

# Double and Iterated Integrals

## The Double Integral

Consider a function  $z = f(x, y)$  that is continuous on a bounded region  $R$  of the  $xy$  plane. Define a partition  $\mathcal{P}$  of  $R$  by drawing a grid of horizontal and vertical lines. This divides the region into  $n$  subregions  $R_1, R_2, \dots, R_n$  of areas  $\Delta_1A, \Delta_2A, \dots, \Delta_nA$ , respectively. (See Fig. 54-1.) In each subregion,  $R_k$ , choose a point  $P_k(x_k, y_k)$  and form the sum

$$\sum_{k=1}^n f(x_k, y_k) \Delta_k A = f(x_1, y_1) \Delta_1 A + \dots + f(x_n, y_n) \Delta_n A \quad (54.1)$$

Define the diameter of a subregion to be the greatest distance between any two points within or on its boundary, and denote by  $d_{\mathcal{P}}$  the maximum diameter of the subregions. Suppose that we select partitions so that  $d_{\mathcal{P}} \rightarrow 0$  and  $n \rightarrow +\infty$ . (In other words, we choose more and more subregions and we make their diameters smaller and smaller.) Then the *double integral* of  $f(x, y)$  over  $R$  is defined as

$$\iint_R f(x, y) dA = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(x_k, y_k) \Delta_k A \quad (54.2)$$

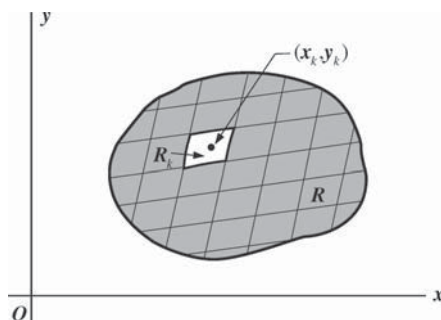


Fig. 54-1

This is not a genuine limit statement. What (54.2) really says is that  $\iint_R f(x, y) dA$  is a number such that, for any  $\epsilon > 0$ , there exists a positive integer  $n_0$  such that, for any  $n \geq n_0$  and any partition with  $d_{\mathcal{P}} < 1/n_0$ , and any corresponding approximating sum  $\sum_{k=1}^n f(x_k, y_k) \Delta_k A$ , we have

$$\left| \sum_{k=1}^n f(x_k, y_k) \Delta_k A - \iint_R f(x, y) dA \right| < \epsilon$$

When  $z = f(x, y)$  is nonnegative on the region  $R$ , as in Fig. 54-2, the double integral (54.2) may be interpreted as a volume. Any term  $f(x_k, y_k) \Delta_k A$  of (54.1) gives the volume of a vertical column whose base is of

area  $\Delta_k A$  and whose altitude is the distance  $z_k = f(x_k, y_k)$  measured along the vertical from the selected point  $P_k(x_k, y_k)$  to the surface  $z = f(x, y)$ . This, in turn, may be taken as an approximation of the volume of the vertical column whose lower base is the subregion  $R_k$  and whose upper base is the projection of  $R_k$  on the surface. Thus, (54.1) is an approximation of the volume “under the surface” (that is, the volume with lower base  $R$  and upper base the surface cut off by moving a line parallel to the  $z$  axis along the boundary of  $R$ ). It is intuitively clear that (54.2) is the measure of this volume.

The evaluation of even the simplest double integral by direct summation is usually very difficult.

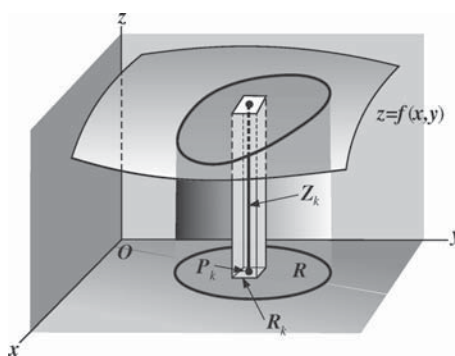


Fig. 54-2

### The Iterated Integral

Consider a volume defined as above, and assume that the boundary of  $R$  is such that no line parallel to the  $x$  axis or to the  $y$  axis cuts it in more than two points. Draw the tangent lines  $x = a$  and  $x = b$  to the boundary with points of tangency  $K$  and  $L$ , and the tangent lines  $y = c$  and  $y = d$  with points of tangency  $M$  and  $N$ . (See Fig. 54-3.) Let the equation of the plane arc  $LMK$  be  $y = g_1(x)$ , and that of the plane arc  $LNK$  be  $y = g_2(x)$ .

Divide the interval  $a \leq x \leq b$  into  $m$  subintervals  $h_1, h_2, \dots, h_m$  of respective lengths  $\Delta_1 x, \Delta_2 x, \dots, \Delta_m x$  by the insertion of points  $\xi_1, \xi_2, \dots, \xi_{m-1}$  so that  $a = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_{m-1} < \xi_m = b$ . Similarly, divide the interval  $c \leq y \leq d$  into  $n$  subintervals  $k_1, k_2, \dots, k_n$  of respective lengths  $\Delta_1 y, \Delta_2 y, \dots, \Delta_n y$  by the insertion points  $\eta_1, \eta_2, \dots, \eta_{n-1}$  so that  $c = \eta_0 < \eta_1 < \eta_2 < \dots < \eta_{n-1} < \eta_n = d$ . Let  $\lambda_m$  be the greatest  $\Delta x$  and let  $\mu_n$  be the greatest  $\Delta y$ . Draw the parallel lines  $x = \xi_1, x = \xi_2, \dots, x = \xi_{m-1}$  and the parallel lines  $y = \eta_1, y = \eta_2, \dots, y = \eta_{n-1}$ , thus dividing the region  $R$  into a set of rectangles  $R_{ij}$  of areas  $\Delta_i x \Delta_j y$ , plus a set of nonrectangles along the boundary (whose areas will be small enough to be safely ignored). In each subinterval  $h_i$  select a point  $x = x_i$  and, in each subinterval  $k_j$  select a point  $y = y_j$ , thereby determining in each subregion  $R_{ij}$  a point  $P_{ij}(x_i, y_j)$ . With each subregion  $R_{ij}$  associate, by means of the equation of the surface, a number  $z_{ij} = f(x_i, y_j)$ , and form the sum

$$\sum_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,n}} f(x_i, y_j) \Delta_i x \Delta_j y \tag{54.3}$$

Now, (54.3) is merely a special case of (54.1). So, if the number of rectangles is indefinitely increased in such a manner that both  $\lambda_m \rightarrow 0$  and  $\mu_n \rightarrow 0$ , the limit of (54.3) should be equal to the double integral (54.2).

In effecting this limit, let us first choose one of the subintervals, say  $h_i$ , and form the sum

$$\left[ \sum_{j=1}^n f(x_i, y_j) \Delta_j y \right] \Delta_i x \quad (i \text{ fixed})$$

of the contributions of all rectangles having  $h_i$  as one dimension, that is, the contributions of all rectangles lying on the  $i$ th column. When  $n \rightarrow +\infty, \mu_n \rightarrow 0$ ,

$$\lim_{n \rightarrow +\infty} \left[ \sum_{j=1}^n f(x_i, y_j) \Delta_j y \right] \Delta_i x = \left[ \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y) dy \right] \Delta_i x = \phi(x_i) \Delta_i x$$

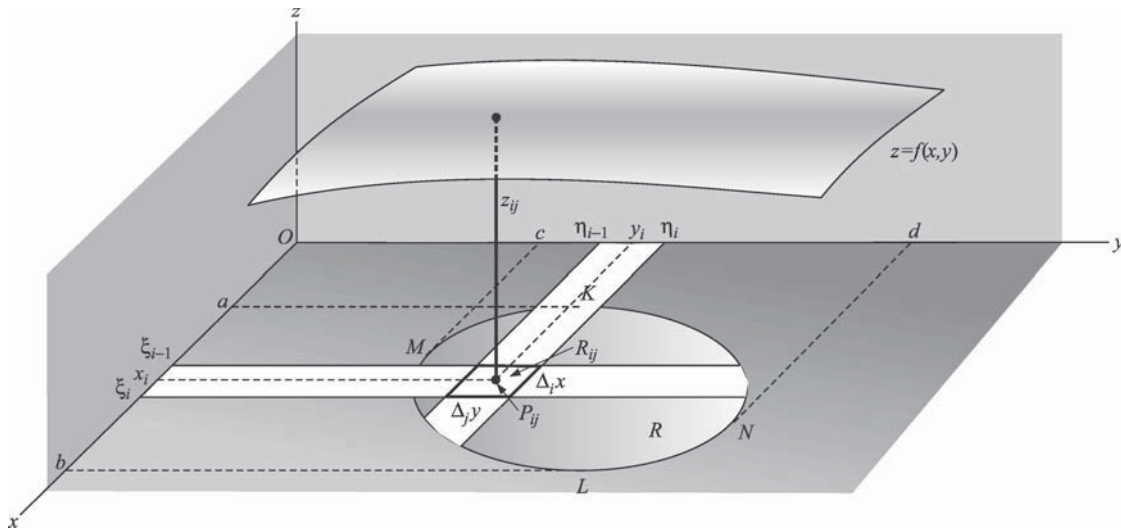


Fig. 54-3

Now summing over the  $m$  columns and letting  $m \rightarrow +\infty$ , we have

$$\begin{aligned} \lim_{m \rightarrow +\infty} \sum_{i=1}^m \phi(x_i) \Delta_i x &= \int_a^b \phi(x) dx = \int_a^b \left[ \int_{g_1(x_i)}^{g_2(x_i)} f(x, y) dy \right] dx \\ &= \int_a^b \int_{g_1(x_i)}^{g_2(x_i)} f(x, y) dy dx \end{aligned} \quad (54.4)$$

Although we shall not use the brackets hereafter, it must be clearly understood that (54.4) calls for the evaluation of two simple definite integrals in a prescribed order: first, the integral of  $f(x, y)$  with respect to  $y$  (considering  $x$  as a constant) from  $y = g_1(x)$ , the lower boundary of  $R$ , to  $y = g_2(x)$ , the upper boundary of  $R$ , and then the integral of this result with respect to  $x$  from the abscissa  $x = a$  of the leftmost point of  $R$  to the abscissa  $x = b$  of the rightmost point of  $R$ . The integral (54.4) is called an *iterated* or *repeated integral*.

It will be left as an exercise to sum first for the contributions of the rectangles lying in each row and then over all the rows to obtain the equivalent iterated integral

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy \quad (54.5)$$

where  $x = h_1(y)$  and  $x = h_2(y)$  are the equations of the plane arcs  $MKN$  and  $MLN$ , respectively.

In Problem 1, it is shown by a different procedure that the iterated integral (54.4) measures the volume under discussion. For the evaluation of iterated integrals, see Problems 2 to 6.

The principal difficulty in setting up the iterated integrals of the next several chapters will be that of inserting the limits of integration to cover the region  $R$ . The discussion here assumed the simplest of regions; more complex regions are considered in Problems 7 to 9.

### SOLVED PROBLEMS

- Let  $z = f(x, y)$  be nonnegative and continuous over the region  $R$  of the  $xy$  plane whose boundary consists of the arcs of two curves  $y = g_1(x)$  and  $y = g_2(x)$  intersecting at the points  $K$  and  $L$ , as in Fig. 54-4. Find a formula for the volume  $V$  under the surface  $z = f(x, y)$ .

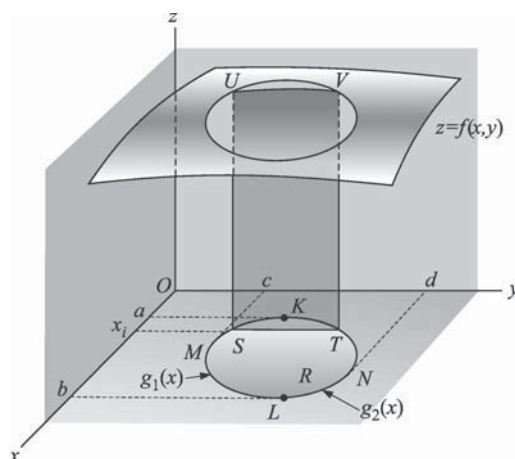


Fig. 54-4

Let the section of this volume cut by a plane  $x = x_i$ , where  $a < x_i < b$ , meet the boundary of  $R$  at the points  $S(x_i, g_1(x_i))$  and  $T(x_i, g_2(x_i))$ , and let it meet the surface  $z = f(x, y)$  in the arc  $UV$  along which  $z = f(x_i, y)$ . The area of this section  $STUV$  is given by

$$A(x_i) = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y) dy$$

Thus, the areas of cross sections of the volume cut by planes parallel to the  $yz$  plane are known functions  $A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy$  of  $x$ , where  $x$  is the distance of the sectioning plane from the origin. By the cross-section formula of Chapter 30, the required volume is given by

$$V = \int_a^b A(x) dx = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$$

This is the iterated integral of (54.4).

In Problems 2–6, evaluate the integral on the left.

2.  $\int_0^1 \int_{x^2}^x dy dx = \int_0^1 [y]_{x^2}^x dx = \int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$
3.  $\int_1^2 \int_y^{3y} (x + y) dx dy = \int_1^2 \left[ \frac{1}{2} x^2 + xy \right]_y^{3y} dy = \int_1^2 6y^2 dy = [2y^3]_1^2 = 14$
4.  $\int_{-1}^2 \int_{2x^2-2}^{x^2+x} x dy dx = \int_{-1}^2 [xy]_{2x^2-2}^{x^2+x} dx = \int_{-1}^2 (x^3 + x^2 - 2x^3 + 2x) dx = \frac{9}{4}$
5.  $\int_0^\pi \int_0^{\cos \theta} \rho \sin \theta d\rho d\theta = \int_0^\pi \left[ \frac{1}{2} \rho^2 \sin \theta \right]_0^{\cos \theta} d\theta = \frac{1}{2} \int_0^\pi \cos^2 \theta \sin \theta d\theta = \left[ -\frac{1}{6} \cos^3 \theta \right]_0^\pi = \frac{1}{3}$
6.  $\int_0^{\pi/2} \int_2^{4 \cos \theta} \rho^3 d\rho d\theta = \int_0^{\pi/2} \left[ \frac{1}{4} \rho^4 \right]_2^{4 \cos \theta} d\theta = \int_0^{\pi/2} (64 \cos^4 \theta - 4) d\theta$   
 $= \left[ 64 \left( \frac{3\theta}{8} + \frac{\sin \theta}{4} + \frac{\sin 4\theta}{32} \right) - 4\theta \right]_0^{\pi/2} = 10\pi$
7. Evaluate  $\iint_R dA$ , where  $R$  is the region in the first quadrant bounded by the semicubical parabola  $y^2 = x^3$  and the line  $y = x$ .

The line and parabola intersect in the points  $(0, 0)$  and  $(1, 1)$ , which establish the extreme values of  $x$  and  $y$  on the region  $R$ .

*Solution 1* (Fig. 54-5): Integrating first over a horizontal strip, that is, with respect to  $x$  from  $x = y$  (the line) to  $x = y^{2/3}$  (the parabola), and then with respect to  $y$  from  $y = 0$  to  $y = 1$ , we get

$$\iint_R dA = \int_0^1 \int_y^{y^{2/3}} dx dy = \int_0^1 (y^{2/3} - y) dy = \left[ \frac{3}{5} y^{5/3} - \frac{1}{2} y^2 \right]_0^1 = \frac{1}{10}$$

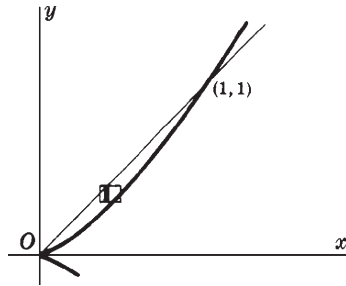


Fig. 54-5

*Solution 2* (Fig. 54-6): Integrating first over a vertical strip, that is, with respect to  $y$  from  $y = x^{3/2}$  (the parabola) to  $y = x$  (the line), and then with respect to  $x$  from  $x = 0$  to  $x = 1$ , we obtain

$$\iint_R dA = \int_0^1 \int_{x^{3/2}}^x dy dx = \int_0^1 (x - x^{3/2}) dx = \left[ \frac{1}{2} x^2 - \frac{2}{5} x^{5/2} \right]_0^1 = \frac{1}{10}$$

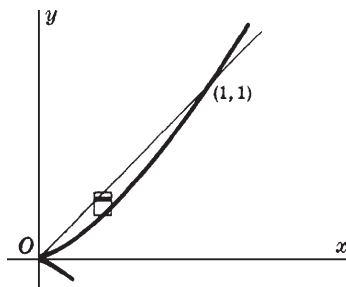


Fig. 54-6

8. Evaluate  $\iint_R dA$  where  $R$  is the region between  $y = 2x$  and  $y = x^2$  lying to the left of  $x = 1$ .  
Integrating first over the vertical strip (see Fig. 54-7), we have

$$\iint_R dA = \int_0^1 \int_{x^2}^{2x} dy dx = \int_0^1 (2x - x^2) dx = \frac{2}{3}$$

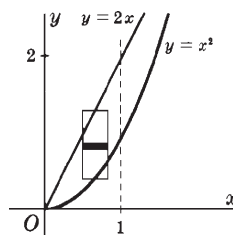


Fig. 54-7

When horizontal strips are used (see Fig. 54-8), two iterated integrals are necessary. Let  $R_1$  denote the part of  $R$  lying below  $AB$ , and  $R_2$  the part above  $AB$ . Then

$$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{y/2}^{\sqrt{y}} dx dy + \int_1^2 \int_{y/2}^1 dx dy = \frac{5}{12} + \frac{1}{4} = \frac{2}{3}$$

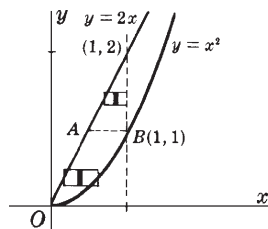


Fig. 54-8

9. Evaluate  $\iint_R x^2 dA$  where  $R$  is the region in the first quadrant bounded by the hyperbola  $xy = 16$  and the lines  $y = x$ ,  $y = 0$ , and  $x = 8$ . (See Fig. 54-9.)

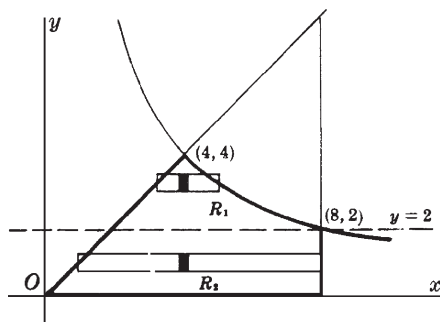


Fig. 54-9

It is evident from Fig. 54-9 that  $R$  must be separated into two regions, and an iterated integral evaluated for each. Let  $R_1$  denote the part of  $R$  lying above the line  $y = 2$ , and  $R_2$  the part below that line. Then

$$\begin{aligned} \iint_R x^2 dA &= \iint_{R_1} x^2 dA + \iint_{R_2} x^2 dA = \int_2^4 \int_y^{16/y} x^2 dx dy + \int_0^2 \int_y^8 x^2 dx dy \\ &= \frac{1}{3} \int_2^4 \left( \frac{16^3}{y^3} - y^3 \right) dy + \frac{1}{3} \int_0^2 (8^3 - y^3) dy = 448 \end{aligned}$$

As an exercise, you might separate  $R$  with the line  $x = 4$  and obtain

$$\iint_R x^2 dA = \int_0^4 \int_0^x x^2 dy dx + \int_4^8 \int_0^{16/x} x^2 dy dx$$

10. Evaluate  $\int_0^1 \int_{3y}^3 e^{x^2} dx dy$  by first reversing the order of integration.

The given integral cannot be evaluated directly, since  $\int e^{x^2} dx$  is not an elementary function. The region  $R$  of integration (see Fig. 54-10) is bounded by the lines  $x = 3y$ ,  $x = 3$ , and  $y = 0$ . To reverse the order of integration, first integrate with respect to  $y$  from  $y = 0$  to  $y = x/3$ , and then with respect to  $x$  from  $x = 0$  to  $x = 3$ . Thus,

$$\begin{aligned} \int_0^1 \int_{3y}^3 e^{x^2} dx dy &= \int_0^3 \int_0^{x/3} e^{x^2} dy dx = \int_0^3 [e^{x^2} y]_0^{x/3} dx \\ &= \frac{1}{3} \int_0^3 e^{x^2} x dx = \left[ \frac{1}{6} e^{x^2} \right]_0^3 = \frac{1}{6} (e^9 - 1) \end{aligned}$$

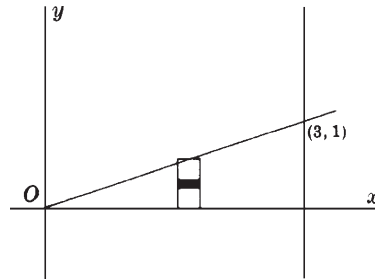


Fig. 54-10

### SUPPLEMENTARY PROBLEMS

11. Evaluate the iterated integral at the left:

(a)  $\int_0^1 \int_1^2 dx dy = 1$

(b)  $\int_0^2 \int_0^3 (x + y) dx dy = 9$

(c)  $\int_2^4 \int_1^2 (x^2 + y^2) dy dx = \frac{70}{3}$

(d)  $\int_0^1 \int_{x^2}^x xy^2 dy dx = \frac{1}{40}$

(e)  $\int_1^2 \int_0^{y^{3/2}} x/y^2 dx dy = \frac{3}{4}$

(f)  $\int_0^1 \int_x^{\sqrt{x}} (x + y^3) dy dx = \frac{7}{60}$

(g)  $\int_0^1 \int_0^{x^2} xe^y dy dx = \frac{1}{2}e - 1$

(h)  $\int_2^4 \int_y^{8-y} y dx dy = \frac{32}{3}$

(i)  $\int_0^{\tan^{-1}(3/2)} \int_0^{2 \sec \theta} \rho d\rho d\theta = 3$

(j)  $\int_0^{\pi/2} \int_0^2 \rho^2 \cos \theta d\rho d\theta = \frac{8}{3}$

(k)  $\int_0^{\pi/4} \int_0^{\tan \theta \sec \theta} \rho^3 \cos^2 \theta d\rho d\theta = \frac{1}{20}$

(l)  $\int_0^{2\pi} \int_0^{1-\cos \theta} \rho^3 \cos^2 \theta d\rho d\theta = \frac{49}{32} \pi$

12. Using an iterated integral, evaluate each of the following double integrals. When feasible, evaluate the iterated integrals in both orders.

(a)  $x$  over the region bounded by  $y = x^2$  and  $y = x^3$

*Ans.*  $\frac{1}{20}$

(b)  $y$  over the region of part (a)

*Ans.*  $\frac{1}{35}$

(c)  $x^2$  over the region bounded by  $y = x$ ,  $y = 2x$ , and  $x = 2$

*Ans.* 4

(d) 1 over each first-quadrant region bounded by  $2y = x^2$ ,  $y = 3x$ , and  $x + y = 4$

*Ans.*  $\frac{8}{3}$ ;  $\frac{46}{3}$

(e)  $y$  over the region above  $y = 0$  bounded by  $y^2 = 4x$  and  $y^2 = 5 - x$

*Ans.* 5

(f)  $\frac{1}{\sqrt{2y - y^2}}$  over the region in the first quadrant bounded by  $x^2 = 4 - 2y$

*Ans.* 4

13. In Problems 11(a) to (h), reverse the order of integration and evaluate the resulting iterated integral.