This theorem may be generalized in the following way. Assume existence and continuity of derivatives as needed and suppose that $f'(x_1) = f''(x_1) = \cdots f^{(2p-1)}(x_1) = 0$ and $f^{(2p)}(x_1) \neq 0$ (p a positive integer). Then:

(*a*) f has a relative minimum at x_1 if $f^{(2p)}(x_1) > 0$,

(*b*) f has a relative maximum at x_1 if $f^{(2p)}(x_1) < 0$.

(Notice that the order of differentiation in each succeeding case is two greater. The nature of the intermediate possibilities is suggested in the next paragraph.)

It is possible that the slope of the tangent line to the graph of f is positive to the left of P_1, zero at the point, and again positive to the right. Then P_1 is called a *point of inflection*. In the simplest case this point of inflection is characterized by $f'(x_1) = 0$, $f''(x_1) = 0$, and $f'''(x_1) \neq 0$.

2. **Particle motion**

The fundamental theories of modern physics are relativity, electromagnetism, and quantum mechanics. Yet Newtonian physics must be studied because it is basic to many of the concepts in these other theories, and because it is most easily applied to many of the circumstances found in everyday life. The simplest aspect of Newtonian mechanics is called *kinematics*, or the *geometry of motion*. In this model of reality, objects are idealized as points and their paths are represented by curves. In the simplest (one-dimensional) case, the curve is a straight line, and it is the speeding up and slowing down of the object that is of importance. The calculus applies to the study in the following way.

If x represents the distance of a particle from the origin and t signifies time, then $x = f(t)$ designates the position of a particle at time t. Instantaneous velocity (or speed in the one-dimensional case) is represented by $\dfrac{dx}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{f(t + \Delta t)}{\Delta t}$ (the limiting case of the formula $\dfrac{\text{change in distance}}{\text{change in time}}$ for speed when the motion is constant). Furthermore, the instantaneous change in velocity is called *acceleration* and represented by $\dfrac{d^2x}{dt^2}$.

Path, velocity, and acceleration of a particle will be represented in three dimensions in Chapter 7 on vectors.

3. **Newton's method**

It is difficult or impossible to solve algebraic equations of higher degree than two. In fact, it has been proved that there are no general formulas representing the roots of algebraic equations of degree five and higher in terms of radicals. However, the graph $y = f(x)$ of an algebraic equation $f(x) = 0$ crosses the x-axis at each single-valued real root. Thus, by trial and error, consecutive integers can be found between which a root lies. Newton's method is a systematic way of using tangents to obtain a better approximation of a specific real root. The procedure is as follows. (See Fig. 4-7.)

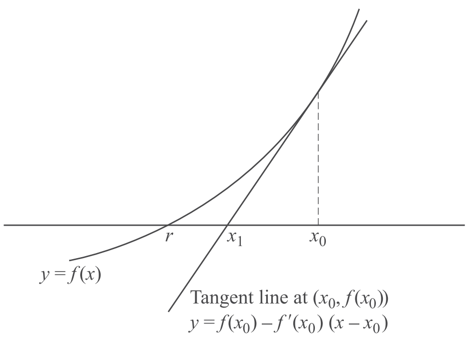


Fig. 4-7

Suppose that f has as many derivatives as required. Let r be a real root of $f(x) = 0$, i.e., $f(r) = 0$. Let x_0 be a value of x near r. For example, the integer preceding or following r. Let $f'(x_0)$ be the slope of the graph of $y = f(x)$ at $P_0[x_0, f(x_0)]$. Let $Q_1(x_1, 0)$ be the x-axis intercept of the tangent line at P_0 then

$$\frac{0 - f(x_0)}{x - x_0} = f'(x_0)$$

where the two representations of the slope of the tangent line have been equated. The solution of this relation for x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Starting with the tangent line to the graph at $P_1[x_1, f(x_1)]$ and repeating the process, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f(x_1)}{f'(x_1)}$$

and in general

$$x_n = x_0 - \sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)}$$

Under appropriate circumstances, the approximation x_n to the root r can be made as good as desired.

Note: Success with Newton's method depends on the shape of the function's graph in the neighborhood of the root. There are various cases which have not been explored here.

Solved Problems

DERIVATIVES

4.1. (*a*) Let $f(x) = \dfrac{3+x}{3-x}$, $x \neq 3$. Evaluate $f'(2)$ from the definition.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{5+h}{1-h} - 5\right) = \lim_{h\to 0} \frac{1}{h} \cdot \frac{6h}{1-h} = \lim_{h\to 0} \frac{6}{1-h} = 6$$

Note: By using rules of differentiation we find

$$f'(x) = \frac{(3-x)\dfrac{d}{dx}(3+x) - (3+x)\dfrac{d}{dx}(3-x)}{(3-x)^2} = \frac{(3-x)(1) - (3+x)(-1)}{(3-x)^2} = \frac{6}{(3-x)^2}$$

at all points x where the derivative exists. Putting $x = 2$, we find $f'(2) = 6$. Although such rules are often useful, one must be careful not to apply them indiscriminately (see Problem 4.5).

(*b*) Let $f(x) = \sqrt{2x-1}$. Evaluate $f'(5)$ from the definition.

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{\sqrt{9+2h} - 3}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{9+2h} - 3}{h} \cdot \frac{\sqrt{9+2h} + 3}{\sqrt{9+2h} + 3} = \lim_{h\to 0} \frac{9+2h-9}{h(\sqrt{9+2h}+3)} = \lim_{h\to 0} \frac{2}{\sqrt{9+2h}+3} = \frac{1}{3}$$

By using rules of differentiation we find $f'(x) = \dfrac{d}{dx}(2x-1)^{1/2} = \frac{1}{2}(2x-1)^{-1/2}\dfrac{d}{dx}(2x-1) = (2x-1)^{-1/2}$. Then $f'(5) = 9^{-1/2} = \frac{1}{3}$.

4.2. (*a*) Show directly from definition that the derivative of $f(x) = x^3$ is $3x^2$.

(*b*) Show from definition that $\dfrac{d}{dx}\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$.

(a) $$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}[(x+h)^3 - x^3]$$
$$= \frac{1}{h}[x^3 + 3x^2h + 3xh^2 + h^3] - x^3] = 3x^2 + 3xh + h^2$$

Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 3x^2$$

(b) $$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

The result follows by multiplying numerator and denominator by $\sqrt{x+h} - \sqrt{x}$ and then letting $h \to 0$.

4.3. If $f(x)$ has a derivative at $x = x_0$, prove that $f(x)$ must be continuous at $x = x_0$.

$$f(x_0+h) - f(x_0) = \frac{f(x_0+h)-f(x_0)}{h} \cdot h, \qquad h \neq 0$$

Then $$\lim_{h \to 0} f(x_0+h) - f(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \cdot \lim_{h \to 0} h = f'(x_0) \cdot 0 = 0$$

since $f'(x_0)$ exists by hypothesis. Thus

$$\lim_{h \to 0} f(x_0+h) - f(x_0) = 0 \qquad \text{or} \qquad \lim_{h \to 0} f(x_0+h) = f(x_0)$$

showing that $f(x)$ is continuous at $x = x_0$.

4.4. Let $f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

(a) Is $f(x)$ continuous at $x = 0$? (b) Does $f(x)$ have a derivative at $x = 0$?

(a) By Problem 3.22(b) of Chapter 3, $f(x)$ is continuous at $x = 0$.

(b) $$f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{f(h)-f(0)}{h} = \lim_{h \to 0} \frac{h \sin 1/h - 0}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$

which does not exist.

This example shows that even though a function is continuous at a point, it need not have a derivative at the point, i.e., the converse of the theorem in Problem 4.3 is not necessarily true.

It is possible to construct a function which is continuous at every point of an interval but has a derivative nowhere.

4.5. Let $f(x) = \begin{cases} x^2 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

(a) Is $f(x)$ differentiable at $x = 0$? (b) Is $f'(x)$ continuous at $x = 0$?

(a) $$f'(0) = \lim_{h \to 0} \frac{f(h)-f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin 1/h - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

by Problem 3.13, Chapter 3. Then $f(x)$ has a derivative (is differentiable) at $x = 0$ and its value is 0.

(b) From elementary calculus differentiation rules, if $x \neq 0$,

$$f'(x) = \frac{d}{dx}\left(x^2 \sin \frac{1}{x}\right) = x^2 \frac{d}{dx}\left(\sin \frac{1}{x}\right) + \left(\sin \frac{1}{x}\right) \frac{d}{dx}(x^2)$$
$$= x^2\left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \left(\sin \frac{1}{x}\right)(2x) = -\cos \frac{1}{x} + 2x \sin \frac{1}{x}$$

Since $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(-\cos\frac{1}{x} + 2x\sin\frac{1}{x}\right)$ does not exist (because $\lim_{x\to 0}\cos 1/x$ does not exist), $f'(x)$ cannot be continuous at $x = 0$ in spite of the fact that $f'(0)$ exists.

This shows that we cannot calculate $f'(0)$ in this case by simply calculating $f'(x)$ and putting $x = 0$, as is frequently supposed in elementary calculus. It is only when the derivative of a function is *continuous* at a point that this procedure gives the right answer. This happens to be true for most functions arising in elementary calculus.

4.6. Present an "ϵ, δ" definition of the derivative of $f(x)$ at $x = x_0$.

$f(x)$ has a derivative $f'(x_0)$ at $x = x_0$ if, given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left|\frac{f(x_0+h) - f(x_0)}{h} - f'(x_0)\right| < \epsilon \qquad \text{when} \qquad 0 < |h| < \delta$$

RIGHT- AND LEFT-HAND DERIVATIVES

4.7. Let $f(x) = |x|$. (*a*) Calculate the right-hand derivatives of $f(x)$ at $x = 0$. (*b*) Calculate the left-hand derivative of $f(x)$ at $x = 0$. (*c*) Does $f(x)$ have a derivative at $x = 0$? (*d*) Illustrate the conclusions in (*a*), (*b*), and (*c*) from a graph.

(*a*) $f'_+(0) = \lim_{h\to 0+}\frac{f(h) - f(0)}{h} = \lim_{h\to 0+}\frac{|h| - 0}{h} = \lim_{h\to 0+}\frac{h}{h} = 1$

since $|h| = h$ for $h > 0$.

(*b*) $f'_-(0) = \lim_{h\to 0-}\frac{f(h) - f(0)}{h} = \lim_{h\to 0-}\frac{|h| - 0}{h} = \lim_{h\to 0-}\frac{-h}{h} = -1$

since $|h| = -h$ for $h < 0$.

(*c*) No. The derivative at 0 does not exist if the right and left hand derivatives are unequal.

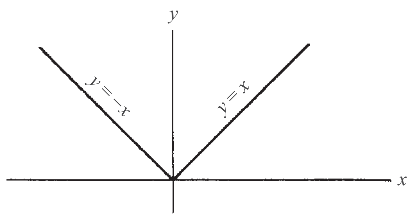


Fig. 4-8

(*d*) The required graph is shown in the adjoining Fig. 4-8.
Note that the slopes of the lines $y = x$ and $y = -x$ are 1 and -1 respectively, representing the right and left hand derivatives at $x = 0$. However, the derivative at $x = 0$ does not exist.

4.8. Prove that $f(x) = x^2$ is differentiable in $0 \leqq x \leqq 1$.

Let x_0 be any value such that $0 < x_0 < 1$. Then

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0}\frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h\to 0}(2x_0 + h) = 2x_0$$

At the end point $x = 0$,

$$f'_+(0) = \lim_{h\to 0+}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0+}\frac{h^2 - 0}{h} = \lim_{h\to 0+} h = 0$$

At the end point $x = 1$,

$$f'_-(1) = \lim_{h\to 0-}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0-}\frac{(1+h)^2 - 1}{h} = \lim_{h\to 0-}(2 + h) = 2$$

Then $f(x)$ is differentiable in $0 \leqq x \leqq 1$. We may write $f'(x) = 2x$ for any x in this interval. It is customary to write $f'_+(0) = f'(0)$ and $f'_-(1) = f'(1)$ in this case.

4.9. Find an equation for the tangent line to $y = x^2$ at the point where (*a*) $x = 1/3$, (*b*) $x = 1$.

(*a*) From Problem 4.8, $f'(x_0) = 2x_0$ so that $f'(1/3) = 2/3$. Then the equation of the tangent line is

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{or} \quad y - \tfrac{1}{9} = \tfrac{2}{3}(x - \tfrac{1}{3}), \qquad \text{i.e., } y = \tfrac{2}{3}x - \tfrac{1}{9}$$

(*b*) As in part (*a*), $y - f(1) = f'(1)(x - 1)$ or $y - 1 = 2(x - 1)$, i.e., $y = 2x - 1$.

DIFFERENTIALS

4.10. If $y = f(x) = x^3 - 6x$, find (*a*) Δy, (*b*) dy, (*c*) $\Delta y - dy$.

(*a*)
$$\begin{aligned}\Delta y = f(x + \Delta x) - f(x) &= \{(x + \Delta x)^3 - 6(x + \Delta x)\} - \{x^3 - 6x\} \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 6x - 6\Delta x - x^3 + 6x \\ &= (3x^2 - 6)\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\end{aligned}$$

(*b*) dy = principal part of $\Delta y = (3x^2 - 6)\Delta x = (3x^2 - 6)dx$, since by definition $\Delta x = dx$.
Note that $f'(x) = 3x^2 - 6$ and $dy = (3x^2 - 6)dx$, i.e., $dy/dx = 3x^2 - 6$. It must be emphasized that dy and dx are not necessarily small.

(*c*) From (*a*) and (*b*), $\Delta y - dy = 3x(\Delta x)^2 + (\Delta x)^3 = \epsilon\Delta x$, where $\epsilon = 3x\Delta x + (\Delta x)^2$.
Note that $\epsilon \to 0$ as $\Delta x \to 0$, i.e., $\dfrac{\Delta y - dy}{\Delta x} \to 0$ as $\Delta x \to 0$. Hence $\Delta y - dy$ is an infinitesimal of higher order than Δx (see Problem 4.83).
In case Δx is small, dy and Δy are approximately equal.

4.11. Evaluate $\sqrt[3]{25}$ approximately by use of differentials.

If Δx is small, $\Delta y = f(x + \Delta x) - f(x) = f'(x)\Delta x$ approximately.
Let $f(x) = \sqrt[3]{x}$. Then $\sqrt[3]{x + \Delta x} - \sqrt[3]{x} \approx \frac{1}{3}x^{-2/3}\Delta x$ (where $\approx$ denotes *approximately equal to*).
If $x = 27$ and $\Delta x = -2$, we have

$$\sqrt[3]{27 - 2} - \sqrt[3]{27} \approx \tfrac{1}{3}(27)^{-2/3}(-2), \qquad \text{i.e., } \sqrt[3]{25} - 3 \approx -2/27$$

Then $\sqrt[3]{25} \approx 3 - 2/27$ or 2.926.
If is interesting to observe that $(2.926)^3 = 25.05$, so that the approximation is fairly good.

DIFFERENTIATION RULES: DIFFERENTIATION OF ELEMENTARY FUNCTIONS

4.12. Prove the formula $\dfrac{d}{dx}\{f(x)g(x)\} = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$, assuming f and g are differentiable.

By definition,

$$\begin{aligned}\frac{d}{dx}\{f(x)g(x)\} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)\{g(x + \Delta x) - g(x)\} + g(x)\{f(x + \Delta x) - f(x)\}}{\Delta x} \\ &= \lim_{\Delta x \to 0} f(x + \Delta x)\left\{\frac{g(x + \Delta x) - g(x)}{\Delta x}\right\} + \lim_{\Delta x \to 0} g(x)\left\{\frac{f(x + \Delta x) - f(x)}{\Delta x}\right\} \\ &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)\end{aligned}$$

Another method:

Let $u = f(x)$, $v = g(x)$. Then $\Delta u = f(x + \Delta x) - f(x)$ and $\Delta v = g(x + \Delta x) - g(x)$, i.e., $f(x + \Delta x) = u + \Delta u$, $g(x + \Delta x) = v + \Delta v$. Thus

$$\begin{aligned}\frac{d}{dx}uv &= \lim_{\Delta x\to 0}\frac{(u+\Delta u)(v+\Delta v)-uv}{\Delta x} = \lim_{\Delta x\to 0}\frac{u\Delta v+v\Delta u+\Delta u\Delta v}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(u\frac{\Delta v}{\Delta x}+v\frac{\Delta u}{\Delta x}+\frac{\Delta u}{\Delta x}\Delta v\right) = u\frac{dv}{dx}+v\frac{du}{dx}\end{aligned}$$

where it is noted that $\Delta v \to 0$ as $\Delta x \to 0$, since v is supposed differentiable and thus continuous.

4.13. If $y = f(u)$ where $u = g(x)$, prove that $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$ assuming that f and g are differentiable.

Let x be given an increment $\Delta x \neq 0$. Then as a consequence u and y take on increments Δu and Δy respectively, where

$$\Delta y = f(u+\Delta u) - f(u), \qquad \Delta u = g(x+\Delta x) - g(x) \tag{1}$$

Note that as $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta u \to 0$.

If $\Delta u \neq 0$, let us write $\epsilon = \dfrac{\Delta y}{\Delta u} - \dfrac{dy}{du}$ so that $\epsilon \to 0$ as $\Delta u \to 0$ and

$$\Delta y = \frac{dy}{du}\Delta u + \epsilon\Delta u \tag{2}$$

If $\Delta u = 0$ for values of Δx, then (*1*) shows that $\Delta y = 0$ for these values of Δx. For such cases, we define $\epsilon = 0$.

It follows that in both cases, $\Delta u \neq 0$ or $\Delta u = 0$, (*2*) holds. Dividing (*2*) by $\Delta x \neq 0$ and taking the limit as $\Delta x \to 0$, we have

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{dy}{du}\frac{\Delta u}{\Delta x}+\epsilon\frac{\Delta u}{\Delta x}\right) = \frac{dy}{du}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}\epsilon\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\\ &= \frac{dy}{du}\frac{du}{dx}+0\cdot\frac{du}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}\end{aligned} \tag{3}$$

4.14. Given $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$, derive the formulas

(*a*) $\dfrac{d}{dx}(\tan x) = \sec^2 x$, (*b*) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

(*a*)
$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x}\\ &= \frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} =^2 x\end{aligned}$$

(*b*) If $y = \sin^{-1} x$, then $x = \sin y$. Taking the derivative with respect to x,

$$1 = \cos y\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

We have supposed here that the principal value $-\pi/2 \leqq \sin^{-1} x \leqq \pi/2$, is chosen so that $\cos y$ is positive, thus accounting for our writing $\cos y = \sqrt{1-\sin^2 y}$ rather than $\cos y = \pm\sqrt{1-\sin^2 y}$.

4.15. Derive the formula $\dfrac{d}{dx}(\log_a u) = \dfrac{\log_a e}{u}\dfrac{du}{dx}$ $(a > 0, a \neq 1)$, where u is a differentiable function of x.

Consider $y = f(u) = \log_a u$. By definition,

$$\begin{aligned}\frac{dy}{du} &= \lim_{\Delta u\to 0}\frac{f(u+\Delta u)-f(u)}{\Delta u} = \lim_{\Delta u\to 0}\frac{\log_a(u+\Delta u)-\log_a u}{\Delta u}\\ &= \lim_{\Delta u\to 0}\frac{1}{\Delta u}\log_a\left(\frac{u+\Delta u}{u}\right) = \lim_{\Delta u\to 0}\frac{1}{u}\log_a\left(1+\frac{\Delta u}{u}\right)^{u/\Delta u}\end{aligned}$$

Since the logarithm is a continuous function, this can be written

$$\frac{1}{u}\log_a\left\{\lim_{\Delta u\to 0}\left(1+\frac{\Delta u}{u}\right)^{u/\Delta u}\right\}=\frac{1}{u}\log_a e$$

by Problem 2.19, Chapter 2, with $x = u/\Delta u$.

Then by Problem 4.13, $\dfrac{d}{dx}(\log_a u) = \dfrac{\log_a e}{u}\dfrac{du}{dx}$.

4.16. Calculate dy/dx if (*a*) $xy^3 - 3x^2 = xy + 5$, (*b*) $e^{xy} + y\ln x = \cos 2x$.

(*a*) Differentiate with respect to x, considering y as a function of x. (We sometimes say that y is an *implicit function of* x, since we cannot solve explicitly for y in terms of x.) Then

$$\frac{d}{dx}(xy^3)-\frac{d}{dx}(3x^2)=\frac{d}{dx}(xy)+\frac{d}{dx}(5) \quad\text{or}\quad (x)(3y^2y')+(y^3)(1)-6x=(x)(y')+(y)(1)+0$$

where $y' = dy/dx$. Solving, $y' = (6x - y^3 + y)/(3xy^2 - x)$.

(*b*) $$\frac{d}{dx}(e^{xy})+\frac{d}{dx}(y\ln x)=\frac{d}{dx}(\cos 2x), \qquad e^{xy}(xy'+y)+\frac{y}{x}+(\ln x)y'=-2\sin 2x.$$

Solving, $$y'=-\frac{2x\sin 2x+xye^{xy}+y}{x^2e^{xy}+x\ln x}$$

4.17. If $y = \cosh(x^2 - 3x + 1)$, find (*a*) dy/dx, (*b*) d^2y/dx^2.

(*a*) Let $y = \cosh u$, where $u = x^2 - 3x + 1$. Then $dy/du = \sinh u$, $du/dx = 2x - 3$, and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=(\sinh u)(2x-3)=(2x-3)\sinh(x^2-3x+1)$$

(*b*) $$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\sinh u\frac{du}{dx}\right)=\sinh u\frac{d^2u}{dx^2}+\cosh u\left(\frac{du}{dx}\right)^2\\&=(\sinh u)(2)+(\cosh u)(2x-3)^2=2\sinh(x^2-3x+1)+(2x-3)^2\cosh(x^2-3x+1)\end{aligned}$$

4.18. If $x^2y + y^3 = 2$, find (*a*) y', (*b*) y'' at the point (1, 1).

(*a*) Differentiating with respect to x, $x^2y' + 2xy + 3y^2y' = 0$ and

$$y'=\frac{-2xy}{x^2+3xy^2}=-\frac{1}{2}\ \text{at}\ (1,1)$$

(*b*) $$y''=\frac{d}{dx}(y')=\frac{d}{dx}\left(\frac{-2xy}{x^2+3y^2}\right)=-\frac{(x^2+3y^2)(2xy'+2y)-(2xy)(2x+6yy')}{(x^2+3y^2)^2}$$

Substituting $x = 1$, $y = 1$, and $y' = -\frac{1}{2}$, we find $y'' = -\frac{3}{8}$.

MEAN VALUE THEOREMS

4.19. Prove Rolle's theorem.

Case 1: $f(x) \equiv 0$ in $[a, b]$. Then $f'(x) = 0$ for all x in (a, b).

Case 2: $f(x) \not\equiv 0$ in $[a, b]$. Since $f(x)$ is continuous there are points at which $f(x)$ attains its maximum and minimum values, denoted by M and m respectively (see Problem 3.34, Chapter 3).

Since $f(x) \not\equiv 0$, at least one of the values M, m is not zero. Suppose, for example, $M \neq 0$ and that $f(\xi) = M$ (see Fig. 4-9). For this case, $f(\xi + h) \leqq f(\xi)$.

If $h > 0$, then $\dfrac{f(\xi+h)-f(\xi)}{h} \leqq 0$ and

$$\lim_{h\to 0+} \frac{f(\xi+h)-f(\xi)}{h} \leqq 0 \qquad (1)$$

If $h < 0$, then $\dfrac{f(\xi+h)-f(\xi)}{h} \geqq 0$ and

$$\lim_{h\to 0-} \frac{f(\xi+h)-f(\xi)}{h} \geqq 0 \qquad (2)$$

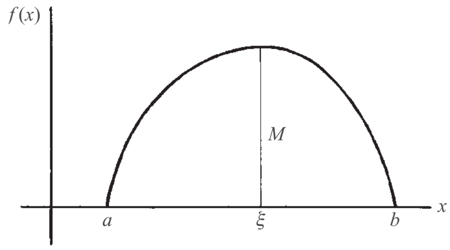


Fig. 4-9

But by hypothesis $f(x)$ has a derivative at all points in (a, b). Then the right-hand derivative (*1*) must be equal to the left-hand derivative (*2*). This can happen only if they are both equal to zero, in which case $f'(\xi) = 0$ as required.

A similar argument can be used in case $M = 0$ and $m \neq 0$.

4.20. Prove the mean value theorem.

Define $F(x) = f(x) - f(a) - (x-a)\dfrac{f(b)-f(a)}{b-a}$.

Then $F(a) = 0$ and $F(b) = 0$.

Also, if $f(x)$ satisfies the conditions on continuity and differentiability specified in Rolle's theorem, then $F(x)$ satisfies them also.

Then applying Rolle's theorem to the function $F(x)$, we obtain

$$F'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a} = 0, \quad a < \xi < b \qquad \text{or} \qquad f'(\xi) = \frac{f(b)-f(a)}{b-a}, \quad a < \xi < b$$

4.21. Verify the mean value theorem for $f(x) = 2x^2 - 7x + 10$, $a = 2$, $b = 5$.

$f(2) = 4, f(5) = 25, f'(\xi) = 4\xi - 7$. Then the mean value theorem states that $4\xi - 7 = (25-4)/(5-2)$ or $\xi = 3.5$. Since $2 < \xi < 5$, the theorem is verified.

4.22. If $f'(x) = 0$ at all points of the interval (a, b), prove that $f(x)$ must be a constant in the interval.

Let $x_1 < x_2$ be any two different points in (a, b). By the mean value theorem for $x_1 < \xi < x_2$,

$$\frac{f(x_2)-f(x_1)}{x_2 - x_1} = f'(\xi) = 0$$

Thus, $f(x_1) = f(x_2) =$ constant. From this it follows that if two functions have the same derivative at all points of (a, b), the functions can only differ by a constant.

4.23. If $f'(x) > 0$ at all points of the interval (a, b), prove that $f(x)$ is strictly increasing.

Let $x_1 < x_2$ be any two different points in (a, b). By the mean value theorem for $x_1 < \xi < x_2$,

$$\frac{f(x_2)-f(x_1)}{x_2 - x_1} = f'(\xi) > 0$$

Then $f(x_2) > f(x_1)$ for $x_2 > x_1$, and so $f(x)$ is strictly increasing.

4.24. (*a*) Prove that $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$ if $a < b$.

(*b*) Show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$.

(*a*) Let $f(x) = \tan^{-1} x$. Since $f'(x) = 1/(1+x^2)$ and $f'(\xi) = 1/(1+\xi^2)$, we have by the mean value theorem

$$\frac{\tan^{-1} b - \tan^{-1} a}{b-a} = \frac{1}{1+\xi^2} \qquad a < \xi < b$$

Since $\xi > a$, $1/(1+\xi^2) < 1/(1+a^2)$. Since $\xi < b$, $1/(1+\xi^2) > 1/(1+b^2)$. Then

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

and the required result follows on multiplying by $b-a$.

(*b*) Let $b = 4/3$ and $a = 1$ in the result of part (*a*). Then since $\tan^{-1} 1 = \pi/4$, we have

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{1}{6} \qquad \text{or} \qquad \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

4.25. Prove Cauchy's generalized mean value theorem.

Consider $G(x) = f(x) - f(a) - \alpha\{g(x) - g(a)\}$, where α is a constant. Then $G(x)$ satisfies the conditions of Rolle's theorem, provided $f(x)$ and $g(x)$ satisfy the continuity and differentiability conditions of Rolle's theorem and if $G(a) = G(b) = 0$. Both latter conditions are satisfied if the constant $\alpha = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

Applying Rolle's theorem, $G'(\xi) = 0$ for $a < \xi < b$, we have

$$f'(\xi) - \alpha g'(\xi) = 0 \qquad \text{or} \qquad \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \qquad a < \xi < b$$

as required.

L'HOSPITAL'S RULE

4.26. Prove L'Hospital's rule for the case of the "indeterminate forms" (*a*) $0/0$, (*b*) ∞/∞.

(*a*) We shall suppose that $f(x)$ and $g(x)$ are differentiable in $a < x < b$ and $f(x_0) = 0$, $g(x_0) = 0$, where $a < x_0 < b$.

By Cauchy's generalized mean value theorem (Problem 25),

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)} \qquad x_0 < \xi < x$$

Then

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = \lim_{x \to x_0+} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to x_0+} \frac{f'(x)}{g'(x)} = L$$

since as $x \to x_0+$, $\xi \to x_0+$.

Modification of the above procedure can be used to establish the result if $x \to x_0-$, $x \to x_0$, $x \to \infty$, $x \to -\infty$.

(*b*) We suppose that $f(x)$ and $g(x)$ are differentiable in $a < x < b$, and $\lim_{x \to x_0+} f(x) = \infty$, $\lim_{x \to x_0+} g(x) = \infty$ where $a < x_0 < b$.

Assume x_1 is such that $a < x_0 < x < x_1 < b$. By Cauchy's generalized mean value theorem,

$$\frac{f(x) - f(x_1)}{g(x) - g(x_1)} = \frac{f'(\xi)}{g'(\xi)} \qquad x < \xi < x_1$$

Hence

$$\frac{f(x) - f(x_1)}{g(x) - g(x_1)} = \frac{f(x)}{g(x)} \cdot \frac{1 - f(x_1)/f(x)}{1 - g(x_1)/g(x)} = \frac{f'(\xi)}{g'(\xi)}$$

from which we see that

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)} \tag{1}$$

Let us now suppose that $\lim_{x \to x_0+} \frac{f'(x)}{g'(x)} = L$ and write (*1*) as

$$\frac{f(x)}{g(x)} = \left(\frac{f'(\xi)}{g'(\xi)} - L\right)\left(\frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)}\right) + L\left(\frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)}\right) \tag{2}$$

We can choose x_1 so close to x_0 that $|f'(\xi)/g'(\xi) - L| < \epsilon$. Keeping x_1 fixed, we see that

$$\lim_{x \to x_0+}\left(\frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)}\right) = 1 \qquad \text{since } \lim_{x \to x_0+} f(x) = \infty \text{ and } \lim_{x \to x_0} g(x) = \infty$$

Then taking the limit as $x \to x_0+$ on both sides of (*2*), we see that, as required,

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = L = \lim_{x \to x_0+} \frac{f'(x)}{g'(x)}$$

Appropriate modifications of the above procedure establish the result if $x \to x_0-$, $x \to x_0$, $x \to \infty$, $x \to -\infty$.

4.27. Evaluate (*a*) $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$ (*b*) $\lim_{x \to 1} \frac{1 + \cos \pi x}{x^2 - 2x + 1}$

All of these have the "indeterminate form" 0/0.

(*a*) $\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2$

(*b*) $\lim_{x \to 1} \frac{1 + \cos \pi x}{x^2 - 2x + 1} = \lim_{x \to 1} \frac{-\pi \sin \pi x}{2x - 2} = \lim_{x \to 1} \frac{-\pi^2 \cos \pi x}{2} = \frac{\pi^2}{2}$

Note: Here L'Hospital's rule is applied twice, since the first application again yields the "indeterminate form" 0/0 and the conditions for L'Hospital's rule are satisfied once more.

4.28. Evaluate (*a*) $\lim_{x \to \infty} \frac{3x^2 - x + 5}{5x^2 + 6x - 3}$ (*b*) $\lim_{x \to \infty} x^2 e^{-x}$

All of these have or can be arranged to have the "indeterminate form" ∞/∞.

(*a*) $\lim_{x \to \infty} \frac{3x^2 - x + 5}{5x^2 + 6x - 3} = \lim_{x \to \infty} \frac{6x - 1}{10x + 6} = \lim_{x \to \infty} \frac{6}{10} = \frac{3}{5}$

(*b*) $\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0$

4.29. Evaluate $\lim_{x \to 0+} x^2 \ln x$.

$$\lim_{x \to 0+} x^2 \ln x = \lim_{x \to 0+} \frac{\ln x}{1/x^2} = \lim_{x \to 0+} \frac{1/x}{-2/x^3} = \lim_{x \to 0+} \frac{-x^2}{2} = 0$$

The given limit has the "indeterminate form" $0 \cdot \infty$. In the second step the form is altered so as to give the indeterminate form ∞/∞ and L'Hospital's rule is then applied.

4.30. Find $\lim_{x \to 0} (\cos x)^{1/x^2}$.

Since $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1/x^2 = \infty$, the limit takes the "indeterminate form" 1^∞.

Let $F(x) = (\cos x)^{1/x^2}$. Then $\ln F(x) = (\ln \cos x)/x^2$ to which L'Hospital's rule can be applied. We have

$$\lim_{x\to 0} \frac{\ln \cos x}{x^2} = \lim_{x\to 0} \frac{(-\sin x)/(\cos x)}{2x} = \lim_{x\to 0} \frac{-\sin x}{2x\cos x} = \lim_{x\to 0} \frac{-\cos x}{-2x\sin x + 2\cos x} = -\frac{1}{2}$$

Thus, $\lim_{x\to 0} \ln F(x) = -\frac{1}{2}$. But since the logarithm is a continuous function, $\lim_{x\to 0} \ln F(x) = \ln(\lim_{x\to 0} F(x))$. Then

$$\ln(\lim_{x\to 0} F(x)) = -\tfrac{1}{2} \qquad \text{or} \qquad \lim_{x\to 0} F(x) = \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

4.31. If $F(x) = (e^{3x} - 5x)^{1/x}$, find (*a*) $\lim_{x\to 0} F(x)$ and (*b*) $\lim_{x\to 0} F(x)$.

The respective indeterminate forms in (*a*) and (*b*) are ∞^0 and 1^∞.

Let $G(x) = \ln F(x) = \dfrac{\ln(e^{3x} - 5x)}{x}$. Then $\lim_{x\to\infty} G(x)$ and $\lim_{x\to 0} G(x)$ assume the indeterminate forms ∞/∞ and $0/0$ respectively, and L'Hospital's rule applies. We have

(*a*) $$\lim_{x\to\infty} \frac{\ln(e^{3x} - 5x)}{x} = \lim_{x\to\infty} \frac{3e^{3x} - 5}{e^{3x} - 5x} = \lim_{x\to 0} \frac{9e^{3x}}{3e^{3x} - 5} = \lim_{x\to\infty} \frac{27e^{3x}}{9e^{3x}} = 3$$

Then, as in Problem 4.30, $\lim_{x\to\infty} (e^{3x} - 5x)^{1/x} = e^3$.

(*b*) $$\lim_{x\to 0} \frac{\ln(e^{3x} - 5x)}{x} = \lim_{x\to 0} \frac{3e^{3x} - 5}{e^{3x} - 5x} = -2 \qquad \text{and} \qquad \lim_{x\to 0} (e^{3x} - 5x)^{1/x} = e^{-2}$$

4.32. Suppose the equation of motion of a particle is $x = \sin(c_1 t + c_2)$, where c_1 and c_2 are constants. (Simple harmonic motion.) (*a*) Show that the acceleration of the particle is proportional to its distance from the origin. (*b*) If $c_1 = 1$, $c_2 = \pi$, and $t \geq 0$, determine the velocity and acceleration at the end points and at the midpoint of the motion.

(*a*) $$\frac{dx}{dt} = c_1 \cos(c_1 t + c_2), \qquad \frac{d^2x}{dt^2} = -c_1^2 \sin(c_1 t + c_2) = -c_1^2 x.$$

This relation demonstrates the proportionality of acceleration and distance.

(*b*) The motion starts at 0 and moves to -1. Then it oscillates between this value and 1. The absolute value of the velocity is zero at the end points, and that of the acceleration is maximum there. The particle coasts through the origin (zero acceleration), while the absolute value of the velocity is maximum there.

4.33. Use Newton's method to determine $\sqrt{3}$ to three decimal points of accuracy.

$\sqrt{3}$ is a solution of $x^2 - 3 = 0$, which lies between 1 and 2. Consider $f(x) = x^2 - 3$ then $f'(x) = 2x$. The graph of f crosses the x-axis between 1 and 2. Let $x_0 = 2$. Then $f(x_0) = 1$ and $f'(x_0) = 1.75$. According to the Newton formula, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - .25 = 1.75$.

Then $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.732$. To verify the three decimal point accuracy, note that $(1.732)^2 = 2.9998$ and $(1.7333)^2 = 3.0033$.

MISCELLANEOUS PROBLEMS

4.34. If $x = g(t)$ and $y = f(t)$ are twice differentiable, find (*a*) dy/dx, (*b*) d^2y/dx^2.

(*a*) Letting primes denote derivatives with respect to t, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)} \qquad \text{if } g'(t) \neq 0$$

(*b*)
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{f'(t)}{g'(t)}\right) = \frac{\frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)}{g'(t)}$$
$$= \frac{1}{g'(t)}\left\{\frac{g'(t)f''(t) - f'(t)g''(t)}{[g'(t)]^2}\right\} = \frac{g'(t)f''(t) - f'(t)g''(t)}{[g'(t)]^3} \qquad \text{if } g'(t) \neq 0$$

4.35. Let $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Prove that (*a*) $f'(0) = 0$, (*b*) $f''(0) = 0$.

(*a*)
$$f'_+(0) = \lim_{h\to 0+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0+} \frac{e^{-1/h^2} - 0}{h} = \lim_{h\to 0+} \frac{e^{-1/h^2}}{h}$$

If $h = 1/u$, using L'Hospital's rule this limit equals

$$\lim_{u\to\infty} ue^{-u^2} = \lim_{u\to\infty} u/e^{u^2} = \lim_{u\to\infty} 1/2ue^{u^2} = 0$$

Similarly, replacing $h \to 0+$ by $h \to 0-$ and $u \to \infty$ by $u \to -\infty$, we find $f'_-(0) = 0$. Thus $f'_+(0) = f'_-(0) = 0$, and so $f'(0) = 0$.

(*b*)
$$f''_+(0) = \lim_{h\to 0+} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0+} \frac{e^{-1/h^2} \cdot 2h^{-3} - 0}{h} = \lim_{h\to 0+} \frac{2e^{-1/h^2}}{h^4} = \lim_{u\to\infty} \frac{2u^4}{e^{u^2}} = 0$$

by successive applications of L'Hospital's rule.

Similarly, $f''_-(0) = 0$ and so $f''(0) = 0$.

In general, $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$

4.36. Find the length of the longest ladder which can be carried around the corner of a corridor, whose dimensions are indicated in the figure below, if it is assumed that the ladder is carried parallel to the floor.

The length of the *longest* ladder is the same as the *shortest* straight line segment AB [Fig. 4-10], which touches both outer walls and the corner formed by the inner walls.

As seen from Fig. 4-10, the length of the ladder AB is

$$L = a\sec\theta + b\csc\theta$$

L is a minimum when

$$dL/d\theta = a\sec\theta\tan\theta - b\csc\theta\cot\theta = 0$$

i.e., $\quad a\sin^3\theta = b\cos^3\theta \quad$ or $\quad \tan\theta = \sqrt[3]{b/a}$

Then $\quad \sec\theta = \dfrac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}}, \quad \csc\theta = \dfrac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}$

so that $\quad L = a\sec\theta + b\csc\theta = (a^{2/3} + b^{2/3})^{3/2}$

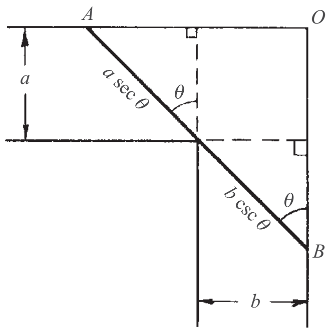


Fig. 4-10

Although it is geometrically evident that this gives the minimum length, we can prove this analytically by showing that $d^2L/d\theta^2$ for $\theta = \tan^{-1}\sqrt[3]{b/a}$ is positive (see Problem 4.78).

Supplementary Problems

DERIVATIVES

4.37. Use the definition to compute the derivatives of each of the following functions at the indicated point: (*a*) $(3x-4)/(2x+3), x=1$; (*b*) $x^3-3x^2+2x-5, x=2$; (*c*) $\sqrt{x}, x=4$; (*d*) $\sqrt[3]{6x-4}$, $x=2$.
Ans. (*a*) 17/25, (*b*) 2, (*c*) $\frac{1}{4}$, (*d*) $\frac{1}{2}$

4.38. Show from definition that (*a*) $\dfrac{d}{dx}x^4=4x^3$, (*b*) $\dfrac{d}{dx}\dfrac{3+x}{3-x}=\dfrac{6}{(3-x)^2}$, $x\neq 3$

4.39. Let $f(x)=\begin{cases} x^3\sin 1/x, & x\neq 0\\ 0, & x=0\end{cases}$. Prove that (*a*) $f(x)$ is continuous at $x=0$, (*b*) $f(x)$ has a derivative at $x=0$, (*c*) $f'(x)$ is continuous at $x=0$.

4.40. Let $f(x)=\begin{cases} xe^{-1/x^2}, & x\neq 0\\ 0, & x=0\end{cases}$. Determine whether $f(x)$ (*a*) is continuous at $x=0$, (*b*) has a derivative at $x=0$.

Ans. (*a*) Yes; (*b*) Yes, 0

4.41. Give an alternative proof of the theorem in Problem 4.3, Page 76, using "ϵ, δ definitions".

4.42. If $f(x)=e^x$, show that $f'(x_0)=e^{x_0}$ depends on the result $\lim\limits_{h\to 0}(e^h-1)/h=1$.

4.43. Use the results $\lim\limits_{h\to 0}(\sin h)/h=1$, $\lim\limits_{h\to 0}(1-\cos h)/h=0$ to prove that if $f(x)=\sin x$, $f'(x_0)=\cos x_0$.

RIGHT- AND LEFT-HAND DERIVATIVES

4.44. Let $f(x)=x|x|$. (*a*) Calculate the right-hand derivative of $f(x)$ at $x=0$. (*b*) Calculate the left-hand derivative of $f(x)$ at $x=0$. (*c*) Does $f(x)$ have a derivative at $x=0$? (*d*) Illustrate the conclusions in (*a*), (*b*), and (*c*) from a graph.
Ans. (*a*) 0; (*b*) 0; (*c*) Yes, 0

4.45. Discuss the (*a*) continuity and (*b*) differentiability of $f(x)=x^p\sin 1/x, f(0)=0$, where p is any positive number. What happens in case p is any real number?

4.46. Let $f(x)=\begin{cases} 2x-3, & 0\leqq x\leqq 2\\ x^2-3, & 2<x\leqq 4\end{cases}$. Discuss the (*a*) continuity and (*b*) differentiability of $f(x)$ in $0\leqq x\leqq 4$.

4.47. Prove that the derivative of $f(x)$ at $x=x_0$ exists if and only if $f'_+(x_0)=f'_-(x_0)$.

4.48. (*a*) Prove that $f(x)=x^3-x^2+5x-6$ is differentiable in $a\leqq x\leqq b$, where a and b are any constants. (*b*) Find equations for the tangent lines to the curve $y=x^3-x^2+5x-6$ at $x=0$ and $x=1$. Illustrate by means of a graph. (*c*) Determine the point of intersection of the tangent lines in (*b*). (*d*) Find $f'(x), f''(x), f'''(x), f^{(\mathrm{IV})}(x), \ldots$.
Ans. (*b*) $y=5x-6, y=6x-7$; (*c*) $(1,-1)$; (*d*) $3x^2-2x+5, 6x-2, 6, 0, 0, 0, \ldots$

4.49. If $f(x)=x^2|x|$, discuss the existence of successive derivatives of $f(x)$ at $x=0$.

DIFFERENTIALS

4.50. If $y=f(x)=x+1/x$, find (*a*) Δy, (*b*) dy, (*c*) $\Delta y-dy$, (*d*) $(\Delta y-dy)/\Delta x$, (*e*) dy/dx.

Ans. (*a*) $\Delta x-\dfrac{\Delta x}{x(x+\Delta x)}$, (*b*) $\left(1-\dfrac{1}{x^2}\right)\Delta x$, (*c*) $\dfrac{(\Delta x)^2}{x^2(x+\Delta x)}$, (*d*) $\dfrac{\Delta x}{x^2(x+\Delta x)}$, (*e*) $1-\dfrac{1}{x^2}$.

Note: $\Delta x=dx$.

4.51. If $f(x) = x^2 + 3x$, find (*a*) Δy, (*b*) dy, (*c*) $\Delta y/\Delta x$, (*d*) dy/dx, and (*e*) $(\Delta y - dy)/\Delta x$, if $x = 1$ and $\Delta x = .01$.
Ans. (*a*) .0501, (*b*) .05, (*c*) 5.01, (*d*) 5, (*e*) .01

4.52. Using differentials, compute approximate values for each of the following: (*a*) $\sin 31^\circ$, (*b*) $\ln(1.12)$, (*c*) $\sqrt[5]{36}$.
Ans. (*a*) 0.515, (*b*) 0.12, (*c*) 2.0125

4.53. If $y = \sin x$, evaluate (*a*) Δy, (*b*) dy. (*c*) Prove that $(\Delta y - dy)/\Delta x \to 0$ as $\Delta x \to 0$.

DIFFERENTIATION RULES AND ELEMENTARY FUNCTIONS

4.54. Prove: (*a*) $\dfrac{d}{dx}\{f(x) + g(x)\} = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$, (*b*) $\dfrac{d}{dx}\{f(x) - g(x)\} = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$,
(*c*) $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

4.55. Evaluate (*a*) $\dfrac{d}{dx}\{x^3 \ln(x^2 - 2x + 5)\}$ at $x = 1$, (*b*) $\dfrac{d}{dx}\{\sin^2(3x + \pi/6)\}$ at $x = 0$.
Ans. (*a*) $3 \ln 4$, (*b*) $\frac{3}{2}\sqrt{3}$

4.56. Derive the formulas: (*a*) $\dfrac{d}{dx}a^u = a^u \ln a \dfrac{du}{dx}$, $a > 0, a \neq 1$; (*b*) $\dfrac{d}{dx}\csc u = -\csc u \cot u \dfrac{du}{dx}$;
(*c*) $\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$ where u is a differentiable function of x.

4.57. Compute (*a*) $\dfrac{d}{dx}\tan^{-1} x$, (*b*) $\dfrac{d}{dx}\csc^{-1} x$, (*c*) $\dfrac{d}{dx}\sinh^{-1} x$, (*d*) $\dfrac{d}{dx}\coth^{-1} x$, paying attention to the use of principal values.

4.58. If $y = x^x$, computer dy/dx. [Hint: Take logarithms before differentiating.]
Ans. $x^x(1 + \ln x)$

4.59. If $y = \{\ln(3x + 2)\}^{\sin^{-1}(2x+.5)}$, find dy/dx at $x = 0$.
Ans. $\left(\dfrac{\pi}{4 \ln 2} + \dfrac{2 \ln \ln 2}{\sqrt{3}}\right)(\ln 2)^{\pi/6}$

4.60. If $y = f(u)$, where $u = g(v)$ and $v = h(x)$, prove that $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$ assuming f, g, and h are differentiable.

4.61. Calculate (*a*) dy/dx and (*b*) d^2y/dx^2 if $xy - \ln y = 1$.
Ans. (*a*) $y^2/(1 - xy)$, (*b*) $(3y^3 - 2xy^4)/(1 - xy)^3$ provided $xy \neq 1$

4.62. If $y = \tan x$, prove that $y''' = 2(1 + y^2)(1 + 3y^2)$.

4.63. If $x = \sec t$ and $y = \tan t$, evaluate (*a*) dy/dx, (*b*) d^2y/dx^2, (*c*) d^3y/dx^3, at $t = \pi/4$.
Ans. (*a*) $\sqrt{2}$, (*b*) -1, (*c*) $3\sqrt{2}$

4.64. Prove that $\dfrac{d^2y}{dx^2} = -\dfrac{d^2x}{dy^2} \Big/ \left(\dfrac{dx}{dy}\right)^3$, stating precise conditions under which it holds.

4.65. Establish formulas (*a*) 7, (*b*) 18, and (*c*) 27, on Page 71.

MEAN VALUE THEOREMS

4.66. Let $f(x) = 1 - (x - 1)^{2/3}$, $0 \leqq x \leqq 2$. (*a*) Construct the graph of $f(x)$. (*b*) Explain why Rolle's theorem is not applicable to this function, i.e., there is no value ξ for which $f'(\xi) = 0$, $0 < \xi < 2$.

4.67. Verify Rolle's theorem for $f(x) = x^2(1-x)^2$, $0 \leqq x \leqq 1$.

4.68. Prove that between any two real roots of $e^x \sin x = 1$ there is at least one real root of $e^x \cos x = -1$. [Hint: Apply Rolle's theorem to the function $e^{-x} - \sin x$.]

4.69. (*a*) If $0 < a < b$, prove that $(1 - a/b) < \ln b/a < (b/a - 1)$
(*b*) Use the result of (*a*) to show that $\frac{1}{6} < \ln 1.2 < \frac{1}{5}$.

4.70. Prove that $(\pi/6 + \sqrt{3}/15) < \sin^{-1} .6 < (\pi/6 + 1/8)$ by using the mean value theorem.

4.71. Show that the function $F(x)$ in Problem 4.20(*a*) represents the difference in ordinants of curve ACB and line AB at any point x in (a, b).

4.72. (*a*) If $f'(x) \leqq 0$ at all points of (a, b), prove that $f(x)$ is monotonic decreasing in (a, b).
(*b*) Under what conditions is $f(x)$ strictly decreasing in (a, b)?

4.73. (*a*) Prove that $(\sin x)/x$ is strictly decreasing in $(0, \pi/2)$. (*b*) Prove that $0 \leqq \sin x \leqq 2x/\pi$ for $0 \leqq x \leqq \pi/2$.

4.74. (*a*) Prove that $\dfrac{\sin b - \sin a}{\cos a - \cos b} = \cot \xi$, where ξ is between a and b.
(*b*) By placing $a = 0$ and $b = x$ in (*a*), show that $\xi = x/2$. Does the result hold if $x < 0$?

L'HOSPITAL'S RULE

4.75. Evaluate each of the following limits.

(*a*) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$ (*e*) $\lim\limits_{x\to 0+} x^3 \ln x$ (*i*) $\lim\limits_{x\to 0}(1/x - \csc x)$ (*m*) $\lim\limits_{x\to\infty} x \ln\left(\dfrac{x+3}{x-3}\right)$

(*b*) $\lim\limits_{x\to 0} \dfrac{e^{2x} - 2e^x + 1}{\cos 3x - 2\cos 2x + \cos x}$ (*f*) $\lim\limits_{x\to 0}(3^x - 2^x)/x$ (*j*) $\lim\limits_{x\to 0} x^{\sin x}$ (*n*) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$

(*c*) $\lim\limits_{x\to 1+} (x^2 - 1)\tan \pi x/2$ (*g*) $\lim\limits_{x\to\infty}(1 - 3/x)^{2x}$ (*k*) $\lim\limits_{x\to 0}(1/x^2 - \cot^2 x)$ (*o*) $\lim\limits_{x\to\infty}(x + e^x + e^{2x})^{1/x}$

(*d*) $\lim\limits_{x\to\infty} x^3 e^{-2x}$ (*h*) $\lim\limits_{x\to\infty}(1 + 2x)^{1/3x}$ (*l*) $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x - \sin^{-1} x}{x(1 - \cos x)}$ (*p*) $\lim\limits_{x\to 0+}(\sin x)^{1/\ln x}$

Ans. (*a*) $\frac{1}{6}$, (*b*) -1, (*c*) $-4/\pi$, (*d*) 0, (*e*) 0, (*f*) $\ln 3/2$, (*g*) e^{-6}, (*h*) 1, (*i*) 0, (*j*) 1, (*k*) $\frac{2}{3}$, (*l*) $\frac{1}{3}$, (*m*) 6, (*n*) $e^{-1/6}$, (*o*) e^2, (*p*) e

MISCELLANEOUS PROBLEMS

4.76. Prove that $\sqrt{\dfrac{1-x}{1+x}} < \dfrac{\ln(1+x)}{\sin^{-1} x} < 1$ if $0 < x < 1$.

4.77. If $\Delta f(x) = f(x + \Delta x) - f(x)$, (*a*) Prove that $\Delta\{\Delta f(x)\} = \Delta^2 f(x) = f(x + 2\Delta x) - 2f(x + \Delta x) + f(x)$, (*b*) derive an expression for $\Delta^n f(x)$ where n is any positive integer, (*c*) show that $\lim\limits_{\Delta x\to 0} \dfrac{\Delta^n f(x)}{(\Delta x)^n} = f^{(n)}(x)$ if this limit exists.

4.78. Complete the analytic proof mentioned at the end of Problem 4.36.

4.79. Find the relative maximum and minima of $f(x) = x^2$, $x > 0$.
Ans. $f(x)$ has a relative minimum when $x = e^{-1}$.

4.80. A train moves according to the rule $x = 5t^3 + 30t$, where t and x are measured in hours and miles, respectively. (*a*) What is the acceleration after 1 minute? (*b*) What is the speed after 2 hours?

4.81. A stone thrown vertically upward has the law of motion $x = -16t^2 + 96t$. (Assume that the stone is at ground level at $t = 0$, that t is measured in seconds, and that x is measured in feet.) (*a*) What is the height of the stone at $t = 2$ seconds? (*b*) To what height does the stone rise? (*c*) What is the initial velocity, and what is the maximum speed attained?

4.82. A particle travels with constant velocities v_1 and v_2 in mediums I and II, respectively (see adjoining Fig. 4-11). Show that in order to go from point P to point Q in the least time, it must follow path PAQ where A is such that

$$(\sin\theta_1)/(\sin\theta_2) = v_1/v_2$$

Note: This is Snell's Law; a fundamental law of optics first discovered experimentally and then derived mathematically.

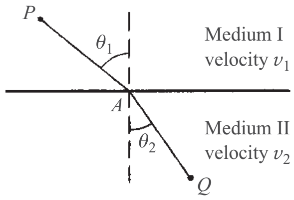


Fig. 4-11

4.83. A variable α is called an *infinitesimal* if it has zero as a limit. Given two infinitesimals α and β, we say that α is an infinitesimal of *higher order* (or the *same order*) if $\lim \alpha/\beta = 0$ (or $\lim \alpha/\beta = l \neq 0$). Prove that as $x \to 0$, (*a*) $\sin^2 2x$ and $(1 - \cos 3x)$ are infinitesimals of the same order, (*b*) $(x^3 - \sin^3 x)$ is an infinitesimal of higher order than $\{x - \ln(1+x) - 1 + \cos x\}$.

4.84. Why can we not use L'Hospital's rule to prove that $\lim\limits_{x\to 0} \dfrac{x^2 \sin 1/x}{\sin x} = 0$ (see Problem 3.91, Chap. 3)?

4.85. Can we use L'Hospital's rule to evaluate the limit of the sequence $u_n = n^3 e^{-n^2}$, $n = 1, 2, 3, \ldots$? Explain.

4.86 (1) Determine decimal approximations with at least three places of accuracy for each of the following irrational numbers. (*a*) $\sqrt{2}$, (*b*) $\sqrt{5}$, (*c*) $7^{1/3}$
(2) The cubic equation $x^3 - 3x^2 + x - 4 = 0$ has a root between 3 and 4. Use Newton's Method to determine it to at least three places of accuracy.

4.87. Using successive applications of Newton's method obtain the positive root of (*a*) $x^3 - 2x^2 - 2x - 7 = 0$, (*b*) $5 \sin x = 4x$ to 3 decimal places.
Ans. (*a*) 3.268, (*b*) 1.131

4.88. If D denotes the operator d/dx so that $Dy \equiv dy/dx$ while $D^k y \equiv d^k y/dx^k$, prove *Leibnitz's formula*

$$D^n(uv) = (D^n u)v + {}_nC_1(D^{n-1}u)(Dv) + {}_nC_2(D^{n-2}u)(D^2 v) + \cdots + {}_nC_r(D^{n-r}u)(D^r v) + \cdots + uD^n v$$

where ${}_nC_r = \binom{n}{r}$ are the binomial coefficients (see Problem 1.95, Chapter 1).

4.89. Prove that $\dfrac{d^n}{dx^n}(x^2 \sin x) = \{x^2 - n(n-1)\}\sin(x + n\pi/2) - 2nx\cos(x + n\pi/2)$.

4.90. If $f'(x_0) = f''(x_0) = \cdots = f^{(2n)}(x_0) = 0$ but $f^{(2n+1)}(x_0) \neq 0$, discuss the behavior of $f(x)$ in the neighborhood of $x = x_0$. The point x_0 in such case is often called a *point of inflection*. This is a generalization of the previously discussed case corresponding to $n = 1$.

4.91. Let $f(x)$ be twice differentiable in (a, b) and suppose that $f'(a) = f'(b) = 0$. Prove that there exists at least one point ξ in (a, b) such that $|f''(\xi)| \geqq \dfrac{4}{(b-a)^2}\{f(b) - f(a)\}$. Give a physical interpretation involving velocity and acceration of a particle.

CHAPTER 5

Integrals

INTRODUCTION OF THE DEFINITE INTEGRAL

The geometric problems that motivated the development of the integral calculus (determination of lengths, areas, and volumes) arose in the ancient civilizations of Northern Africa. Where solutions were found, they related to concrete problems such as the measurement of a quantity of grain. Greek philosophers took a more abstract approach. In fact, Eudoxus (around 400 B.C.) and Archimedes (250 B.C.) formulated ideas of integration as we know it today.

Integral calculus developed independently, and without an obvious connection to differential calculus. The calculus became a "whole" in the last part of the seventeenth century when Isaac Barrow, Isaac Newton, and Gottfried Wilhelm Leibniz (with help from others) discovered that the integral of a function could be found by asking what was differentiated to obtain that function.

The following introduction of integration is the usual one. It displays the concept geometrically and then defines the integral in the nineteenth-century language of limits. This form of definition establishes the basis for a wide variety of applications.

Consider the area of the region bound by $y = f(x)$, the x-axis, and the joining vertical segments (ordinates) $x = a$ and $x = b$. (See Fig. 5-1.)

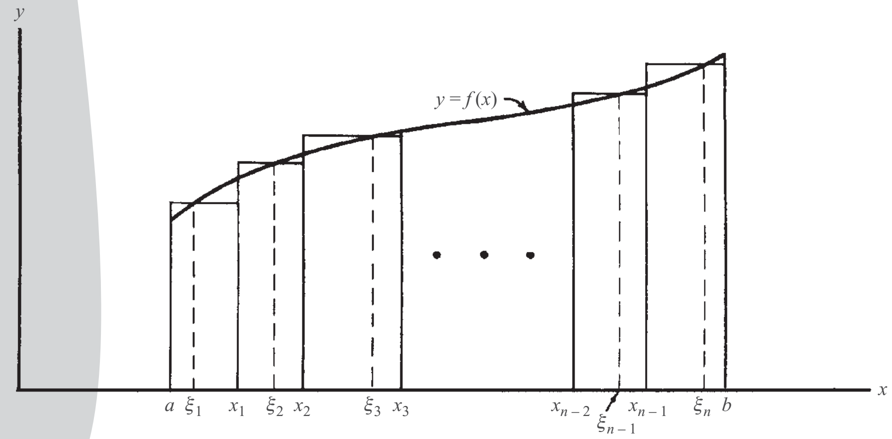


Fig. 5-1

Subdivide the interval $a \leqq x \leqq b$ into n sub-intervals by means of the points $x_1, x_2, \ldots, x_{n-1}$ chosen arbitrarily. In each of the new intervals $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$ choose points $\xi_1, \xi_2, \ldots, \xi_n$ arbitrarily. Form the sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + f(\xi_3)(x_3 - x_2) + \cdots + f(\xi_n)(b - x_{n-1}) \tag{1}$$

By writing $x_0 = a$, $x_n = b$, and $x_k - x_{k-1} = \Delta x_k$, this can be written

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k \tag{2}$$

Geometrically, this sum represents the total area of all rectangles in the above figure.

We now let the number of subdivisions n increase in such a way that each $\Delta x_k \to 0$. If as a result the sum (*1*) or (*2*) approaches a limit which does not depend on the mode of subdivision, we denote this limit by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k \tag{3}$$

This is called the *definite integral of $f(x)$ between a and b*. In this symbol $f(x)\,dx$ is called the *integrand*, and $[a, b]$ is called the *range of integration*. We call a and b the limits of integration, a being the lower limit of integration and b the upper limit.

The limit (*3*) exists whenever $f(x)$ is continuous (or piecewise continuous) in $a \leqq x \leqq b$ (see Problem 5.31). When this limit exists we say that f is *Riemann integrable* or simply *integrable* in $[a, b]$.

The definition of the definite integral as the limit of a sum was established by Cauchy around 1825. It was named for Riemann because he made extensive use of it in this 1850 exposition of integration.

Geometrically the value of this definite integral represents the area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ only if $f(x) \geqq 0$. If $f(x)$ is sometimes positive and sometimes negative, the definite integral represents the algebraic sum of the areas above and below the x-axis, treating areas above the x-axis as positive and areas below the x-axis as negative.

MEASURE ZERO

A set of points on the x-axis is said to have *measure zero* if the sum of the lengths of intervals enclosing all the points can be made arbitrary small (less than any given positive number ϵ). We can show (see Problem 5.6) that any countable set of points on the real axis has measure zero. In particular, the set of rational numbers which is countable (see Problems 1.17 and 1.59, Chapter 1), has measure zero.

An important theorem in the theory of Riemann integration is the following:

Theorem. If $f(x)$ is bounded in $[a, b]$, then a necessary and sufficient condition for the existence of $\int_a^b f(x)\,dx$ is that the set of discontinuities of $f(x)$ have measure zero.

PROPERTIES OF DEFINITE INTEGRALS

If $f(x)$ and $g(x)$ are integrable in $[a, b]$ then

1. $\displaystyle\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

2. $\displaystyle\int_a^b A f(x)\,dx = A\int_a^b f(x)\,dx$ where A is any constant

3. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided $f(x)$ is integrable in $[a, c]$ and $[c, b]$.

4. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. $\displaystyle\int_a^a f(x)\,dx = 0$

6. If in $a \leqq x \leqq b$, $m \leqq f(x) \leqq M$ where m and M are constants, then

$$m(b-a) \leqq \int_a^b f(x)\,dx \leqq M(b-a)$$

7. If in $a \leqq x \leqq b$, $f(x) \leqq g(x)$ then

$$\int_a^b f(x)\,dx \leqq \int_a^b g(x)\,dx$$

8. $\displaystyle\left|\int_a^b f(x)\,dx\right| \leqq \int_a^b |f(x)|\,dx \qquad \text{if } a < b$

MEAN VALUE THEOREMS FOR INTEGRALS

As in differential calculus the mean value theorems listed below are existence theorems. The first one generalizes the idea of finding an arithmetic mean (i.e., an average value of a given set of values) to a continuous function over an interval. The second mean value theorem is an extension of the first one that defines a weighted average of a continuous function.

By analogy, consider determining the arithmetic mean (i.e., average value) of temperatures at noon for a given week. This question is resolved by recording the 7 temperatures, adding them, and dividing by 7. To generalize from the notion of arithmetic mean and ask for the average temperature for the week is much more complicated because the spectrum of temperatures is now continuous. However, it is reasonable to believe that there exists a time at which the *average* temperature takes place. The manner in which the integral can be employed to resolve the question is suggested by the following example.

Let f be continuous on the closed interval $a \leqq x \leqq b$. Assume the function is represented by the correspondence $y = f(x)$, with $f(x) > 0$. Insert points of equal subdivision, $a = x_0, x_1, \ldots, x_n = b$. Then all $\Delta x_k = x_k - x_{k-1}$ are equal and each can be designated by Δx. Observe that $b - a = n\Delta x$. Let ξ_k be the midpoint of the interval Δx_k and $f(\xi_k)$ the value of f there. Then the average of these functional values is

$$\frac{f(\xi_1) + \cdots + f(\xi_n)}{n} = \frac{[f(\xi_1) + \cdots + f(\xi_n)]\Delta x}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(\xi_\kappa)\Delta\xi_\kappa$$

This sum specifies the average value of the n functions at the midpoints of the intervals. However, we may abstract the last member of the string of equalities (dropping the special conditions) and define

$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{k=1}^{n} f(\xi_\kappa)\Delta\xi_\kappa = \frac{1}{b-a}\int_a^b f(x)\,dx$$

as the average value of f on $[a, b]$.

Of course, the question of for what value $x = \xi$ the average is attained is not answered; and, in fact, in general, only existence not the value can be demonstrated. To see that there is a point $x = \xi$ such that $f(\xi)$ represents the average value of f on $[a, b]$, recall that a continuous function on a closed interval has maximum and minimum values, M and m, respectively. Thus (think of the integral as representing the area under the curve). (See Fig. 5-2.)

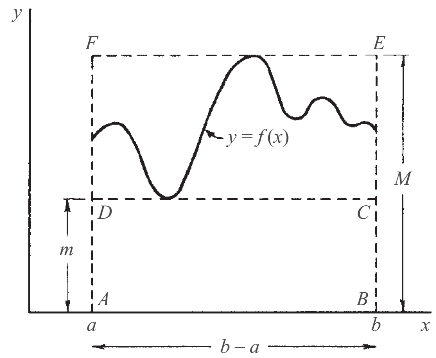


Fig. 5-2

$$m(b - a) \leqq \int_a^b f(x)\,dx \leqq M(b - a)$$

or

$$m \leqq \frac{1}{b - a}\int_a^b f(x)\,dx \leqq M$$

Since f is a continuous function on a closed interval, there exists a point $x = \xi$ in (a, b) intermediate to m and M such that

$$f(\xi) = \frac{1}{b - a}\int_a^b f(x)\,dx$$

While this example is not a rigorous proof of the first mean value theorem, it motivates it and provides an interpretation. (See Chapter 3, Theorem 10.)

1. **First mean value theorem.** If $f(x)$ is continuous in $[a, b]$, there is a point ξ in (a, b) such that

$$\int_a^b f(x)\,dx = (b - a)f(\xi) \tag{4}$$

2. **Generalized first mean value theorem.** If $f(x)$ and $g(x)$ are continuous in $[a, b]$, and $g(x)$ does not change sign in the interval, then there is a point ξ in (a, b) such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx \tag{5}$$

   This reduces to (*4*) if $g(x) = 1$.

CONNECTING INTEGRAL AND DIFFERENTIAL CALCULUS

In the late seventeenth century the key relationship between the derivative and the integral was established. The connection which is embodied in the fundamental theorem of calculus was responsible for the creation of a whole new branch of mathematics called analysis.

Definition: Any function F such that $F'(x) = f(x)$ is called an *antiderivative*, *primitive*, or *indefinite integral* of f.

The antiderivative of a function is not unique. This is clear from the observation that for any constant c

$$(F(x) + c)' = F'(x) = f(x)$$

The following theorem is an even stronger statement.

Theorem. Any two primitives (i.e., antiderivatives), F and G of f differ at most by a constant, i.e., $F(x) - G(x) = C$.

(See the problem set for the proof of this theorem.)

EXAMPLE. If $F'(x) = x^2$, then $F(x) = \int x^2 dx = \dfrac{x^3}{3} + c$ is an indefinite integral (antiderivative or primitive) of x^2.

The indefinite integral (which is a function) may be expressed as a definite integral by writing

$$\int f(x)\,dx = \int_c^x f(t)\,dt$$

The functional character is expressed through the upper limit of the definite integral which appears on the right-hand side of the equation.

This notation also emphasizes that the definite integral of a given function only depends on the limits of integration, and thus any symbol may be used as the variable of integration. For this reason, that variable is often called a *dummy* variable. The indefinite integral notation on the left depends on continuity of f on a domain that is not described. One can visualize the definite integral on the right by thinking of the dummy variable t as ranging over a subinterval $[c, x]$. (There is nothing unique about the letter t; any other convenient letter may represent the dummy variable.)

The previous terminology and explanation set the stage for the fundamental theorem. It is stated in two parts. The first states that the antiderivative of f is a new function, the integrand of which is the derivative of that function. Part two demonstrates how that primitive function (antiderivative) enables us to evaluate definite integrals.

THE FUNDAMENTAL THEOREM OF THE CALCULUS

Part 1 Let f be integrable on a closed interval $[a, b]$. Let c satisfy the condition $a \leqq c \leqq b$, and define a new function

$$F(x) = \int_c^x f(t)\,dt \quad \text{if} \quad a \leqq x \leqq b$$

Then the derivative $F'(x)$ exists at each point x in the open interval (a, b), where f is continuous and $F'(x) = f(x)$. (See Problem 5.10 for proof of this theorem.)

Part 2 As in Part 1, assume that f is integrable on the closed interval $[a, b]$ and continuous in the open interval (a, b). Let F be any antiderivative so that $F'(x) = f(x)$ for each x in (a, b). If $a < c < b$, then for any x in (a, b)

$$\int_c^x f(t)\,dt = F(x) - F(c)$$

If the open interval on which f is continuous includes a and b, then we may write

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad \text{(See Problem 5.11)}$$

This is the usual form in which the theorem is used.

EXAMPLE. To evaluate $\int_1^2 x^2\,dx$ we observe that $F'(x) = x^2$, $F(x) = \dfrac{x^3}{3} + c$ and $\int_1^2 x^2\,dx = \left(\frac{2^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right) = \frac{7}{3}$. Since c subtracts out of this evaluation it is convenient to exclude it and simply write $\dfrac{2^3}{3} - \dfrac{1^3}{3}$.

GENERALIZATION OF THE LIMITS OF INTEGRATION

The upper and lower limits of integration may be variables. For example:

$$\int_{\sin x}^{\cos x} t\,dt = \left[\frac{t^2}{2}\right]_{\sin x}^{\cos x} = (\cos^2 x - \sin^2 x)/2$$

In general, if $F'(x) = f(x)$ then

$$\int_{u(x)}^{v(x)} f(t)\,dt = F[v(x)] = F[u(x)]$$

CHANGE OF VARIABLE OF INTEGRATION

If a determination of $\int f(x)\,dx$ is not immediately obvious in terms of elementary functions, useful results may be obtained by changing the variable from x to t according to the transformation $x = g(t)$. (This change of integrand that follows is suggested by the differential relation $dx = g'(t)\,dt$.) The fundamental theorem enabling us to do this is summarized in the statement

$$\int f(x)\,dx = \int f\{g(t)\}g'(t)\,dt \tag{6}$$

where after obtaining the indefinite integral on the right we replace t by its value in terms of x, i.e., $t = g^{-1}(x)$. This result is analogous to the chain rule for differentiation (see Page 69).

The corresponding theorem for definite integrals is

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\{g(t)\}g'(t)\,dt \tag{7}$$

where $g(\alpha) = a$ and $g(\beta) = b$, i.e., $\alpha = g^{-1}(a)$, $\beta = g^{-1}(b)$. This result is certainly valid if $f(x)$ is continuous in $[a, b]$ and if $g(t)$ is continuous and has a continuous derivative in $\alpha \leqq t \leqq \beta$.

INTEGRALS OF ELEMENTARY FUNCTIONS

The following results can be demonstrated by differentiating both sides to produce an identity. In each case an arbitrary constant c (which has been omitted here) should be added.

1. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} \quad n \neq -1$

2. $\int \dfrac{du}{u} = \ln|u|$

3. $\int \sin u\,du = -\cos u$

4. $\int \cos u\,du = \sin u$

5. $\int \tan u\,du = \ln|\sec u| = -\ln|\cos u|$

6. $\int \cot u\,du = \ln|\sin u|$

7. $\int \sec u\,du = \ln|\sec u + \tan u| = \ln|\tan(u/2 + \pi/4)|$

8. $\int \csc u\,du = \ln|\csc u - \cot u| = \ln|\tan u/2|$

9. $\int \sec^2 u\,du = \tan u$

10. $\int \csc^2 u\,du = -\cot u$

11. $\int \sec u \tan u\,du = \sec u$

12. $\int \csc u \cot u\,du = -\csc u$

13. $\int a^u\,du = \dfrac{a^u}{\ln a} \quad a > 0, a \neq 1$

14. $\int e^u\,du = e^u$

15. $\int \sinh u\,du = \cosh u$

16. $\int \cosh u\,du = \sinh u$

17. $\int \tanh u\,du = \ln \cosh u$

18. $\int \coth u\,du = \ln|\sinh u|$

19. $\int \operatorname{sech} u\,du = \tan^{-1}(\sinh u)$

20. $\int \operatorname{csch} u\,du = -\coth^{-1}(\cosh u)$

21. $\int \operatorname{sech}^2 u\,du = \tanh u$

22. $\int \operatorname{csch}^2 u\,du = -\coth u$

23. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u$

24. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u$

25. $\int \dfrac{du}{\sqrt{s^2 - u^2}} = \sin^{-1}\dfrac{u}{a} \quad \text{or} \quad -\cos^{-1}\dfrac{u}{a}$

26. $\int \dfrac{du}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}\,|$

27. $\int \dfrac{du}{u^2 + a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} \quad \text{or} \quad -\dfrac{1}{a}\cot^{-1}\dfrac{u}{a}$

28. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right|$

29. $\int \dfrac{du}{u\sqrt{a^2 \pm u^2}} = \dfrac{1}{a}\ln\left|\dfrac{u}{a + \sqrt{a^2 \pm u^2}}\right|$

30. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\cos^{-1}\dfrac{a}{u} \quad \text{or} \quad \dfrac{1}{a}\sec^{-1}\dfrac{u}{a}$

31. $\int \sqrt{u^2 \pm a^2}\,du = \dfrac{u}{2}\sqrt{u^2 \pm a^2} \pm \dfrac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}\,|$

32. $\int \sqrt{a^2 - u^2}\,du = \dfrac{u}{2}\sqrt{a^2 - u^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{u}{a}$

33. $\int e^{au}\sin bu\,du = \dfrac{e^{au}(a\sin bu - b\cos bu)}{a^2 + b^2}$

34. $\int e^{au}\cos bu\,du = \dfrac{e^{au}(a\cos bu + b\sin bu)}{a^2 + b^2}$

SPECIAL METHODS OF INTEGRATION

1. **Integration by parts.**
   Let u and v be differentiable functions. According to the product rule for differentials

$$d(uv) = u\,dv + v\,du$$

   Upon taking the antiderivative of both sides of the equation, we obtain

$$uv = \int u\,dv + \int v\,du$$

   This is the formula for integration by parts when written in the form

$$\int u\,dv = uv - \int v\,du \quad \text{or} \quad \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

   where $u = f(x)$ and $v = g(x)$. The corresponding result for definite integrals over the interval $[a, b]$ is certainly valid if $f(x)$ and $g(x)$ are continuous and have continuous derivatives in $[a, b]$. See Problems 5.17 to 5.19.

2. **Partial fractions.** Any rational function $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, with the degree of $P(x)$ less than that of $Q(x)$, can be written as the sum of rational functions having the form $\dfrac{A}{(ax+b)^r}, \dfrac{Ax+B}{(ax^2+bx+c)^r}$ where $r = 1, 2, 3, \ldots$ which can always be integrated in terms of elementary functions.

EXAMPLE 1. $\dfrac{3x-2}{(4x-3)(2x+5)^3} = \dfrac{A}{4x-3} + \dfrac{B}{(2x+5)^3} + \dfrac{C}{(2x+5)^2} + \dfrac{D}{2x+5}$

EXAMPLE 2. $\dfrac{5x^2-x+2}{(x^2+2x+4)^2(x-1)} = \dfrac{Ax+B}{(x^2+2x+4)^2} + \dfrac{Cx+D}{x^2+2x+4} + \dfrac{E}{x-1}$

The constants, A, B, C, etc., can be found by clearing of fractions and equating coefficients of like powers of x on both sides of the equation or by using special methods (see Problem 5.20).

3. **Rational functions of $\sin x$ and $\cos x$** can always be integrated in terms of elementary functions by the substitution $\tan x/2 = u$ (see Problem 5.21).
4. **Special devices** depending on the particular form of the integrand are often employed (see Problems 5.22 and 5.23).

IMPROPER INTEGRALS

If the range of integration $[a, b]$ is not finite or if $f(x)$ is not defined or not bounded at one or more points of $[a, b]$, then the integral of $f(x)$ over this range is called an *improper integral*. By use of appropriate limiting operations, we may define the integrals in such cases.

EXAMPLE 1. $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \lim_{M\to\infty} \int_0^M \frac{dx}{1+x^2} = \lim_{M\to\infty} \tan^{-1} x\Big|_0^M = \lim_{M\to\infty} \tan^{-1} M = \pi/2$

EXAMPLE 2. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0+} 2\sqrt{x}\Big|_\epsilon^1 = \lim_{\epsilon\to 0+} (2 - 2\sqrt{\epsilon}) = 2$

EXAMPLE 3. $\displaystyle\int_0^1 \frac{dx}{x} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{x} = \lim_{\epsilon\to 0+} \ln x\Big|_\epsilon^1 = \lim_{\epsilon\to 0+} (-\ln \epsilon)$

Since this limit does not exist we say that the integral diverges (i.e., does not converge).

For further examples, see Problems 5.29 and 5.74 through 5.76. For further discussion of improper integrals, see Chapter 12.

NUMERICAL METHODS FOR EVALUATING DEFINITE INTEGRALS

Numerical methods for evaluating definite integrals are available in case the integrals cannot be evaluated exactly. The following special numerical methods are based on subdividing the interval $[a, b]$ into n equal parts of length $\Delta x = (b - a)/n$. For simplicity we denote $f(a + k\Delta x) = f(x_k)$ by y_k, where $k = 0, 1, 2, \ldots, n$. The symbol $\approx$ means "approximately equal." In general, the approximation improves as n increases.

1. **Rectangular rule**.

$$\int_a^b f(x)\,dx \approx \Delta x\{y_0 + y_1 + y_2 + \cdots + y_{n-1}\} \quad \text{or} \quad \Delta x\{y_1 + y_2 + y_3 + \cdots + y_n\} \tag{8}$$

The geometric interpretation is evident from the figure on Page 90. When left endpoint function values $y_0, y_1, \ldots, y_{n-1}$ are used, the rule is called "the left-hand rule." Similarly, when right endpoint evaluations are employed, it is called "the right-hand rule."

2. **Trapezoidal rule.**

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\{y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\} \tag{9}$$

This is obtained by taking the mean of the approximations in (*8*). Geometrically this replaces the curve $y = f(x)$ by a set of approximating line segments.

3. **Simpson's rule.**

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\} \tag{10}$$

The above formula is obtained by approximating the graph of $y = g(x)$ by a set of parabolic arcs of the form $y = ax^2 + bx + c$. The correlation of two observations lead to 10. First,

$$\int_{-h}^{h} [ax^2 + bx + c]\,dx = \frac{h}{3}[2ah^2 + 6c]$$

The second observation is related to the fact that the vertical parabolas employed here are determined by three nonlinear points. In particular, consider $(-h, y_0)$, $(0, y_1)$, (h, y_2) then $y_0 = a(-h)^2 + b(-h) + c$, $y_1 = c$, $y_2 = ah^2 + bh + c$. Consequently, $y_0 + 4y_1 + y_2 = 2ah^2 + 6c$. Thus, this combination of ordinate values (corresponding to equally space domain values) yields the area bound by the parabola, vertical segments, and the x-axis. Now these ordinates may be interpreted as those of the function, f, whose integral is to be approximated. Then, as illustrated in Fig. 5-3:

$$\sum_{k=1}^{n} \frac{h}{3}[y_{k-1} + 4y_k + y_{k+1}] = \frac{\Delta x}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]$$

The Simpson rule is likely to give a better approximation than the others for smooth curves.

APPLICATIONS

The use of the integral as a limit of a sum enables us to solve many physical or geometrical problems such as determination of areas, volumes, arc lengths, moments of intertia, centroids, etc.

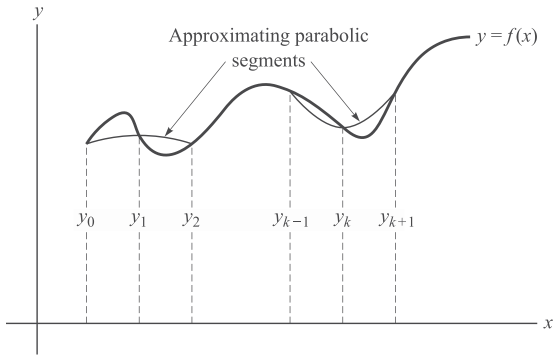


Fig. 5-3

ARC LENGTH

As you walk a twisting mountain trail, it is possible to determine the distance covered by using a pedometer. To create a geometric model of this event, it is necessary to describe the trail and a method of measuring distance along it. The trail might be referred to as a *path*, but in more exacting geometric terminology the word, *curve* is appropriate. That segment to be measured is an arc of the curve. The arc is subject to the following restrictions:

1. It does not intersect itself (i.e., it is a simple arc).
2. There is a tangent line at each point.
3. The tangent line varies continuously over the arc.

These conditions are satisfied with a parametric representation $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \leqq t \leqq b$, where the functions f, g, and h have continuous derivatives that do not simultaneously vanish at any point. This arc is in Euclidean three space and will be discussed in Chapter 10. In this introduction to curves and their arc length, we let $z = 0$, thereby restricting the discussion to the plane.

A careful examination of your walk would reveal movement on a sequence of straight segments, each changed in direction from the previous one. This suggests that the length of the arc of a curve is obtained as the limit of a sequence of lengths of polygonal approximations. (The polygonal approximations are characterized by the number of divisions $n \to \infty$ and no subdivision is bound from zero. (See Fig. 5-4.)

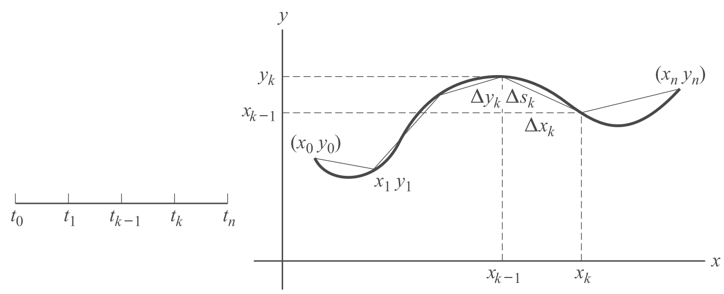


Fig. 5-4

Geometrically, the measurement of the kth segment of the arc, $0 \leqq t \leqq s$, is accomplished by employing the Pythagorean theorem, and thus, the measure is defined by

$$\lim_{n\to\infty}\sum_{k=1}^{n}\{(\Delta x_k)^2 + (\Delta y_k)^2\}^{1/2}$$

or equivalently

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left\{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right\}^{1/2}(\Delta x_k)$$

where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$.

Thus, the length of the arc of a curve in rectangular Cartesian coordinates is

$$L = \int_a^b \{[f'(t)^2] + [g'(t)]^2\}^{1/2}\, dt = \int\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{1/2} dt$$

(This form may be generalized to any number of dimensions.)

Upon changing the variable of integration from t to x we obtain the planar form

$$L = \int_{f(a)}^{f(b)}\left\{1 + \left[\frac{dy}{dx}\right]^2\right\}^{1/2}$$

(This form is only appropriate in the plane.)

The generic differential formula $ds^2 = dx^2 + dy^2$ is useful, in that various representations algebraically arise from it. For example,

$$\frac{ds}{dt}$$

expresses instantaneous speed.

AREA

Area was a motivating concept in introducing the integral. Since many applications of the integral are geometrically interpretable in the context of area, an extended formula is listed and illustrated below.

Let f and g be continuous functions whose graphs intersect at the graphical points corresponding to $x = a$ and $x = b$, $a < b$. If $g(x) \geqq f(x)$ on $[a, b]$, then the area bounded by $f(x)$ and $g(x)$ is

$$A = \int_a^b \{g(x) - f(x)\}\, dx$$

If the functions intersect in (a, b), then the integral yields an algebraic sum. For example, if $g(x) = \sin x$ and $f(x) = 0$ then:

$$\int_0^{2\pi} \sin x\, dx = \cos x\Big|_0^{2\pi} = 0$$

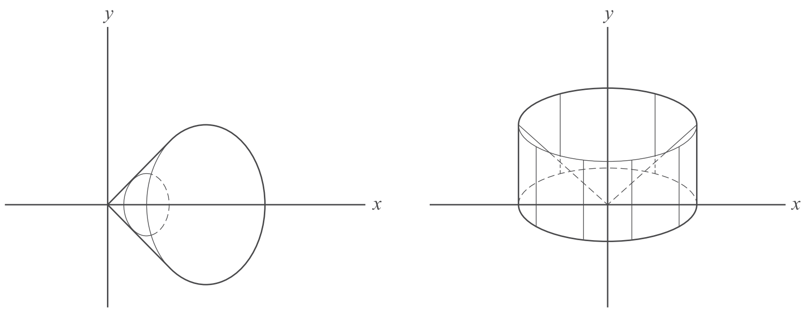
VOLUMES OF REVOLUTION

Disk Method

Assume that f is continuous on a closed interval $a \leqq x \leqq b$ and that $f(x) \geqq 0$. Then the solid realized through the revolution of a plane region R (bound by $f(x)$, the x-axis, and $x = a$ and $x = b$) about the x-axis has the volume

$$V = \pi\int_a^b [f(x)]^2\, dx$$

This method of generating a volume is called the *disk method* because the cross sections of revolution are circular disks. (See Fig. 5-5(a).)



(a) **Fig. 5-5** (b)

EXAMPLE. A solid cone is generated by revolving the graph of $y = kx$, $k > 0$ and $0 \leqq x \leqq b$, about the x-axis. Its volume is

$$V = \pi \int_0^b k^2 x^2 \, dx = \pi \left. \frac{k^3 x^3}{3} \right|_0^b = \pi \frac{k^3 b^3}{3}$$

Shell Method

Suppose f is a continuous function on $[a, b]$, $a \geqq 0$, satisfying the condition $f(x) \geqq 0$. Let R be a plane region bound by $f(x)$, $x = a$, $x = b$, and the x-axis. The volume obtained by orbiting R about the y-axis is

$$V = \int_a^b 2\pi x f(x) \, dx$$

This method of generating a volume is called the *shell method* because of the cylindrical nature of the vertical lines of revolution. (See Fig. 5-5(b).)

EXAMPLE. If the region bounded by $y = kx$, $0 \leqq x \leqq b$ and $x = b$ (with the same conditions as in the previous example) is orbited about the y-axis the volume obtained is

$$V = 2\pi \int_0^b x(kx) \, dx = 2\pi k \left. \frac{x^3}{3} \right|_0^b = 2\pi k \frac{b^3}{3}$$

By comparing this example with that in the section on the disk method, it is clear that for the same plane region the disk method and the shell method produce different solids and hence different volumes.

Moment of Inertia

Moment of inertia is an important physical concept that can be studied through its idealized geometric form. This form is abstracted in the following way from the physical notions of kinetic energy, $K = \frac{1}{2} m v^2$, and angular velocity, $v = \omega r$. (m represents mass and v signifies linear velocity). Upon substituting for v

$$K = \tfrac{1}{2} m \omega^2 r^2 = \tfrac{1}{2} (mr^2) \omega^2$$

When this form is compared to the original representation of kinetic energy, it is reasonable to identify mr^2 as rotational mass. It is this quantity, $I = mr^2$ that we call the *moment of inertia*.

Then in a purely geometric sense, we denote a plane region R described through continuous functions f and g on $[a, b]$, where $a > 0$ and $f(x)$ and $g(x)$ intersect at a and b only. For simplicity, assume $g(x) \geqq f(x) > 0$. Then

$$I = \int_a^b x^2[g(x) - f(x)]\,dx$$

By idealizing the plane region, R, as a volume with uniform density *one*, the expression $[f(x) - g(x)]\,dx$ stands in for mass and r^2 has the coordinate representation x^2. (See Problem 5.25(*b*) for more details.)

Solved Problems

DEFINITION OF A DEFINITE INTEGRAL

5.1. If $f(x)$ is continuous in $[a, b]$ prove that

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx$$

Since $f(x)$ is continuous, the limit exists independent of the mode of subdivision (see Problem 5.31). Choose the subdivision of $[a, b]$ into n equal parts of equal length $\Delta x = (b-a)/n$ (see Fig. 5-1, Page 90). Let $\xi_k = a + k(b-a)/n$, $k = 1, 2, \ldots, n$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx$$

5.2. Express $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$ as a definite integral.

Let $a = 0$, $b = 1$ in Problem 1. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$$

5.3. (*a*) Express $\displaystyle\int_0^1 x^2\,dx$ as a limit of a sum, and use the result to evaluate the given definite integral.
(*b*) Interpret the result geometrically.

(*a*) If $f(x) = x^2$, then $f(k/n) = (k/n)^2 = k^2/n^2$. Thus by Problem 5.2,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} = \int_0^1 x^2\,dx$$

This can be written, using Problem 1.29 of Chapter 1,

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right) = \lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3} \\ &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\ &= \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6} = \frac{1}{3}\end{aligned}$$

which is the required limit.

Note: By using the fundamental theorem of the calculus, we observe that $\int_0^1 x^2\,dx = (x^3/3)|_0^1 = 1^3/3 - 0^3/3 = 1/3$.

(*b*) The area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$ is equal to $\frac{1}{3}$.

5.4. Evaluate $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\}$.

The required limit can be written

$$\lim_{n\to\infty}\frac{1}{n}\left\{\frac{1}{1+1/n}+\frac{1}{1+2/n}+\cdots+\frac{1}{1+n/n}\right\}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+k/n}$$
$$=\int_0^1\frac{dx}{1+x}=\ln(1+x)|_0^1=\ln 2$$

using Problem 5.2 and the fundamental theorem of the calculus.

5.5. Prove that $\lim_{n\to\infty}\frac{1}{n}\left\{\sin\frac{t}{n}+\sin\frac{2t}{n}+\cdots+\sin\frac{(n-1)t}{n}\right\}=\frac{1-\cos t}{t}$.

Let $a=0, b=t, f(x)=\sin x$ in Problem 1. Then

$$\lim_{n\to\infty}\frac{t}{n}\sum_{k=1}^{n}\sin\frac{kt}{n}=\int_0^t\sin x\,dx=1-\cos t$$

and so

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\sin\frac{kt}{n}=\frac{1-\cos t}{t}$$

using the fact that $\lim_{n\to\infty}\frac{\sin t}{n}=0$.

MEASURE ZERO

5.6. Prove that a countable point set has measure zero.

Let the point set be denoted by $x_1, x_2, x_3, x_4, \ldots$ and suppose that intervals of lengths less than $\epsilon/2, \epsilon/4, \epsilon/8, \epsilon/16, \ldots$ respectively enclose the points, where ϵ is any positive number. Then the sum of the lengths of the intervals is less than $\epsilon/2+\epsilon/4+\epsilon/8+\cdots=\epsilon$ (let $a=\epsilon/2$ and $r=\frac{1}{2}$ in Problem 2.25(a) of Chapter 2), showing that the set has measure zero.

PROPERTIES OF DEFINITE INTEGRALS

5.7. Prove that $\left|\int_a^b f(x)\,dx\right| \leqq \int_a^b |f(x)|\,dx$ if $a<b$.

By absolute value property 2, Page 3,

$$\left|\sum_{k=1}^{n}f(\xi_k)\Delta x_k\right| \leqq \sum_{k=1}^{n}|f(\xi_k)\Delta x_k|=\sum_{k=1}^{n}|f(\xi_k)|\Delta x_k$$

Taking the limit as $n\to\infty$ and each $\Delta x_k\to 0$, we have the required result.

5.8. Prove that $\lim_{n\to\infty}\int_0^{2\pi}\frac{\sin nx}{x^2+n^2}\,dx=0$.

$$\left|\int_0^{2\pi}\frac{\sin nx}{x^2+n^2}\,dx\right| \leqq \int_0^{2\pi}\left|\frac{\sin nx}{x^2+n^2}\right|dx \leqq \int_0^{2\pi}\frac{dx}{n^2}=\frac{2\pi}{n^2}$$

Then $\lim_{n\to\infty}\left|\int_0^{2\pi}\frac{\sin nx}{x^2+n^2}\,dx\right|=0$, and so the required result follows.

MEAN VALUE THEOREMS FOR INTEGRALS

5.9. Given the right triangle pictured in Fig. 5-6: (*a*) Find the average value of h. (*b*) At what point does this average value occur? (*c*) Determine the average value of $f(x) = \sin^{-1} x, 0 \leqq x \leqq \frac{1}{2}$. (Use integration by parts.) (*d*) Determine the average value of $f(x) = \cos^2 x, 0 \leqq x \leqq \dfrac{\pi}{2}$.

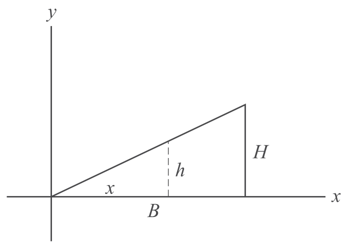


Fig. 5-6

(*a*) $h(x) = \dfrac{H}{B}x$. According to the mean value theorem for integrals, the average value of the function h on the interval $[0, B]$ is

$$A = \frac{1}{B}\int_0^B \frac{H}{B} x\,dx = \frac{H}{2}$$

(*b*) The point, ξ, at which the average value of h occurs may be obtained by equating $f(\xi)$ with that average value, i.e., $\dfrac{H}{B}\xi = \dfrac{H}{2}$. Thus, $\xi = \dfrac{B}{2}$.

FUNDAMENTAL THEOREM OF THE CALCULUS

5.10. If $F(x) = \displaystyle\int_a^x f(t)\,dt$ where $f(x)$ is continuous in $[a, b]$, prove that $F'(x) = f(x)$.

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left\{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right\} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$
$$= f(\xi) \qquad \xi \text{ between } x \text{ and } x+h$$

by the first mean value theorem for integrals (Page 93).

Then if x is any point interior to $[a, b]$,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} f(\xi) = f(x)$$

since f is continuous.

If $x = a$ or $x = b$, we use right- or left-hand limits, respectively, and the result holds in these cases as well.

5.11. Prove the fundamental theorem of the calculus, Part 2 (Pages 94 and 95).

By Problem 5.10, if $F(x)$ is *any* function whose derivative is $f(x)$, we can write

$$F(x) = \int_a^x f(t)\,dt + c$$

where c is any constant (see last line of Problem 22, Chapter 4).

Since $F(a) = c$, it follows that $F(b) = \displaystyle\int_a^b f(t)\,dt + F(a)$ or $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$.

5.12. If $f(x)$ is continuous in $[a, b]$, prove that $F(x) = \displaystyle\int_a^x f(t)\,dt$ is continuous in $[a, b]$.

If x is any point interior to $[a, b]$, then as in Problem 5.10,

$$\lim_{h\to 0} F(x+h) - F(x) = \lim_{h\to 0} hf(\xi) = 0$$

and $F(x)$ is continuous.

If $x = a$ and $x = b$, we use right- and left-hand limits, respectively, to show that $F(x)$ is continuous at $x = a$ and $x = b$.

Another method:

By Problem 5.10 and Problem 4.3, Chapter 4, it follows that $F'(x)$ exists and so $F(x)$ must be continuous.

CHANGE OF VARIABLES AND SPECIAL METHODS OF INTEGRATION

5.13. Prove the result (*7*), Page 95, for changing the variable of integration.

Let $F(x) = \int_a^x f(x)\,dx$ and $G(t) = \int_\alpha^t f\{g(t)\}\,g'(t)\,dt$, where $x = g(t)$.

Then $dF = f(x)\,dx$, $dG = f\{g(t)\}\,g'(t)\,dt$.

Since $dx = g'(t)\,dt$, it follows that $f(x)\,dx = f\{g(t)\}\,g'(t)\,dt$ so that $dF(x) = dG(t)$, from which $F(x) = G(t) + c$.

Now when $x = a$, $t = \alpha$ or $F(a) = G(\alpha) + c$. But $F(a) = G(\alpha) = 0$, so that $c = 0$. Hence $F(x) = G(t)$. Since $x = b$ when $t = \beta$, we have

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\{g(t)\}\,g'(t)\,dt$$

as required.

5.14. Evaluate:

(*a*) $\displaystyle\int (x+2)\sin(x^2+4x-6)\,dx$ (*c*) $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}}$ (*e*) $\displaystyle\int_0^{1/\sqrt{2}} \frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}\,dx$

(*b*) $\displaystyle\int \frac{\cot(\ln x)}{x}\,dx$ (*d*) $\displaystyle\int 2^{-x}\tanh 2^{1-x}\,dx$ (*f*) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+x+1}}$

(*a*) **Method 1:** Let $x^2 + 4x - 6 = u$. Then $(2x+4)\,dx = du$, $(x+2)\,dx = \frac{1}{2}\,du$ and the integral becomes

$$\frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(x^2+4x-6) + c$$

Method 2:

$$\int (x+2)\,\sin(x^2+4x-6)\,dx = \frac{1}{2}\int \sin(x^2+4x-6)d(x^2+4x-6) = -\frac{1}{2}\cos(x^2+4x-6) + c$$

(*b*) Let $\ln x = u$. Then $(dx)/x = du$ and the integral becomes

$$\int \cot u\,du = \ln|\sin u| + c = \ln|\sin(\ln x)| + c$$

(*c*) **Method 1:** $\displaystyle\int \frac{dx}{\sqrt{(x+2)(3-x)}} = \int \frac{dx}{\sqrt{6+x-x^2}} = \int \frac{dx}{\sqrt{6-(x^2-x)}} = \int \frac{dx}{\sqrt{25/4-(x-\frac{1}{2})^2}}$

Letting $x - \frac{1}{2} = u$, this becomes

$$\int \frac{du}{\sqrt{25/4-u^2}} = \sin^{-1}\frac{u}{5/2} + c = \sin^{-1}\left(\frac{2x-1}{5}\right) + c$$

Then

$$\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}} = \sin^{-1}\left(\frac{2x-1}{5}\right)\Bigg|_{-1}^{1} = \sin^{-1}\left(\frac{1}{5}\right) - \sin^{-1}\left(-\frac{3}{5}\right)$$
$$= \sin^{-1}.2 + \sin^{-1}.6$$

Method 2: Let $x - \frac{1}{2} = u$ as in Method 1. Now when $x = -1$, $u = -\frac{3}{2}$; and when $x = 1$, $u = \frac{1}{2}$. Thus by Formula 25, Page 96.

$$\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}} = \int_{-1}^{1} \frac{dx}{\sqrt{25/4 - (x-\frac{1}{2})^2}} = \int_{-3/2}^{1/2} \frac{du}{\sqrt{25/4 - u^2}} = \sin^{-1} \frac{u}{5/2}\Big|_{-3/2}^{1/2}$$
$$= \sin^{1} .2 + \sin^{-1} .6$$

(*d*) Let $2^{1-x} = u$. Then $-2^{1-x}(\ln 2)dx = du$ and $2^{-x}dx = -\dfrac{du}{2\ln 2}$, so that the integral becomes

$$-\frac{1}{2\ln 2}\int \tanh u\, du = -\frac{1}{2\ln 2}\ln\cosh 2^{1-x} + c$$

(*e*) Let $\sin^{-1} x^2 = u$. Then $du = \dfrac{1}{\sqrt{1-(x^2)^2}} 2x\,dx = \dfrac{2x\,dx}{\sqrt{1-x^4}}$ and the integral becomes

$$\frac{1}{2}\int u\,du = \frac{1}{4}u^2 + c = \frac{1}{4}(\sin^{-1} x^2)^2 + c$$

Thus $\displaystyle\int_0^{1/\sqrt{2}} \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}}\,dx = \frac{1}{4}(\sin^{-1} x^2)^2\Big|_0^{1/\sqrt{2}} = \frac{1}{4}\left(\sin^{-1}\frac{1}{2}\right)^2 = \frac{\pi^2}{144}.$

(*f*)
$$\int \frac{x\,dx}{\sqrt{x^2+x+1}} = \frac{1}{2}\int \frac{2x+1-1}{\sqrt{x^2+x+1}}\,dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\frac{dx}{\sqrt{x^2+x+1}}$$
$$= \frac{1}{2}\int (x^2+x+1)^{-1/2} d(x^2+x+1) - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}}$$
$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln|x + \tfrac{1}{2} + \sqrt{(x+\tfrac{1}{2})^2 + \tfrac{3}{4}}| + c$$

5.15. Show that $\displaystyle\int_1^2 \frac{dx}{(x^2-2x+4)^{3/2}} = \frac{1}{6}.$

Write the integral as $\displaystyle\int_1^2 \frac{dx}{[(x-1)^2+3]^{3/2}}$. Let $x - 1 = \sqrt{3}\tan u$, $dx = \sqrt{3}\sec^2 u\,du$. When $x = 1$, $u = \tan^{-1} 0 = 0$; when $x = 2$, $u = \tan^{-1} 1/\sqrt{3} = \pi/6$. Then the integral becomes

$$\int_0^{\pi/6} \frac{\sqrt{3}\sec^2 u\,du}{[3+3\tan^2 u]^{3/2}} = \int_0^{\pi/6} \frac{\sqrt{3}\sec^2 u\,du}{[3\sec^2 u]^{3/2}} = \frac{1}{3}\int_0^{\pi/6} \cos u\,du = \frac{1}{3}\sin u\Big|_0^{\pi/6} = \frac{1}{6}$$

5.16. Determine $\displaystyle\int_e^{e^2} \frac{dx}{x(\ln x)^3}.$

Let $\ln x = y$, $(dx)/x = dy$. When $x = e$, $y = 1$; when $x = e^2$, $y = 2$. Then the integral becomes

$$\int_1^2 \frac{dy}{y^3} = \frac{y^{-2}}{-2}\Big|_1^2 = \frac{3}{8}$$

5.17. Find $\displaystyle\int x^n \ln x\,dx$ if (*a*) $n \neq -1$, (*b*) $n = -1$.

(*a*) Use integration by parts, letting $u = \ln x$, $dv = x^n\,dx$, so that $du = (dx)/x$, $v = x^{n+1}/(n+1)$. Then

$$\int x^n \ln x\,dx = \int u\,dv = uv - \int v\,du = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}\cdot\frac{dx}{x}$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

(*b*) $\int x^{-1}\ln x\,dx = \int \ln x\,d(\ln x) = \frac{1}{2}(\ln x)^2 + c.$

5.18. Find $\int 3^{\sqrt{2x+1}}\,dx$.

Let $\sqrt{2x+1} = y$, $2x+1 = y^2$. Then $dx = y\,dy$ and the integral becomes $\int 3^y \cdot y\,dy$.

Integrate by parts, letting $u = y$, $dv = 3^y\,dy$; then $du = dy$, $v = 3^y/(\ln 3)$, and we have

$$\int 3^y\cdot y\,dy = \int u\,dv = uv - \int v\,du = \frac{y\cdot 3^y}{\ln 3} - \int \frac{3^y}{\ln 3}\,dy = \frac{y\cdot 3^y}{\ln 3} - \frac{3^y}{(\ln 3)^2} + c$$

5.19. Find $\int_0^1 x\ln(x+3)\,dx$.

Let $u = \ln(x+3)$, $dv = x\,dx$. Then $du = \dfrac{dx}{x+3}$, $v = \dfrac{x^2}{2}$. Hence on integrating by parts,

$$\int x\ln(x+3)\,dx = \frac{x^2}{2}\ln(x+3) - \frac{1}{2}\int\frac{x^2\,dx}{x+3} = \frac{x^2}{2}\ln(x+3) - \frac{1}{2}\int\left(x - 3 + \frac{9}{x+3}\right)dx$$
$$= \frac{x^2}{2}\ln(x+3) - \frac{1}{2}\left\{\frac{x^2}{2} - 3x + 9\ln(x+3)\right\} + c$$

Then
$$\int_0^1 x\ln(x+3)\,dx = \frac{5}{4} - 4\ln 4 + \frac{9}{2}\ln 3$$

5.20. Determine $\int \dfrac{6-x}{(x-3)(2x+5)}\,dx$.

Use the method of *partial fractions*. Let $\dfrac{6-x}{(x-3)(2x+5)} = \dfrac{A}{x-3} + \dfrac{B}{2x+5}$.

Method 1: To determine the constants A and B, multiply both sides by $(x-3)(2x+5)$ to obtain

$$6 - x = A(2x+5) + B(x-3) \quad \text{or} \quad 6 - x = 5A - 3B + (2A+B)x \qquad (1)$$

Since this is an identity, $5A - 3B = 6$, $2A + B = -1$ and $A = 3/11$, $B = -17/11$. Then

$$\int\frac{6-x}{(x-3)(2x+5)}\,dx = \int\frac{3/11}{x-3}\,dx + \int\frac{-17/11}{2x+5}\,dx = \frac{3}{11}\ln|x-3| - \frac{17}{22}\ln|2x+5| + c$$

Method 2: Substitute suitable values for x in the identity (*1*). For example, letting $x = 3$ and $x = -5/2$ in (*1*), we find at once $A = 3/11$, $B = -17/11$.

5.21. Evaluate $\int \dfrac{dx}{5 + 3\cos x}$ by using the substitution $\tan x/2 = u$.

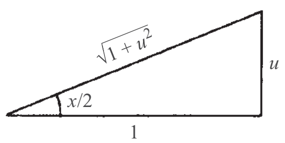


Fig. 5-7

From Fig. 5-7 we see that

$$\sin x/2 = \frac{u}{\sqrt{1+u^2}}, \qquad \cos x/2 = \frac{1}{\sqrt{1+u^2}}$$

Then $\cos x = \cos^2 x/2 - \sin^2 x/2 = \dfrac{1-u^2}{1+u^2}$.

Also $du = \dfrac{1}{2}\sec^2 x/2\,dx$ or $dx = 2\cos^2 x/2\,du = \dfrac{2\,du}{1+u^2}$.

Thus the integral becomes $\displaystyle\int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1} u/2 + c = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan x/2\right) + c.$

5.22. Evaluate $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$.

Let $x = \pi - y$. Then

$$\begin{aligned} I &= \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{(\pi-y)\sin y}{1+\cos^2 y}\,dy = \pi\int_0^\pi \frac{\sin y}{1+\cos^2 y}\,dy - \int_0^\pi \frac{y\sin y}{1+\cos^2 y}\,dy \\ &= -\pi\int_0^\pi \frac{d(\cos y)}{1+\cos^2 y} - I = -\pi\tan^{-1}(\cos y)|_0^\pi - I = \pi^2/2 - I \end{aligned}$$

i.e., $I = \pi^2/2 - I$ or $I = \pi^2/4$.

5.23. Prove that $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$.

Letting $x = \pi/2 - y$, we have

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos y}}{\sqrt{\cos y}+\sqrt{\sin y}}\,dy = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

Then

$$\begin{aligned} I + I &= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \\ &= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \end{aligned}$$

from which $2I = \pi/2$ and $I = \pi/4$.

The same method can be used to prove that for all real values of m,

$$\int_0^{\pi/2} \frac{\sin^m x}{\sin^m x + \cos^m x}\,dx = \frac{\pi}{4}$$

(see Problem 5.89).

Note: This problem and Problem 5.22 show that some definite integrals can be evaluated without first finding the corresponding indefinite integrals.

NUMERICAL METHODS FOR EVALUATING DEFINITE INTEGRALS

5.24. Evaluate $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ approximately, using (*a*) the trapezoidal rule, (*b*) Simpson's rule, where the interval [0, 1] is divided into $n = 4$ equal parts.

Let $f(x) = 1/(1+x^2)$. Using the notation on Page 98, we find $\Delta x = (b-a)/n = (1-0)/4 = 0.25$. Then keeping 4 decimal places, we have: $y_0 = f(0) = 1.0000$, $y_1 = f(0.25) = 0.9412$, $y_2 = f(0.50) = 0.8000$, $y_3 = f(0.75) = 0.6400$, $y_4 = f(1) = 0.50000$.

(*a*) The trapezoidal rule gives

$$\frac{\Delta x}{2}\{y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\} = \frac{0.25}{2}\{1.0000 + 2(0.9412) + 2(0.8000) + 2(0.6400) + 0.500\}$$
$$= 0.7828.$$

(*b*) Simpson's rule gives

$$\frac{\Delta x}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\} = \frac{0.25}{3}\{1.0000 + 4(0.9412) + 2(0.8000) + 4(0.6400) + 0.5000\}$$
$$= 0.7854.$$

The true value is $\pi/4 \approx 0.7854$.

APPLICATIONS (AREA, ARC LENGTH, VOLUME, MOMENT OF INTERTIA)

5.25. Find the (*a*) area and (*b*) moment of inertia about the y-axis of the region in the xy plane bounded by $y = 4 - x^2$ and the x-axis.

(*a*) Subdivide the region into rectangles as in the figure on Page 90. A typical rectangle is shown in the adjoining Fig. 5-8. Then

$$\text{Required area} = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}(4 - \xi_k^2)\,\Delta x_k$$
$$= \int_{-2}^{2}(4 - x^2)\,dx = \frac{32}{3}$$

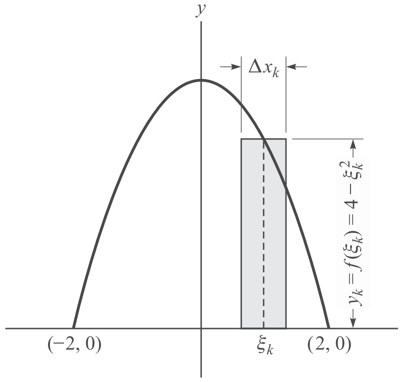


Fig. 5-8

(*b*) Assuming unit density, the moment of inertia about the y-axis of the typical rectangle shown above is $\xi_k^2 f(\xi_k)\,\Delta x_k$. Then

$$\text{Required moment of inertia} = \lim_{n\to\infty}\sum_{k=1}^{n}\xi_k^2 f(\xi_k)\,\Delta x_k = \lim_{n\to\infty}\sum_{k=1}^{n}\xi_k^2(4 - \xi_k^2)\,\Delta x_k$$
$$= \int_{-2}^{2} x^2(4 - x^2)\,dx = \frac{128}{15}$$

5.26. Find the length of arc of the parabola $y = x^2$ from $x = 0$ to $x = 1$.

$$\text{Required arc length} = \int_0^1 \sqrt{1 + (dy/dx)^2}\,dx = \int_0^1 \sqrt{1 + (2x)^2}\,dx$$
$$= \int_0^1 \sqrt{1 + 4x^2}\,dx = \frac{1}{2}\int_0^2 \sqrt{1 + u^2}\,du$$
$$= \tfrac{1}{2}\{\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln(u + \sqrt{1 + u^2})\}|_0^2 = \tfrac{1}{2}\sqrt{5} + \tfrac{1}{4}\ln(2 + \sqrt{5})$$

5.27. (*a*) (Disk Method) Find the volume generated by revolving the region of Problem 5.25 about the x-axis.

$$\text{Required volume} = \lim_{n\to\infty}\sum_{k=1}^{n}\pi y_k^2 \Delta x_k = \pi\int_{-2}^{2}(4 - x^2)^2\,dx = 512\pi/15.$$

(*b*) (Disk Method) Find the volume of the frustrum of a paraboloid obtained by revolving $f(x) = \sqrt{kx}$, $0 < a \leqq x \leqq b$ about the x-axis.

$$V = \pi \int_a^b kx\,dx = \frac{\pi k}{2}(b^2 - a^2).$$

(*c*) (Shell Method) Find the volume obtained by orbiting the region of part (*b*) about the *y*-axis. Compare this volume with that obtained in part (*b*).

$$V = 2\pi \int_0^b x(kx)\,dx = 2\pi k b^3/3$$

The solids generated by the two regions are different, as are the volumes.

MISCELLANEOUS PROBLEMS

5.28 If $f(x)$ and $g(x)$ are continuous in $[a, b]$, prove *Schwarz's inequality for integrals*:

$$\left(\int_a^b f(x)\,g(x)\,dx\right)^2 \leqq \int_a^b \{f(x)\}^2\,dx \int_a^b \{g(x)\}^2\,dx$$

We have

$$\int_a^b \{f(x) + \lambda g(x)\}^2\,dx = \int_a^b \{f(x)\}^2\,dx + 2\lambda \int_a^b f(x)\,g(x)\,dx + \lambda^2 \int_a^b \{g(x)\}^2\,dx \geqq 0$$

for all real values of λ. Hence by Problem 1.13 of Chapter 1, using (*1*) with

$$A^2 = \int_a^b g(x)\}^2\,dx, \qquad B^2 = \int_a^b \{f(x)\}^2\,dx, \qquad C = \int_a^b f(x)\,g(x)\,dx$$

we find $C^2 \leqq A^2B^2$, which gives the required result.

5.29. Prove that $\lim\limits_{M\to\infty} \int_0^M \frac{dx}{x^4+4} = \frac{\pi}{8}$.

We have $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2+2)^2 - (2x)^2 = (x^2+2+2x)(x^2+2-2x)$.
According to the method of partial fractions, assume

$$\frac{1}{x^4+4} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{x^2-2x+2}$$

Then $1 = (A+C)x^3 + (B - 2A + 2C + D)x^2 + (2A - 2B + 2C + 2D)x + 2B + 2D$
so that $A + C = 0,\ B - 2A + 2C + D = 0,\ 2A - 2B + 2C + 2D = 0,\ 2B + 2D = 1$
Solving simultaneously, $A = \frac{1}{8},\ B = \frac{1}{4},\ C = -\frac{1}{8},\ D = \frac{1}{4}$. Thus

$$\begin{aligned}
\int \frac{dx}{x^4+4} &= \frac{1}{8}\int \frac{x+2}{x^2+2x+2}\,dx - \frac{1}{8}\int \frac{x-2}{x^2-2x+2}\,dx \\
&= \frac{1}{8}\int \frac{x+1}{(x+1)^2+1}\,dx + \frac{1}{8}\int \frac{dx}{(x+1)^2+1} - \frac{1}{8}\int \frac{x-1}{(x-1)^2+1}\,dx + \frac{1}{8}\int \frac{dx}{(x-1)^2+1} \\
&= \frac{1}{16}\ln(x^2+2x+2) + \frac{1}{8}\tan^{-1}(x+1) - \frac{1}{16}\ln(x^2-2x+2) + \frac{1}{8}\tan^{-1}(x-1) + C
\end{aligned}$$

Then

$$\lim_{M\to\infty} \int_0^M \frac{dx}{x^4+4} = \lim_{M\to\infty} \left\{\frac{1}{16}\ln\left(\frac{M^2+2M+2}{M^2-2M+2}\right) + \frac{1}{8}\tan^{-1}(M+1) + \frac{1}{8}\tan^{-1}(M-1)\right\} = \frac{\pi}{8}$$

We denote this limit by $\int_0^\infty \frac{dx}{x^4+4}$, called an *improper integral of the first kind*. Such integrals are considered further in Chapter 12. See also Problem 5.74.

5.30. Evaluate $\lim_{x\to 0} \frac{\int_0^x \sin t^3\, dt}{x^4}$.

The conditions of L'Hospital's rule are satisfied, so that the required limit is

$$\lim_{x\to 0} \frac{\frac{d}{dx}\int_0^x \sin t^3\, dt}{\frac{d}{dx}(x^4)} = \lim_{x\to 0} \frac{\sin x^3}{4x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x^3)}{\frac{d}{dx}(4x^3)} = \lim_{x\to 0} \frac{3x^2 \cos x^3}{12x^2} = \frac{1}{4}$$

5.31. Prove that if $f(x)$ is continuous in $[a, b]$ then $\int_a^b f(x)\, dx$ exists.

Let $\sigma = \sum_{k=1}^{n} f(\xi_k)\, \Delta x_k$, using the notation of Page 91. Since $f(x)$ is continuous we can find numbers M_k and m_k representing the l.u.b. and g.l.b. of $f(x)$ in the interval $[x_{k-1}, x_k]$, i.e., such that $m_k \leqq f(x) \leqq M_k$. We then have

$$m(b-a) \leqq s = \sum_{k=1}^{n} m_k \Delta x_k \leqq \sigma \leqq \sum_{k=1}^{n} M_k \Delta x_k = S \leqq M(b-a) \qquad (1)$$

where m and M are the g.l.b. and l.u.b. of $f(x)$ in $[a, b]$. The sums s and S are sometimes called the *lower* and *upper sums*, respectively.

Now choose a second mode of subdivision of $[a, b]$ and consider the corresponding lower and upper sums denoted by s' and S' respectively. We have must

$$s' \leqq S \qquad \text{and} \qquad S' \geqq s \qquad (2)$$

To prove this we choose a third mode of subdivision obtained by using the division points of both the first and second modes of subdivision and consider the corresponding lower and upper sums, denoted by t and T, respectively. By Problem 5.84, we have

$$s \leqq t \leqq T \leqq S' \qquad \text{and} \qquad s' \leqq t \leqq T \leqq S \qquad (3)$$

which proves (*2*).

From (*2*) it is also clear that as the number of subdivisions is increased, the upper sums are monotonic decreasing and the lower sums are monotonic increasing. Since according to (*1*) these sums are also bounded, it follows that they have limiting values which we shall call $\bar{s}$ and $\underline{S}$ respectively. By Problem 5.85, $\bar{s} \leqq \underline{S}$. In order to prove that the integral exists, we must show that $\bar{s} = \underline{S}$.

Since $f(x)$ is continuous in the closed interval $[a, b]$, it is uniformly continuous. Then given any $\epsilon > 0$, we can take each Δx_k so small that $M_k - m_k < \epsilon/(b-a)$. It follows that

$$S - s = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \frac{\epsilon}{b-a} \sum_{k=1}^{n} \Delta x_k = \epsilon \qquad (4)$$

Now $S - s = (S - \underline{S}) + (\underline{S} - \bar{s}) + (\bar{s} - s)$ and it follows that each term in parentheses is positive and so is less than ϵ by (*4*). In particular, since $\underline{S} - \bar{s}$ is a definite number it must be zero, i.e., $\underline{S} = \bar{s}$. Thus, the limits of the upper and lower sums are equal and the proof is complete.

Supplementary Problems

DEFINITION OF A DEFINITE INTEGRAL

5.32. (*a*) Express $\int_0^1 x^3\, dx$ as a limit of a sum. (*b*) Use the result of (*a*) to evaluate the given definite integral. (*c*) Interpret the result geometrically.
Ans. (*b*) $\frac{1}{4}$

5.33. Using the definition, evaluate (*a*) $\int_0^2 (3x+1)\, dx$, (*b*) $\int_3^6 (x^2 - 4x)\, dx$.
Ans. (*a*) 8, (*b*) 9

5.34. Prove that $\lim_{n\to\infty}\left\{\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right\}=\frac{\pi}{4}$.

5.35. Prove that $\lim_{n\to\infty}\left\{\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}=\frac{1}{p+1}\right\}$ if $p>-1$.

5.36. Using the definition, prove that $\int_a^b e^x\,dx=e^b-e^a$.

5.37. Work Problem 5.5 directly, using Problem 1.94 of Chapter 1.

5.38. Prove that $\lim_{n\to\infty}\left\{\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right\}=\ln(1+\sqrt{2})$.

5.39. Prove that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2x^2}=\frac{\tan^{-1}x}{x}$ if $x\neq 0$.

PROPERTIES OF DEFINITE INTEGRALS

5.40. Prove (*a*) Property 2, (*b*) Property 3 on Pages 91 and 92.

5.41. If $f(x)$ is integrable in (a,c) and (c,b), prove that $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$.

5.42. If $f(x)$ and $g(x)$ are integrable in $[a,b]$ and $f(x)\leqq g(x)$, prove that $\int_a^b f(x)\,dx \leqq \int_a^b g(x)\,dx$.

5.43. Prove that $1-\cos x \geqq x^2/\pi$ for $0\leqq x\leqq \pi/2$.

5.44. Prove that $\left|\int_0^1 \frac{\cos nx}{x+1}\,dx\right| \leqq \ln 2$ for all n.

5.45. Prove that $\left|\int_1^{\sqrt{3}} \frac{e^{-x}\sin x}{x^2+1}\,dx\right| \leqq \frac{\pi}{12e}$.

MEAN VALUE THEOREMS FOR INTEGRALS

5.46. Prove the result (5), Page 92. [Hint: If $m\leqq f(x)\leqq M$, then $mg(x)\leqq f(x)g(x)\leqq Mg(x)$. Now integrate and divide by $\int_a^b g(x)\,dx$. Then apply Theorem 9 in Chapter 3.

5.47. Prove that there exist values ξ_1 and ξ_2 in $0\leqq x\leqq 1$ such that

$$\int_0^1 \frac{\sin \pi x}{x^2+1}\,dx=\frac{2}{\pi(\xi_1^2+1)}=\frac{\pi}{4}\sin \pi\xi_2$$

Hint: Apply the first mean value theorem.

5.48. (*a*) Prove that there is a value ξ in $0\leqq x\leqq \pi$ such that $\int_0^{\pi} e^{-x}\cos x\,dx=\sin\xi$. (*b*) Suppose a wedge in the shape of a right triangle is idealized by the region bound by the x-axis, $f(x)=x$, and $x=L$. Let the weight distribution for the wedge be defined by $W(x)=x^2+1$. Use the generalized mean value theorem to show that the point at which the weighted value occurs is $\frac{3L}{4}\frac{L^2+2}{L^2+3}$.

CHANGE OF VARIABLES AND SPECIAL METHODS OF INTEGRATION

5.49. Evaluate: (*a*) $\int x^2 e^{\sin x^3} \cos x^3\, dx$, (*b*) $\int_0^1 \frac{\tan^{-1} t}{1+t^2}\, dt$, (*c*) $\int_1^3 \frac{dx}{\sqrt{4x-x^2}}$, (*d*) $\int \frac{\operatorname{csch}^2 \sqrt{u}}{\sqrt{u}}\, du$, (*e*) $\int_{-2}^2 \frac{dx}{16-x^2}$.

Ans. (*a*) $\frac{1}{3} e^{\sin x^3} + c$, (*b*) $\pi^2/32$, (*c*) $\pi/3$, (*d*) $-2\coth\sqrt{u} + c$, (*e*) $\frac{1}{4}\ln 3$.

5.50. Show that (*a*) $\int_0^1 \frac{dx}{(3+2x-x^2)^{3/2}} = \frac{\sqrt{3}}{12}$, (*b*) $\int \frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} + c$.

5.51. Prove that (*a*) $\int \sqrt{u^2 \pm a^2}\, du = \frac{1}{2} u\sqrt{u^2 \pm a^2} \pm \frac{1}{2} a^2 \ln|u + \sqrt{u^2 \pm a^2}|$

(*b*) $\int \sqrt{a^2 - u^2}\, du = \frac{1}{2} u \sqrt{a^2 - u^2} + \frac{1}{2} a^2 \sin^{-1} u/a + c, \quad a > 0.$

5.52. Find $\int \frac{x\, dx}{\sqrt{x^2+2x+5}}$. *Ans.* $\sqrt{x^2+2x+5} - \ln|x+1+\sqrt{x^2+2x+5}| + c$.

5.53. Establish the validity of the method of integration by parts.

5.54. Evaluate (*a*) $\int_0^\pi x\cos 3x\, dx$, (*b*) $\int x^3 e^{-2x}\, dx$. *Ans.* (*a*) $-2/9$, (*b*) $-\frac{1}{3} e^{-2x}(4x^3 + 6x^2 + 6x + 3) + c$

5.55. Show that (*a*) $\int_0^1 x^2 \tan^{-1} x\, dx = \frac{1}{12}\pi - \frac{1}{6} + \frac{1}{6}\ln 2$

(*b*) $\int_{-2}^2 \sqrt{x^2+x+1}\, dx = \frac{5\sqrt{7}}{4} + \frac{3\sqrt{3}}{4} + \frac{3}{8}\ln\left(\frac{5+2\sqrt{7}}{2\sqrt{3}-3}\right)$.

5.56. (*a*) If $u = f(x)$ and $v = g(x)$ have continuous nth derivatives, prove that

$$\int uv^{(n)}\, dx = uv^{(n-1)} - u'v^{(n-2)} + u''v^{(n-3)} - \cdots - (-1)^n \int u^{(n)} v\, dx$$

called *generalized integration by parts.* (*b*) What simplifications occur if $u^{(n)} = 0$? Discuss. (*c*) Use (*a*) to evaluate $\int_0^\pi x^4 \sin x\, dx$. *Ans.* (*c*) $\pi^4 - 12\pi^2 + 48$

5.57. Show that $\int_0^1 \frac{x\, dx}{(x+1)^2(x^2+1)} = \frac{\pi - 2}{8}$.

[Hint: Use partial fractions, i.e., assume $\frac{x}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$ and find A, B, C, D.]

5.58. Prove that $\int_0^\pi \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}, \quad \alpha > 1.$

NUMERICAL METHODS FOR EVALUATING DEFINITE INTEGRALS

5.59. Evaluate $\int_0^1 \frac{dx}{1+x}$ approximately, using (*a*) the trapezoidal rule, (*b*) Simpson's rule, taking $n = 4$. Compare with the exact value, $\ln 2 = 0.6931$.

5.60. Using (*a*) the trapezoidal rule, (*b*) Simpson's rule evaluate $\int_0^{\pi/2} \sin^2 x\, dx$ by obtaining the values of $\sin^2 x$ at $x = 0^\circ, 10^\circ, \ldots, 90^\circ$ and compare with the exact value $\pi/4$.

5.61. Prove the (*a*) rectangular rule, (*b*) trapezoidal rule, i.e., (*16*) and (*17*) of Page 98.

5.62. Prove Simpson's rule.

5.63. Evaluate to 3 decimal places using numerical integration: (*a*) $\int_1^2 \frac{dx}{1+x^2}$, (*b*) $\int_0^1 \cosh x^2\, dx$.
Ans. (*a*) 0.322, (*b*) 1.105.

APPLICATIONS

5.64. Find the (*a*) area and (*b*) moment of inertia about the *y*-axis of the region in the *xy* plane bounded by $y = \sin x$, $0 \leqq x \leqq \pi$ and the *x*-axis, assuming unit density.
Ans. (*a*) 2, (*b*) $\pi^2 - 4$

5.65. Find the moment of inertia about the *x*-axis of the region bounded by $y = x^2$ and $y = x$, if the density is proportional to the distance from the *x*-axis.
Ans. $\frac{1}{8}M$, where M = mass of the region.

5.66. (*a*) Show that the arc length of the *catenary* $y = \cosh x$ from $x = 0$ to $x = \ln 2$ is $\frac{3}{4}$. (*b*) Show that the length of arc of $y = x^{3/2}$, $2 \leqq x \leqq 5$ is $\frac{343}{27} - 2\sqrt{2}\,11^{3/2}$.

5.67. Show that the length of one arc of the *cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $(0 \leqq \theta \leqq 2\pi)$ is $8a$.

5.68. Prove that the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab.

5.69. (*a*) (Disk Method) Find the volume of the region obtained by revolving the curve $y = \sin x$, $0 \leqq x \leqq \pi$, about the *x*-axis. *Ans.* (*a*) $\pi^2/2$
(*b*) (Disk Method) Show that the volume of the frustrum of a paraboloid obtained by revolving $f(x) = \sqrt{kx}$, $0 < a \leqq x \leqq b$, about the *x*-axis is $\pi \int_a^b kx\, dx = \frac{\pi k}{2}(b^2 - a^2)$. (*c*) Determine the volume obtained by rotating the region bound by $f(x) = 3$, $g(x) = 5 - x^2$ on $-\sqrt{2} \leqq x \leqq \sqrt{2}$. (*d*) (Shell Method) A spherical bead of radius *a* has a circular cylindrical hole of radius *b*, $b < a$, through the center. Find the volume of the remaining solid by the shell method. (*e*) (Shell Method) Find the volume of a solid whose outer boundary is a torus (i.e., the solid is generated by orbiting a circle $(x - a)^2 + y^2 = b^2$ about the *y*-axis $(a > b)$.

5.70. Prove that the centroid of the region bounded by $y = \sqrt{a^2 - x^2}$, $-a \leqq x \leqq a$ and the *x*-axis is located at $(0, 4a/3\pi)$.

5.71. (*a*) If $\rho = f(\phi)$ is the equation of a curve in polar coordinates, show that the area bounded by this curve and the lines $\phi = \phi_1$ and $\phi = \phi_2$ is $\frac{1}{2}\int_{\phi_1}^{\phi_2} \rho^2 d\phi$. (*b*) Find the area bounded by one loop of the *lemniscate* $\rho^2 = a^2 \cos 2\phi$.
Ans. (*b*) a^2

5.72. (*a*) Prove that the arc length of the curve in Problem 5.71(*a*) is $\int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + (d\rho/d\phi)^2}\, d\phi$. (*b*) Find the length of arc of the *cardioid* $\rho = a(1 - \cos\phi)$.
Ans. (*b*) $8a$

MISCELLANEOUS PROBLEMS

5.73. Establish the mean value theorem for derivatives from the first mean value theorem for integrals. [Hint: Let $f(x) = F'(x)$ in (*4*), Page 93.]

5.74. Prove that (*a*) $\lim_{\epsilon \to 0+} \int_0^{4-\epsilon} \frac{dx}{\sqrt{4-x}} = 4$, (*b*) $\lim_{\epsilon \to 0+} \int_\epsilon^3 \frac{dx}{\sqrt[3]{x}} = 6$, (*c*) $\lim_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$ and give a geometric interpretation of the results.

[These limits, denoted usually by $\int_0^4 \frac{dx}{\sqrt{4-x}}$, $\int_0^3 \frac{dx}{\sqrt[3]{x}}$ and $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ respectively, are called *improper integrals of the second kind* (see Problem 5.29) since the integrands are not bounded in the range of integration. For further discussion of improper integrals, see Chapter 12.]

5.75. Prove that (*a*) $\lim_{M \to \infty} \int_0^M x^5 e^{-x}\, dx = 4! = 24$, (*b*) $\lim_{\epsilon \to 0+} \int_1^{2-\epsilon} \frac{dx}{\sqrt{x(2-x)}} = \frac{\pi}{2}$.

5.76. Evaluate (*a*) $\int_0^\infty \frac{dx}{1+x^3}$, (*b*) $\int_0^{\pi/2} \frac{\sin 2x}{(\sin x)^{4/3}}\,dx$, (*c*) $\int_0^\infty \frac{dx}{x+\sqrt{x^2+1}}$.

Ans. (*a*) $\frac{2\pi}{3\sqrt{3}}$ (*b*) 3 (*c*) does not exist

5.77. Evaluate $\lim_{x\to\pi/2} \frac{ex^2/\pi - e\pi/4 + \int_x^{\pi/2} e^{\sin t}\,dt}{1+\cos 2x}$. *Ans.* $e/2\pi$

5.78. Prove: (*a*) $\frac{d}{dx}\int_{x^2}^{x^3} (t^2+t+1)\,dt = 3x^3+x^5-2x^3+3x^2-2x$, (*b* $\frac{d}{dx}\int_x^{x^2} \cos t^2\,dt = 2x\cos x^4 - \cos x^2$.

5.79. Prove that (*a*) $\int_0^\pi \sqrt{1+\sin x}\,dx = 4$, (*b*) $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \sqrt{2}\ln(\sqrt{2}+1)$.

5.80. Explain the fallacy: $I = \int_{-1}^{1} \frac{dx}{1+x^2} = -\int_{-1}^{1} \frac{dy}{1+y^2} = -I$, using the transformation $x = 1/y$. Hence $I = 0$. But $I = \tan^{-1}(1) - \tan^{-1}(-1) = \pi/4 - (-\pi/4) = \pi/2$. Thus $\pi/2 = 0$.

5.81. Prove that $\int_0^{1/2} \frac{\cos \pi x}{\sqrt{1+x^2}}\,dx \leqq \frac{1}{4}\tan^{-1}\frac{1}{2}$.

5.82. Evaluate $\lim_{n\to\infty} \left\{\frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{2n-1}}{n^{3/2}}\right\}$. *Ans.* $\frac{2}{3}(2\sqrt{2}-1)$

5.83. Prove that $f(x) = \begin{cases} 1 \text{ if } x \text{ is irrational} \\ 0 \text{ if } x \text{ is rational} \end{cases}$ is not Riemann integrable in [0, 1].

[Hint: In (*2*), Page 91, let ξ_k, $k = 1, 2, 3, \ldots, n$ be first rational and then irrational points of subdivision and examine the lower and upper sums of Problem 5.31.]

5.84. Prove the result (*3*) of Problem 5.31. [Hint: First consider the effect of only one additional point of subdivision.]

5.85. In Problem 5.31, prove that $\bar{s} \leqq \underline{S}$. [Hint: Assume the contrary and obtain a contradiction.]

5.86. If $f(x)$ is sectionally continuous in $[a, b]$, prove that $\int_a^b f(x)\,dx$ exists. [Hint: Enclose each point of discontinuity in an interval, noting that the sum of the lengths of such intervals can be made arbitrarily small. Then consider the difference between the upper and lower sums.

5.87. If $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 3 & x = 1 \\ 6x-1 & 1 < x < 2 \end{cases}$, find $\int_0^2 f(x)\,dx$. Interpret the result graphically. *Ans.* 9

5.88. Evaluate $\int_0^3 \{x - [x] + \frac{1}{2}\}\,dx$ where $[x]$ denotes the greatest integer less than or equal to x. Interpret the result graphically. *Ans.* 3

5.89. (*a*) Prove that $\int_0^{\pi/2} \frac{\sin^m x}{\sin^m x + \cos^m x}\,dx = \frac{\pi}{4}$ for all real values of m.

(*b*) Prove that $\int_0^{2\pi} \frac{dx}{1+\tan^4 x} = \pi$.

5.90. Prove that $\int_0^{\pi/2} \frac{\sin x}{x}\,dx$ exists.

5.91. Show that $\int_0^{0.5} \frac{\tan^{-1} x}{x}\,dx = 0.4872$ approximately.

5.92. Show that $\int_0^\pi \frac{x\,dx}{1+\cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$.