CHAPTER 26

Exponential and Logarithmic Functions

From Chapter 25, we know that the natural logarithm $\ln x$ is an increasing differentiable function with domain the set of all positive real numbers and range the set of all real numbers. Since it is increasing, it is a one-to-one function and, therefore, has an inverse function, which we shall denote by e^x.

Definition

e^x is the inverse of $\ln x$.

It follows that the domain of e^x is the set of all real numbers and its range is the set of all positive real numbers. Since e^x is the inverse of $\ln x$, the graph of e^x can be obtained from that of $\ln x$ by reflection in the line $y = x$. See Fig. 26-1.

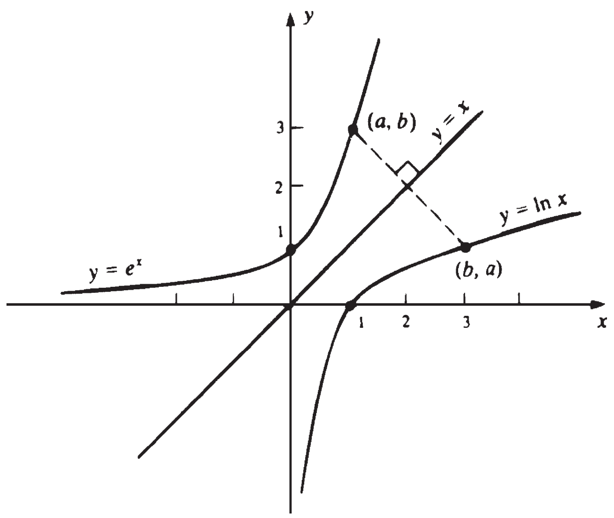

Fig 26-1

Our notation may be confusing. It should not be assumed from the notation that e^x is an ordinary power of base e with exponent x. Later in this chapter, we will find out that this is indeed true, but we do not know it yet.

Properties of e^x

(26.1) $e^x > 0$ for all x

The range of e^x is the set of positive real numbers.

(26.2) $\ln (e^x) = x$

(26.3) $e^{\ln x} = x$

Properties (26.2) and (26.3) follow from the fact that e^x and $\ln x$ are inverses of each other.

(26.4) e^x is an increasing function.
Assume $u < v$. Since $u = \ln(e^u)$ and $v = \ln(e^v)$, $\ln(e^u) < \ln(e^v)$. But, since $\ln x$ is increasing, $e^u < e^v$. [For, if $e^v \le e^u$, then $\ln(e^v) \le \ln(e^u)$.]

(26.5) $D_x(e^x) = e^x$
Let $y = e^x$. Then $\ln y = x$. By implicit differentiation, $\frac{1}{y}y' = 1$ and, therefore, $y' = y = e^x$. For a more rigorous argument, let $f(x) = \ln x$ and $f^{-1}(y) = e^y$. Note that $f'(x) = \frac{1}{x}$. By Theorem 10.2(b),

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}, \qquad \text{that is,} \quad D_y(e^y) = \frac{1}{1/e^y} = e^y$$

EXAMPLE 26.1: $D_x(e^{\sin x}) = D_u(e^u)D_x(u)$ (Chain Rule, with $u = \sin x$)

$$= e^u(\cos x) = e^{\sin x}(\cos x)$$

(26.6) $\int e^x\,dx = e^x + C$

EXAMPLE 26.2: To find $\int xe^{x^2}\,dx$, let $u = x^2$, $du = 2x\,dx$. Then

$$\int xe^{x^2}\,dx = \tfrac{1}{2}\int e^u\,du = \tfrac{1}{2}e^u + C = \tfrac{1}{2}e^{x^2} + C$$

(26.7) $\int e^{-x}dx = -e^{-x} + C$
Let $u = -x$, $du = -dx$. Then $\int e^{-x}dx = -\int e^u du = -e^u + C = -e^{-x} + C$.

(26.8) $e^0 = 1$
By (26.3), $1 = e^{\ln 1} = e^0$.

(26.9) $e^{u+v} = e^u e^v$
$\ln(e^{u+v}) = u + v = \ln(e^u) + \ln(e^v) = \ln(e^u e^v)$ by (25.6). Hence, $e^{u+v} = e^u e^v$ because $\ln x$ is a one-to-one function.

(26.10) $e^{u-v} = \dfrac{e^u}{e^v}$
By (26.9), $e^{u-v}e^v = e^{(u-v)+v} = e^u$. Now divide by e^v.

(26.11) $e^{-v} = \dfrac{1}{e^v}$
Replace u by 0 in (26.10) and use (26.8).

(26.12) $x < e^x$ for all x
By Problem 7 of Chapter 25, $\ln x \le x - 1 < x$. By (26.3) and (26.4), $x = e^{\ln x} < e^x$.

(26.13) $\lim_{x\to+\infty} e^x = +\infty$
This follows from (26.4) and (26.12).

(26.14) $\lim_{x\to-\infty} e^x = 0$
Let $u = -x$. As $x \to \infty$, $u \to +\infty$ and, by (26.13), $e^u \to +\infty$. Then, by (26.11), $e^x = e^{-u} = \frac{1}{e^u} \to 0$.

The mystery of the letter e in the expression e^x can now be cleared up.

Definition

Let e be the number such that $\ln e = 1$.

Since $\ln x$ is a one-to-one function from the set of positive real numbers onto the set of all real numbers, there must be exactly one number x such that $\ln x = 1$. That number is designated e.

Since, by (25.12), $\ln 2 < 1 < 2\ln 2 = \ln 4$, we know that $2 < e < 4$.

(26.15) (GC) $e \sim 2.718281828$
This estimate can be obtained from a graphing calculator. Later we will find out how to approximate e to any degree of accuracy.

Now we can show that the notation e^x is not misleading, that is, that e^x actually is a power of e. First of all, this can be proved for positive integers x by mathematical induction. [In fact, by (26.3), $e = e^{\ln e} = e^1$. So, by (26.9), $e^{n+1} = e^n e^1 = e^n e$ for any positive integer n and therefore, if we assume by inductive hypothesis that e^n represents the produce of e by itself n times, then e^{n+1} is the product of e by itself $n + 1$ times.] By (26.8) $e^0 = 1$, which corresponds to the standard definition of e^0. If n is a positive integer, e^{-n} would ordinarily be defined by $1/e^n$ and this is identical to the function value given by (26.11). If k and n are positive integers, then the power $e^{k/n}$ is ordinarily defined as $\sqrt[n]{e^k}$. Now, in fact, by (26.9), the product $e^{k/n}e^{k/n}\ldots e^{k/n}$, where there are n factors, is equal to $e^{k/n+k/n+\cdots+k/n} = e^k$. Thus, the function value $e^{k/n}$ is identical to the nth root of e^k. For negative fractions, we again apply (26.11) to see that the function value is identical to the value specified by the usual definition. Hence, the function value e^x is the usual power of e when x is any rational number. Since our function e^x is continuous, the value of e^x when x is irrational is the desired limit of e^r for rational numbers r approaching x.

The graph of $y = e^x$ is shown in Fig. 26-2. By (26.13), the graph rises without bound on the right and, by (26.14), the negative x axis is a horizontal asymptote on the left. Since $D_x^2(e^x) = D_x(e^x) = e^x > 0$, the graph is concave upward everywhere. The graph of $y = e^{-x}$ is also shown in Fig. 26-2. It is obtained from the graph of $y = e^x$ by reflection in the y axis.

(26.16) $e^x = \lim\limits_{n\to+\infty}\left(1+\frac{x}{n}\right)^n$

For a proof, see Problem 5.

(26.17) $e = \lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$

This is a special case of (26.16) when $x = 1$. We can use this formula to approximate e, although the convergence to e is rather slow. For example, when $n = 100$, we get 2.7169 and, when $n = 10\,000$, we get 2.7181, which is correct only to three decimal places.

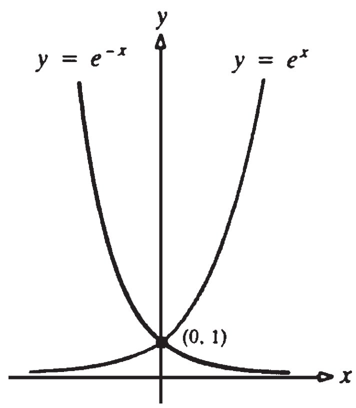

Fig. 26-2

The General Exponential Function

Let $a > 0$. Then we can define a^x as follows:

Definition

$a^x = e^{x \ln a}$

Note that this is consistent with the definition of e^x since, when $a = e$, $\ln a = 1$.

(26.18) $D_x(a^x) = (\ln a)a^x$

In fact,

$$D_x(e^{x\ln a}) = D_u(e^u)D_x u \qquad \text{(chain rule with } u = x\ln a)$$

$$= e^u(\ln a) = e^{x\ln a}(\ln a) = a^x(\ln a)$$

EXAMPLE 26.3: $D_x(2^x) = (\ln 2)2^x$.

(26.19) $\int a^x dx = \frac{1}{\ln a} a^x + C$

This is a direct consequence of (26.18).

EXAMPLE 26.4: $\int 10^x = \frac{1}{\ln 10} 10^x + C$

We can derive the usual properties of powers.

(**26.20**) $a^0 = 1$

$a^0 = e^{0 \ln a} = e^0 = 1$

(**26.21**) $a^{u+v} = a^u a^v$

$a^{u+v} = e^{(u+v)\ln a} = e^{u \ln a + v \ln a} = e^{u \ln a} e^{v \ln a} = a^u a^v$

(**26.22**) $a^{u-v} = \frac{a^u}{a^v}$

By (26.21), $a^{u-v}a^v = a^{(u-v)+v} = a^u$. Now divide by a^v.

(**26.23**) $a^{-v} = \frac{1}{a^v}$

Replace u by 0 in (26.22) and use (26.20).

(**26.24**) $a^{uv} = (a^u)^v$

$(a^u)^v = e^{v\ln(a^u)} = e^{v(u(\ln a))} = e^{(uv)\ln a} = a^{uv}$

(**26.25**) $(ab)^u = a^u b^u$

$a^u b^u = e^{u\ln a} e^{u \ln b} = e^{u \ln a + u \ln b} = e^{u(\ln a + \ln b)} = e^{u \ln(ab)} = (ab)^u$

Recall that we know that $D_x(x^r) = rx^{r-1}$ for rational numbers r. Now we are able to prove that formula for any real number r.

(**26.26**) $D_x(x^r) = rx^{r-1}$

Since $x^r = e^{r \ln x}$,

$$D_x(x^r) = D_x(e^{r\ln x}) = D_u(e^u)D_x(u) \quad \text{(Chain Rule with } u = r\ln x)$$

$$= e^u\left(r\left(\frac{1}{x}\right)\right) = r(x^r)\left(\frac{1}{x}\right) = r\frac{x^r}{x^1} = rx^{r-1}$$

General Logarithmic Functions

Let $a > 0$. We want to define a function $\log_a x$ that plays the role of the traditional logarithm to the base a. If $y = \log_a x$, then $a^y = x$ and, therefore, $\ln(a^y) = \ln x$, $y \ln a = \ln x$, $y = \frac{\ln x}{\ln a}$.

Definition

$$\log_a x = \frac{\ln x}{\ln a}.$$

(**26.27**) $y = \log_a x$ is equivalent to $a^y = x$

$$y = \log_a x \Leftrightarrow y = \frac{\ln x}{\ln a} \Leftrightarrow y \ln a = \ln x$$

$\Leftrightarrow \ln(a^y) = \ln x \Leftrightarrow a^y = x$ (The symbol $\Leftrightarrow$ is the symbol for equivalence, that is, *if and only if*.)

Thus, the general logarithmic function with base a is the inverse of the general exponential function with base a.

(**26.28**) $a^{\log_a x} = x$

(26.29) $\log_a(a^x) = x$

These follow from (26.27). See Problem 6.

The usual properties of logarithm can easily be derived. See Problem 7.

Notice that $\log_e x = \dfrac{\ln x}{\ln e} = \dfrac{\ln x}{1} = \ln x$. Thus, the natural logarithm turns out to be a logarithm in the usual sense, with base e.

SOLVED PROBLEMS

1. Evaluate: (*a*) $\ln(e^3)$; (*b*) $e^{7\ln 2}$; (*c*) $e^{(\ln 3)-2}$; (*e*) 1^u.

(a) $\ln(e^3) = 3$ by (26.2)

(b) $e^{7\ln 2} = (e^{\ln 2})^7 = 2^7 = 128$ by (26.24) and (26.3)

(c) $e^{(\ln 3)-2} = \dfrac{e^{\ln 3}}{e^2} = \dfrac{3}{e^2}$ by (26.10)

(d) $1^u = e^{u\ln 1} = e^{u(0)} = e^0 = 1$ by (26.8)

2. Find the derivatives of: (a) e^{3x+1}; (b)5^{3x}; (c)$3x^\pi$; (d) x^2e^x.

(a) $D_x(e^{3x+1}) = e^{3x+1}(3) = 3e^{3x+1}$ by the Chain Rule

(b) $D_x(5^{3x}) = D_u(5^u)D(u)$ (chain rule with $u = 3x$)

$= (\ln 5)5^u(3)$ by (26.18)

$= 3(\ln 5)5^{3x}$

(c) $D_x(3x^\pi) = 3(\pi x^{\pi-1}) = 3\pi x^{\pi-1}$ by (26.26)

(d) $D_x(x^2e^x) = x^2D_x(e^x) + e^xD_x(x^2)$ by the product rule

$= x^2e^x + e^x(2x) = xe^x(x+2)$

3. Find the following antiderivative: (a) $\int 3(2^x)\,dx$; (b) $\int x^2e^{x^3}\,dx$.

(a) $\int 3(2^x)\,dx = 3\int 2^x\,dx = 3\dfrac{1}{\ln 2}2^x + C = \dfrac{3}{\ln 2}2^x + C$

(b) Let $u = x^3$, $du = 3x^2\,dx$. Then $\int x^2e^{x^3}dx = \dfrac{1}{3}\int e^u du = \dfrac{1}{3}e^u + C = \dfrac{1}{3}e^{x^3} + C$

4. Solve the following equations for x: (a) $\ln x^3 = 2$; (b) $\ln(\ln x) = 0$; (c) $e^{2x-1} = 3$; (d) $e^x - 3e^{-x} = 2$.

In general, $\ln A = B$ is equivalent to $A = e^B$, and $e^C = D$ is equivalent to $C = \ln D$.

(a) $\ln x^3 = 3\ln x$. Hence, $\ln x^3 = 2$ yields $3\ln x = 2$, $\ln x = \frac{2}{3}$, $x = e^{2/3}$.

(b) $\ln(\ln x) = 0$ is equivalent to $\ln x = e^0 = 1$, which, in turn, is equivalent to $x = e^1 = e$.

(c) $e^{2x-1} = 3$ is equivalent to $2x - 1 = \ln 3$, and then to $x = \dfrac{\ln 3 + 1}{2}$.

(d) Multiply both sides by e^x: $e^{2x} - 3 = 2e^x$, $e^{2x} - 2e^x - 3 = 0$. Letting $u = e^x$ yields the quadratic equation $u^2 - 2u - 3 = 0$; $(u-3)(u+1) = 0$, with solutions $u = 3$ and $u = -1$. Hence, $e^x = 3$ or $e^x = -1$. The latter is impossible since e^x is always positive. Hence, $e^x = 3$ and, therefore, $x = \ln 3$.

5. Prove (26.16): $e^u = \lim\limits_{n\to+\infty}\left(1+\dfrac{u}{n}\right)^n$.

Let $a_n = \left(1+\dfrac{u}{n}\right)^n$. Then

$$\ln a_n = n\ln\left(1+\frac{u}{n}\right) = u\left(\frac{\ln(1+u/n) - \ln 1}{u/n}\right)$$

The expression $\left(\dfrac{\ln(1+u/n)-\ln 1}{u/n}\right)$ is a difference quotient for $D_x(\ln x)$ at $x = 1$, with $\Delta x = u/n$. As $n \to +\infty$, $u/n \to 0$. So, that difference quotient approaches $D_x(\ln x)\big|_{x=1} = (1/x)\big|_{x=1} = 1$. Hence, $\lim\limits_{n\to+\infty}\ln a_n = u(1) = u$. So, $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} e^{\ln a_n} = e^u$.

6. Prove (26.28) $a^{\log_a x} = x$ and (26.29) $\log_a (a^x) = x$.

Substituting $\log_a x$ for y in (26.27), we get $a^{\log_a x} = x$.

Substituting a^y for x in (26.27), we get $y = \log_a (a^y)$.

7. Derive the following properties of $\log_a x$:

(a) $\log_a 1 = 0$.

$$\log_a 1 = \frac{\ln 1}{\ln a} = \frac{0}{\ln a} = 0$$

(b) $\log_a a = 1$.

$$\log_a a = \frac{\ln a}{\ln a} = 1$$

(c) $\log_a uv = \log_a u + \log_a v$.

$$\log_a uv = \frac{\ln uv}{\ln a} = \frac{\ln u + \ln v}{\ln a} = \frac{\ln u}{\ln a} + \frac{\ln v}{\ln a} = \log_a u + \log_a v$$

(d) $\log_a \dfrac{u}{v} = \log_a u - \log_a v$.

Replace u in (c) by $\dfrac{u}{v}$.

(e) $\log_a \dfrac{1}{v} = -\log_a v$.

Replace u by 1 in (d).

(f) $\log_a (u^r) = r \log_a u$.

$$\log_a (u^r) = \frac{\ln (u^r)}{\ln a} = \frac{r \ln u}{\ln a} = r \log_a u$$

(g) $D_x(\log_a x) = \dfrac{1}{\ln a}\dfrac{1}{x}$.

$$D_x(\log_a x) = D_x\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} D_x(\ln x) = \frac{1}{\ln a}\frac{1}{x}$$

SUPPLEMENTARY PROBLEMS

8. Calculate the derivatives of the following functions:

(a) $y = e^{5x}$ *Ans.* $y' = 5e^{5x}$

(b) $y = e^{\tan 3x}$ *Ans.* $y' = 3\sec^2(3x) e^{\tan 3x}$

(c) $y = e^{-x \cos x}$ *Ans.* $y' = -e^{-x}(\cos x + \sin x)$

(d) $y = 3^{-x^2}$ *Ans.* $y' = -2x(\ln 3)3^{-x^2}$

(e) $y = \sin^{-1}(e^x)$ *Ans.* $y' = \dfrac{e^x}{\sqrt{1 - e^{2x}}}$

(f) $y = e^{e^x}$ *Ans.* $y' = e^{x+e^x}$

(g) $y = x^x$ *Ans.* $y' = x^x(1 + \text{In } x)$

(h) $y = \log_{10}(3x^2 - 5)$ *Ans.* $y' = \dfrac{1}{\ln 10}\dfrac{6x}{3x^2 - 5}$

9. Find the following antiderivatives:

(a) $\int 3^{2x} dx$ *Ans.* $\dfrac{1}{2 \ln 3} 3^{2x} + C$

(b) $\int \dfrac{e^{1/x}}{x^2} dx$ *Ans.* $-e^{1/x} + C$

(c) $\int (e^x + 1)^3 e^x dx$ *Ans.* $\dfrac{(e^x + 1)^4}{4} + C$

(d) $\int \dfrac{dx}{e^x + 1}$ *Ans.* $x - \ln(e^x + 1) +$

(e) $\int \dfrac{e^{1/x^2}}{x^3} dx$ *Ans.* $-\tfrac{1}{2} e^{1/x^2} + C$

(f) $\int e^{-x^2+2} x dx$ *Ans.* $-\dfrac{1}{2} e^{-x^2+2} + C$

(g) $\int (e^x+1)^2\,dx$ *Ans.* $\frac{1}{2}e^{2x}+2e^x+x+C$

(h) $\int (e^x - x^e)\,dx$ *Ans.* $e^x - \dfrac{x^{e+1}}{e+1}+C$

(i) $\int \dfrac{e^{2x}}{e^{2x}+3}\,dx$ *Ans.* $\frac{1}{2}\ln(e^{2x}+3)+C$

(j) $\int \dfrac{e^x\,dx}{\sqrt{1-e^{2x}}}$ *Ans.* $\sin^{-1}(e^x)+C$

(k) $\int x^3(5^{x^4+1})\,dx$ *Ans.* $\dfrac{1}{4\ln 5}5^{x^4+1}+C$

(l) $\int \dfrac{\log_{10} x}{x}\,dx$ *Ans.* $\dfrac{1}{2\ln 10}(\ln x)^2+C = \dfrac{\ln 10}{2}(\log_{10} x)^2+C$

10. (**Hyperbolic Functions**) Define

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}, \quad \operatorname{sec}h = \frac{1}{\cosh x}$$

Derive the following results:

(a) $D_x(\sinh x) = \cosh x$ and $D_x(\cosh x) = \sinh x$.
(b) $D_x(\tanh x) = \operatorname{sech}^2 x$ and $D_x(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$.
(c) $\cosh^2 x - \sinh^2 x = 1$.
(d) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.
(e) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
(f) $\sinh 2x = 2 \sinh x \cosh x$.
(g) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$.
(h) (GC) Sketch the graph of $y = 2\cosh(x/2)$ (called a "catenary"), and find its minimum point.

Ans. (0, 2)

11. Solve the following equations for x.

(a) $e^{3x} = 2$ *Ans.* $\frac{1}{3}\ln 2$
(b) $\ln(x^4) = -1$ *Ans.* $e^{-1/4}$
(c) $\ln(\ln x) = 2$ *Ans.* e^{e^2}
(d) $e^x - 4e^{-x} = 3$ *Ans.* $2\ln 2$
(e) $e^x + 12e^{-x} = 7$ *Ans.* $2\ln 2$ and $\ln 3$
(f) $5^x = 7$ *Ans.* $\dfrac{\ln 7}{\ln 5} = \log_5 7$
(g) $\log_2(x+3) = 5$ *Ans.* 29
(h) $\log_2 x^2 + \log_2 x = 4$ *Ans.* $\sqrt[3]{16}$
(i) $\log_2(2^{4x}) = 20$ *Ans.* 5
(j) $e^{-2x} - 7e^{-x} = 8$ *Ans.* $-3\ln 2$
(k) $x^x = x^3$ *Ans.* 1 and 3

12. Evaluate (a) $\lim_{h\to 0}\dfrac{e^h - 1}{h}$; (b) $\lim_{h\to 0}\dfrac{e^{h^2}-1}{h}$.

Ans. (a) 1; (b) 0

13. Evaluate: (a) $\int_0^{\ln 2}\dfrac{e^x}{e^x+2}\,dx$; (b) $\int_1^e \dfrac{2+\ln x}{x}\,dx$

Ans. (a) $\ln\frac{4}{3}$; (b) $\frac{5}{2}$

14. (GC) Use Newton's method to approximate (to four decimal places) a solution of $e^x = \dfrac{1}{x}$.

Ans. 0.5671

15. (GC) Use Simpson's rule with $n = 4$ to approximate $\int_0^1 e^{-x^2/2}\,dx$ to four decimal places.

Ans. 0.8556

16. If interest is paid at r percent per year and is compounded n times per year, then P dollars become $P\left(1+\frac{r}{100n}\right)^n$ dollars after 1 year. If $n \to +\infty$, then the interest is said to be *compounded continuously.*

(a) If compounded continuously at r percent per year, show that P dollars becomes $Pe^{r/100}$ dollars after 1 year, and $Pe^{rt/100}$ dollars after t years.
(b) At r percent compounded continuously, how many years does it take for a given amount of money to double?
(c) (GC) Estimate to two decimal places how many years it would take to double a given amount of money compounded continuously at 6% per year?
(d) (GC) Compare the result of compounding continuously at 5% with that obtained by compounding once a year.

Ans. (b) $\frac{100(\ln 2)}{r} \sim \frac{69.31}{r}$; (c) about 11.55 years;
(d) After 1 year, \$1 becomes \$1.05 when compounded once a year, and about \$1.0512 when compounded continuously.

17. Find $(\log_{10} e) \cdot \ln 10$.

Ans. 1

18. Write as a single logarithm with base a: $3 \log_a 2 + \log_a 40 - \log_a 16$

Ans. $\log_a 20$

19. (GC) Estimate $\log_2 7$ to eight decimal places.

Ans. 2.80735492

20. Show that $\log_b x = (\log_a x)(\log_b a)$.

21. (GC) Graph $y = e^{-x^2/2}$. Indicate absolute extrema, inflection points, asymptotes, and any symmetry.

Ans. Absolute maximum at (0, 1), inflection points at $x = \pm 1$, x axis is a horizontal asymptote on the left and right, symmetric with respect to the y axis.

22. Given $e^{xy} - x + y^2 = 1$, find $\frac{dy}{dx}$ by implicit differentiation.

Ans. $\frac{1 - ye^{xy}}{2y + xe^{xy}}$

23. (GC) Graph $y = \sinh x = \frac{e^x - e^{-x}}{2}$.

24. Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.

Ans. $\ln(e^x + e^{-x}) + C$

25. Use logarithmic differentiation to find the derivative of $y = x^{3/x}$.

Ans. $\frac{3y(1 - \ln x)}{x^2}$