

# Applications of Integration I: Area and Arc Length

# Area Between a Curve and the y Axis

We already know how to find the area of a region like that shown in Fig. 29-1, bounded below by the x axis, above by a curve y = f(x), and lying between x = a and x = b. The area is the definite integral  $\int_{a}^{b} f(x) dx$ .

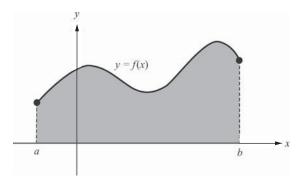


Fig. 29-1

Now consider a region like that shown in Fig. 29-2, bounded on the left by the *y* axis, on the right by a curve x = g(y), and lying between y = c and y = d. Then, by an argument similar to that for the case shown in Fig. 29-1, the area of the region is the definite integral  $\int_{c}^{d} g(y) dy$ .

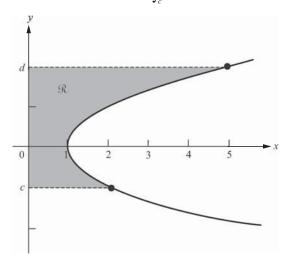


Fig. 29-2

**EXAMPLE 29.1:** Consider the region bounded on the right by the parabola  $x = 4 - y^2$ , on the left by the y axis, and above and below by y = 2 and y = -1. See Fig. 29-3. Then the area of this region is  $\int_{-1}^{2} (4 - y^2) dy$ . By the Fundamental Theorem of Calculus, this is

$$(4y - \frac{1}{3}y^3)_{-1}^2 = (8 - \frac{8}{3}) - (-4 - (-\frac{1}{3})) = 12 - \frac{9}{3} = 12 - 3 = 9$$

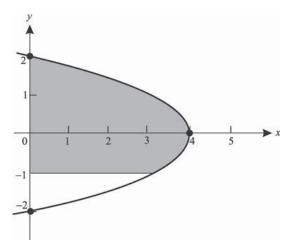


Fig. 29-3

## **Areas Between Curves**

Assume that f and g are continuous functions such that  $g(x) \le f(x)$  for  $a \le x \le b$ . Then the curve y = f(x) lies above the curve y = g(x) between x = a and x = b. The area A of the region between the two curves and lying between x = a and x = b is given by the formula

$$A = \int_{a}^{b} (f(x) - g(x)) dx$$
 (29.1)

To see why this formula holds, first look at the special case where  $0 \le g(x) \le f(x)$  for  $a \le x \le b$ . (See Fig. 29-4.) Clearly, the area is the difference between two areas, the area  $A_f$  of the region under the curve y = f(x) and above the x axis, and the area  $A_g$  of the region under the curve y = g(x) and above the x axis. Since  $A_f = \int_a^b f(x) dx$  and  $A_g = \int_a^b g(x) dx$ ,

$$A = A_f - A_g = \int_a^b f(x) dx - \int_a^b g(x) dx$$
$$= \int_a^b (f(x) - g(x)) dx \quad \text{by (23.6)}$$

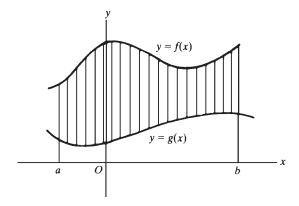
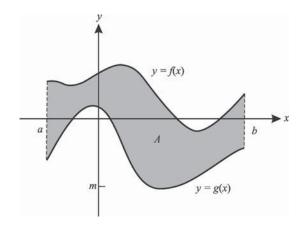


Fig. 29-4

Now look at the general case (see Fig. 29-5), when one or both of the curves y = f(x) and y = g(x) may lie below the x axis. Let m < 0 be the absolute minimum of g on [a, b]. Raise both curves by |m| units. The new graphs, shown in Fig. 29-6, are on or above the x axis and enclose the same area A as the original graphs. The upper curve is the graph of y = f(x) + |m| and the lower curve is the graph of y = g(x) + |m|. Hence, by the special case above,

$$A = \int_{a}^{b} ((f(x) + |m| - (g(x) + |m|)) dx = \int_{a}^{b} (f(x) - g(x)) dx$$



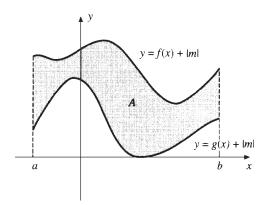


Fig. 29-5

Fig. 29-6

**EXAMPLE 29.2:** Find the area *A* of the region  $\mathcal{R}$  under the line  $y = \frac{1}{2}x + 2$ , above the parabola  $y = x^2$ , and between the *y* axis and x = 1. (See the shaded region in Fig. 29-7.) By (29.1),

$$A = \int_0^1 \left( \left( \frac{1}{2}x + 2 \right) - x^2 \right) dx = \left( \frac{1}{4}x^2 + 2x - \frac{1}{3}x^3 \right) \Big|_0^1 = \left( \frac{1}{4} + 2 - \frac{1}{3} \right) - (0 + 0 - 0) = \frac{3}{12} + \frac{24}{12} - \frac{4}{12} = \frac{23}{12}$$

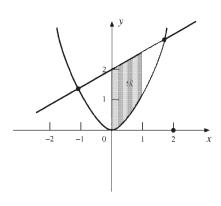


Fig. 29-7

#### **Arc Length**

Let f be differentiable on [a, b]. Consider the part of the graph of f from (a, f(a)) to (b, f(b)). Let us find a formula for the length L of this curve. Divide [a, b] into n equal subintervals, each of length  $\Delta x$ . To each point  $x_k$  in this subdivision there corresponds a point  $P_k(x_k, f(x_k))$  on the curve. (See Fig. 29-8.) For large n, the sum  $\overline{P_0P_1} + \overline{P_1P_2} + \ldots + \overline{P_{n-1}P_n} = \sum_{k=1}^n \overline{P_{k-1}P_k}$  of the lengths of the line segments  $P_{k-1}P_k$  is an approximation to the length of the curve.

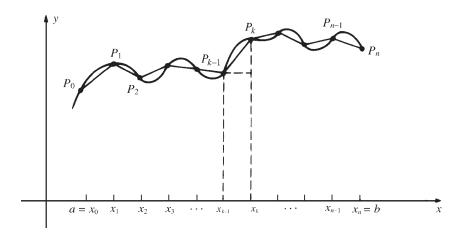


Fig. 29-8

By the distance formula (2.1),

$$\overline{P_{k-1}P_k} = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

Now,  $x_k - x_{k-1} = \Delta x$  and, by the law of the mean (Theorem 13.4),

$$f(x_k) - f(x_{k-1}) = (x_k - x_{k-1})f'(x_k^*) = (\Delta x)f'(x_k^*)$$

for some  $x_k^*$  in  $(x_{k-1}, x_k)$ . Thus,

$$\overline{P_{k-1}P_k} = \sqrt{(\Delta x)^2 + (\Delta x)^2 (f'(x_k^*))^2} = \sqrt{(1 + (f'(x_k^*))^2)(\Delta x)^2}$$

$$= \sqrt{1 + (f'(x_k^*))^2} \sqrt{(\Delta x)^2} = \sqrt{1 + (f'(x_k^*))^2} \Delta x$$

$$\sum_{k=1}^n \overline{P_{k-1}P_k} = \sum_{k=1}^n \sqrt{1 + (f'(x_k^*))^2} \Delta x$$

So,

The right-hand sum is an approximating sum for the definite integral  $\int_a^b \sqrt{1 + (f'(x))^2} dx$ . Therefore, letting  $n \to +\infty$ , we get the *arc length formula*:

$$L = \int_{a}^{b} \sqrt{1 + (f'(x))^{2}} dx = \int_{a}^{b} \sqrt{1 + (y')^{2}} dx$$
 (29.2)

**EXAMPLE 29.3:** Find the arc length *L* of the curve  $y = x^{3/2}$  from x = 0 to x = 5.

By (29.2), since 
$$y' = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$$
,

$$L = \int_0^5 \sqrt{1 + (y')^2} \, dx = \int_0^5 \sqrt{1 + \frac{9}{4}x} \, dx$$

$$= \frac{4}{9} \int_0^5 (1 + \frac{9}{4}x)^{1/2} \left(\frac{9}{4}\right) dx = \frac{4}{9} \frac{2}{3} (1 + \frac{9}{4}x)^{3/2} \right]_0^5 \qquad \text{(by Quick Formula I and the Fundamental Theorem of Calculus)}$$

$$= \frac{8}{27} ((\frac{49}{4})^{3/2} - 1^{3/2}) = \frac{8}{27} (\frac{343}{8} - 1) = \frac{335}{27}$$

## SOLVED PROBLEMS

1. Find the area bounded by the parabola  $x = 8 + 2y - y^2$ , the y axis, and the lines y = -1 and y = 3. Note, by completing the square, that  $x = -(y^2 - 2y - 8) = -((y - 1)^2 - 9) = 9 - (y - 1)^2 = (4 - y)(2 + y)$ . Hence, the vertex of the parabola is (9, 1) and the parabola cuts the y axis at y = 4 and y = -2. We want the area of the shaded region in Fig. 29-9, which is given by

$$\int_{-1}^{3} (8 + 2y - y^2) dy = (8y + y^2 - \frac{1}{3}y^3) \Big]_{-1}^{3} = (24 + 9 - 9) - (-8 + 1 - \frac{1}{3}) = \frac{92}{3}$$

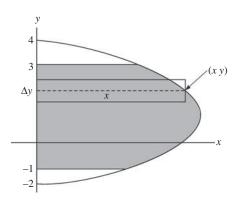


Fig. 29-9

2. Find the area of the region between the curves  $y = \sin x$  and  $y = \cos x$  from x = 0 to  $x = \pi/4$ . The curves intersect at  $(\pi/4, \sqrt{2}/2)$ , and  $0 \le \sin x < \cos x$  for  $0 \le x < \pi/4$ . (See Fig. 29-10.) Hence, the area is

$$\int_0^{\pi/4} (\cos x - \sin x) \, dx = (\sin x + \cos x) \Big|_0^{\pi/4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1$$

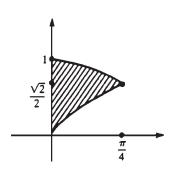


Fig. 29-10

3. Find the area of the region bounded by the parabolas  $y = 6x - x^2$  and  $y = x^2 - 2x$ .

By solving  $6x - x^2 = x^2 - 2x$ , we see that the parabolas intersect when x = 0 and x = 4, that is, at (0, 0) and (4, 8). (See Fig. 29-11.) By completing the square, the first parabola has the equation  $y = 9 - (x - 3)^2$ ; therefore, it has its vertex at (3, 9) and opens downward. Likewise, the second parabola has the equation  $y = (x - 1)^2 - 1$ ; therefore, its vertex is at (1, -1) and it opens upward. Note that the first parabola lies above the second parabola in the given region. By (29.1), the required area is

$$\int_0^4 ((6x - x^2) - (x^2 - 2x)) dx = \int_0^4 (8x - 2x^2) dx = (4x^2 - \frac{2}{3}x^3) \Big|_0^4 = (64 - \frac{128}{3}) = \frac{64}{3}$$

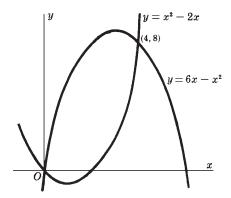


Fig. 29-11

**4.** Find the area of the region bounded by the parabola  $y^2 = 4x$  and the line y = 2x - 4.

Solving the equations simultaneously, we get  $(2x-4)^2 = 4x$ ,  $x^2 - 4x + 4 = x$ ,  $x^2 - 5x + 4 = 0$ , (x-1)(x-4) = 0. Hence, the curves intersect when x = 1 or x = 4, that is, at (1, -2) and (4, 4). (See Fig. 29-12.) Note that neither curve is above the other throughout the region. Hence, it is better to take y as the independent variable and rewrite the curves as  $x = \frac{1}{4}y^2$  and  $x = \frac{1}{2}(y+4)$ . The line is always to the right of the parabola.

The area is obtained by integrating along the y axis:

$$\int_{-2}^{4} \left(\frac{1}{2}(y+4) - \frac{1}{4}y^{2}\right) dy = \frac{1}{4} \int_{-2}^{4} (2y+8-y^{2}) dy$$

$$= \frac{1}{4} \left(y^{2} + 8y - \frac{1}{3}y^{3}\right) \Big|_{-2}^{4} = \frac{1}{4} \left( (16 + 32 - \frac{64}{3}) - (4 - 16 + \frac{8}{3}) \right) = 9$$

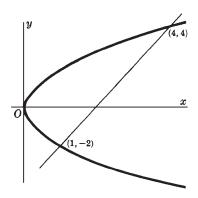


Fig. 29-12

5. Find the area of the region between the curve  $y = x^3 - 6x^2 + 8x$  and the x axis.

Since  $x^3 - 6x^2 + 8x = x(x^2 - 6x + 8) = x(x - 2)(x - 4)$ , the curve crosses the x axis at x = 0, x = 2, and x = 4. The graph looks like the curve shown in Fig. 29-13. (By applying the quadratic formula to y', we find that the maximum and minimum values occur at  $x = 2 \pm \frac{2}{3}\sqrt{3}$ .) Since the part of the region with  $2 \le x \le 4$  lies below the x axis, we must calculate two separate integrals, one with respect to y between x = 0 and x = 2, and the other with respect to -y between x = 2 and x = 4. Thus, the required area is

$$\int_{0}^{2} (x^{3} - 6x^{2} + 8x) dx - \int_{2}^{4} (x^{3} - 6x^{2} + 8x) dx = \left(\frac{1}{4}x^{4} - 2x^{3} + 4x^{2}\right)\Big|_{0}^{2} - \left(\frac{1}{4}x^{4} - 2x^{3} + 4x^{2}\right)\Big|_{2}^{4} = 4 + 4 = 8$$

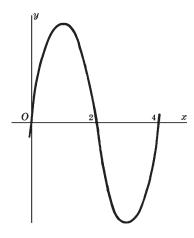


Fig. 29-13

Note that, if we had made the mistake of simply calculating the integral  $\int_0^4 (x^3 - 6x^2 + 8x) dx$ , we would have got the incorrect answer 0.

**6.** Find the area enclosed by the curve  $y^2 = x^2 - x^4$ .

The curve is symmetric with respect to the coordinate axes. Hence the required area is four times the portion lying in the first quadrant. (See Fig. 29-14.) In the first quadrant,  $y = \sqrt{x^2 - x^4} = x\sqrt{1 - x^2}$  and the curve intersects the x axis at x = 0 and x = 1. So, the required area is

$$4\int_0^1 x\sqrt{1-x^2} dx = -2\int_0^1 (1-x^2)^{1/2} (-2x) dx$$
$$= -2\left(\frac{2}{3}\right) (1-x^2)^{3/2} \Big]_0^1 \quad \text{(by Quick Formula I)}$$
$$= -\frac{4}{3} (0-1^{3/2}) = -\frac{4}{3} (-1) = \frac{4}{3}$$

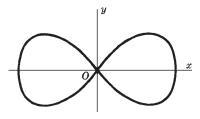


Fig. 29-14

7. Find the arc length of the curve  $x = 3y^{3/2} - 1$  from y = 0 to y = 4. We can reverse the roles of x and y in the arc length formula (29.2):  $L = \int_{c}^{d} \sqrt{1 + \left(\frac{dx}{dy}\right)^{2}} dy$ . Since  $\frac{dx}{dy} = \frac{9}{2}y^{1/2}$ ,

$$L = \int_0^4 \sqrt{1 + \frac{81}{4}y} \, dy = \frac{4}{81} \int_0^4 (1 + \frac{81}{4}y)^{1/2} (\frac{81}{4}) \, dy = \frac{4}{81} (\frac{2}{3}) (1 + \frac{81}{4}y)^{3/2} \bigg|_0^4 = \frac{8}{243} ((82)^{3/2} - 1)^{3/2}) = \frac{8}{243} (82\sqrt{82} - 1)$$

**8.** Find the arc length of the curve  $24xy = x^4 + 48$  from x = 2 to x = 4.  $y = \frac{1}{24}x^3 + 2x^{-1}$ . Hence,  $y' = \frac{1}{8}x^2 - 2/x^2$ . Thus,

$$(y')^2 = \frac{1}{64}x^4 - \frac{1}{2} + \frac{4}{x^4}$$
$$1 + (y')^2 = \frac{1}{64}x^4 + \frac{1}{2} + \frac{4}{x^4} = \left(\frac{1}{8}x^2 + \frac{2}{x^2}\right)^2$$

So,  

$$L = \int_{2}^{4} \sqrt{1 + (y')^{2}} dx = \int_{2}^{4} \left(\frac{1}{8}x^{2} + \frac{2}{x^{2}}\right) dx = \int_{2}^{4} \left(\frac{1}{8}x^{2} + 2x^{-2}\right) dx$$

$$= \left(\frac{1}{24}x^{3} - 2x^{-1}\right) \Big]_{2}^{4} = \left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right) = \frac{17}{6}$$

**9.** Find the arc length of the catenary  $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$  from x = 0 to x = a.  $y' = \frac{1}{2}(e^{x/a} + e^{-x/a})$  and, therefore,

$$1 + (y')^2 = 1 + \frac{1}{4}(e^{2x/a} - 2 + e^{-2x/a}) = \frac{1}{4}(e^{x/a} + e^{-x/a})^2$$

So,  $L = \frac{1}{2} \int_0^a (e^{x/a} + e^{-x/a}) dx = \frac{a}{2} (e^{x/a} - e^{-x/a}) \bigg]_0^a = \frac{a}{2} (e - e^{-1})$ 

# **SUPPLEMENTARY PROBLEMS**

10. Find the area of the region lying above the x axis and under the parabola  $y = 4x - x^2$ .

Ans.  $\frac{32}{3}$ 

11. Find the area of the region bounded by the parabola  $y = x^2 - 7x + 6$ , the x axis, and the lines x = 2 and x = 6.

Ans.  $\frac{56}{3}$ 

12. Find the area of the region bounded by the given curves.

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(a) $y = x^2$ , $y = 0$ , $x = 2$ , $x = 5$	Ans.	39
(b) $y = x^3, y = 0, x = 1, x = 3$	Ans.	20
(c) $y = 4x - x^2$ , $y = 0$ , $x = 1$ , $x = 3$	Ans.	<u>22</u> 3
(d) $x = 1 + y^2, x = 10$	Ans.	36
(e) $x = 3y^2 - 9$ , $x = 0$ , $y = 0$ , $y = 1$	Ans.	8
(f) $x = y^2 + 4y, x = 0$	Ans.	32
(g) $y = 9 - x^2$ , $y = x + 3$	Ans.	125 6
(h) $y = 2 - x^2y = -x$	Ans.	$\frac{9}{2}$
(i) $y = x^2 - 4$ , $y = 8 - 2x^2$	Ans.	32
(j) $y = x^4 - 4x^2$ , $y = 4x^2$	Ans.	$\frac{512}{15}\sqrt{2}$
(k) $y = e^x$ , $y = e^{-x}$ , $x = 0$ , $x = 2$	Ans.	$\frac{e^2+1}{e^2-2}$
(1) $y = e^{x/a} + e^{-x/a}, y = 0, x = \pm a$	Ans.	$2a\left(\frac{e^{-\epsilon}}{\epsilon}\right)$
(m) $xy = 12, y = 0, x = 1, x = e^2$	Ans.	24

- (n)  $y = \frac{1}{1+x^2}$ , y = 0,  $x = \pm 1$ (o)  $y = \tan x$ , x = 0,  $x = \frac{\pi}{4}$ (p)  $y = 25 - x^2$ ,  $256x = 3y^2$ ,  $16y = 9x^2$
- 13. Find the length of the indicated arc of the given curve.
  - (a)  $y^3 = 8x^2$  from x = 1 to x = 8 Ans.  $(104\sqrt{13} 125)/27$ (b)  $6xy = x^4 + 3$  from x = 1 to x = 2 Ans.  $\frac{17}{12}$ (c)  $27y^2 = 4(x-2)^3$  from (2, 0) to (11,  $6\sqrt{3}$  Ans. 14

 $\frac{1}{2} \ln 2$ 

Ans.

(d) 
$$y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$$
 from  $x = 1$  to  $x = e$ 

Ans. 
$$\frac{1}{2}e^2 - \frac{1}{4}$$

(e) 
$$y = \ln \cos x$$
 from  $x = \frac{\pi}{6}$  to  $x = \frac{\pi}{4}$   
(f)  $x^{2/3} + y^{2/3} = 4$  from  $x = 1$  to  $x = 8$ 

Ans. 
$$\ln\left(\frac{1+\sqrt{2}}{\sqrt{3}}\right)$$

(f) 
$$x^{2/3} + y^{2/3} = 4$$
 from  $x = 1$  to  $x = 8$ 

14. (GC) Estimate the arc length of  $y = \sin x$  from x = 0 to  $x = \pi$  to an accuracy of four decimal places. (Use Simpson's Rule with n = 10.)

Ans. 3.8202