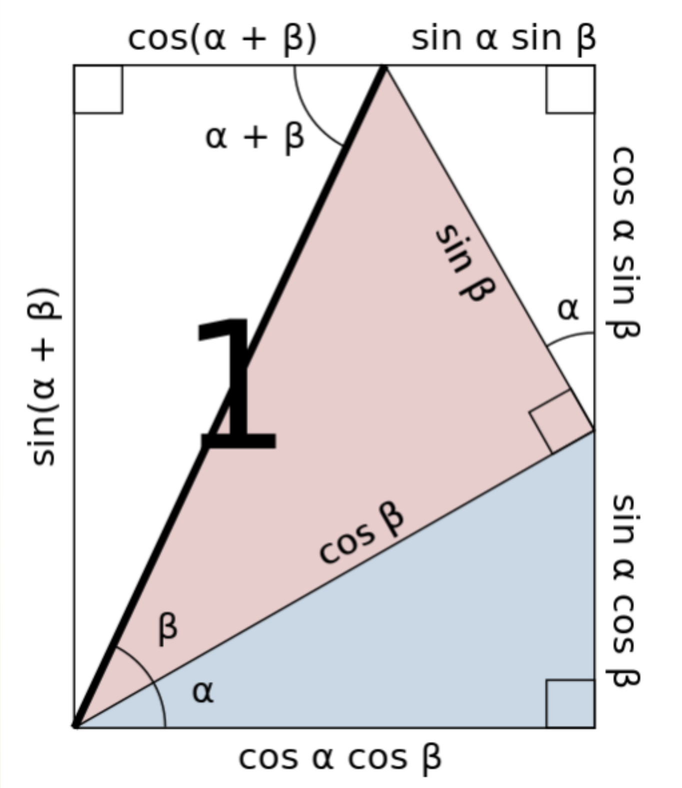
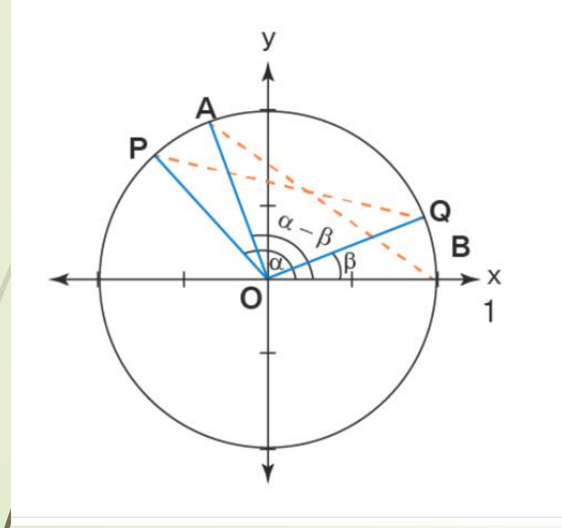
# Chapter 5
# Trigonometric Identities (Part 2)



# Angle Subtraction Identity of Cosine

- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$




In the left figure, label two points $A$ and $B$ on the unit circle such that the coordinate of point $B$ is (1, 0) on the x-axis and the coordinate of point $A$ is $(\cos(\alpha - \beta), \sin(\alpha - \beta))$. Draw $\overline{PQ}$ and $\overline{AB}$, note that $\Delta\, POQ$ and $\Delta\, AOB$ are rotations of one another and $PQ = AB$. Using the distance formula in analytic geometry to find $PQ$ and $AB$ as follows:

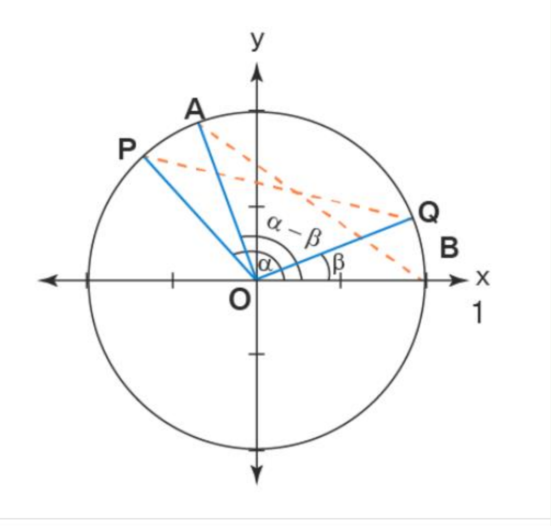
# Angle Subtraction Identity of Cosine (Cont.)

Proof:

$PQ = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$ .......... (eq. 1)

$AB = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2}$

$= \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta))^2}$ ..... (eq. 2)




Since, $PQ = AB$ and $PQ^2 = AB^2$

$PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

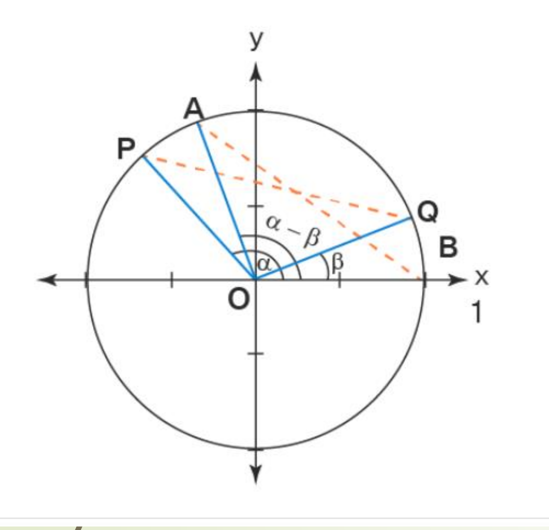
$= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$

$= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$

$= 1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$

$= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$ ...................... (eq.3)

# Angle Subtraction Identity of Cosine (Cont.)




$$AB^2 = (\cos(\alpha-\beta)-1)^2 + (\sin(\alpha-\beta))^2$$

$$= \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)$$

$$= \cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1$$

$$= 1 + 1 - 2\cos(\alpha-\beta)$$

$$= 2 - 2\cos(\alpha-\beta) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(eq. 4)}$$

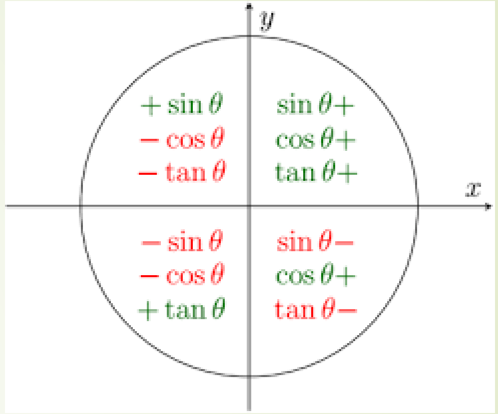
eq. 3 = eq. 4;

$$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha-\beta)$$

Using algebraic properties, $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

Therefore, $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ #

# Angle Addition Identity of Cosine




- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

Proof: Replace $\beta = -(-\beta)$ as follows:

$$\cos(\alpha+\beta) = \cos\big(\alpha-(-\beta)\big) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

Since, $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$

Therefore, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ #

# Angle Addition Identity of Sine

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

Proof: $\sin(\alpha+\beta) = \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right)$

$$= \cos\left(\frac{\pi}{2} - \alpha - \beta\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right)$$

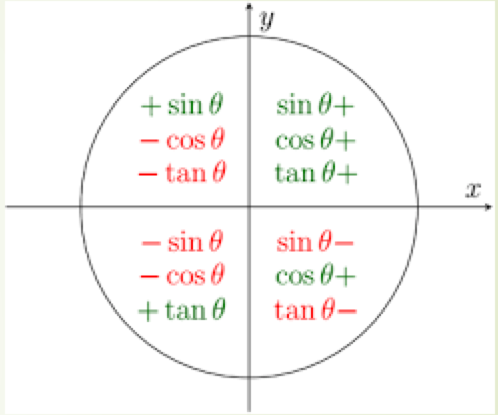
$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Therefore, $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ #

# Angle Subtraction Identity of Sine

- $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$




Proof: $\sin(\alpha-\beta)=\sin\big(\alpha+(-\beta)\big)$

$$=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$

Since, $\cos(-\beta)=\cos\beta$ and $\sin(-\beta)=-\sin\beta$

Therefore, $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ #

# Angle Addition Identity of Tangent

- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

Proof: $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

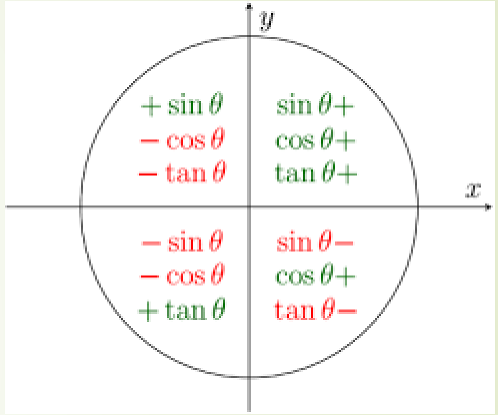
Dividing the numerator and denominator by $\cos\alpha\cos\beta$,

$$\tan(\alpha+\beta) = \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Therefore, $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$ #

# Angle Subtraction Identity of Tangent

- $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$




**Proof**:

$$\tan(\alpha - \beta) = \tan\big(\alpha + (-\beta)\big) = \frac{\tan(\alpha) + \tan(-\beta)}{1 - \tan(\alpha)\tan(-\beta)}$$

Since, $\tan(-\beta) = -\tan\beta$

Therefore, $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$ #

# Example

Prove the following complementary and supplementary trigonometric identities by using sum and difference trigonometric identities.

(a) $sin\left(\frac{\pi}{2} - \theta\right) = cos\,\theta$ (b) $cos(180° - \theta) = -\,cos\,\theta$

(c) $\tan(2\pi - \theta) = -\tan\theta$

# Example (Cont.)

Proof:

(a) $\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta - \cos\left(\frac{\pi}{2}\right)\sin\theta$

$= (1)(\cos\theta) - (0)(\sin\theta)$

$= \cos\theta - 0$

$= \cos\theta$ #

# Example (Cont.)

Proof:

(b) $\cos(180° - \theta) = \cos(180°)\cos\theta + \sin(180°)\sin\theta$

$$= (-1)\cos\theta + (0)\sin\theta$$

$$= -\cos\theta + 0$$

$$= -\cos\theta \qquad \#$$

# Example (Cont.)

Proof:

$$\text{(c) } \tan(2\pi - \theta) = \frac{\tan(2\pi) - \tan\theta}{1 + \tan(2\pi)\tan\theta} = \frac{\frac{\sin(2\pi)}{\cos(2\pi)} - \tan\theta}{1 + \frac{\sin(2\pi)}{\cos(2\pi)}\tan\theta}$$

$$= \frac{0 - \tan\theta}{1 + (0)\tan\theta} = \frac{-\tan\theta}{1 + 0}$$

$$= -\tan\theta \qquad \#$$

# Double Angle Identities

- $\sin 2A = 2\sin A\cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
- $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

**Note:** There are ***common alternate forms*** for double-angle identities of cosine as follows:

- $\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$ #
- $\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A$ #

# Half Angle Identities

- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1 - \cos\theta)}$ ………………………………….(p. 14)
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1 + \cos\theta)}$ ………………………………….(p. 14)
- $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}$ ………………………………….(p. 15)

# Product Identities

*Product to Sum Identities*

- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ ............................(p. 15)
- $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

# Product Identities (Cont.)

*Sum to Product Identities*

- $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
- $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

# Example

Write the following difference of sines expression as a product:

$\sin(4\theta) - \sin(2\theta)$.

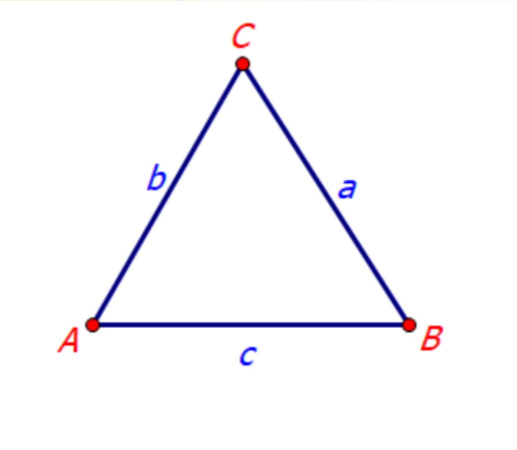
Solution: Formula $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Substitute the values into the formula:

$$\sin(4\theta) - \sin(2\theta) = 2\cos\left(\frac{4\theta+2\theta}{2}\right)\sin\left(\frac{4\theta-2\theta}{2}\right)$$

$$= 2\cos(3\theta)\sin(\theta) \quad \#$$

# Triangle Identities (Sine, Cosine, Tangent Rules)




The *Sine Rule* (also known as the *Law of Sines*) is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The *Cosine Rule* (also known as the *Law of Cosines*) is an extension of the *Pythagorean Theorem* to any triangle:

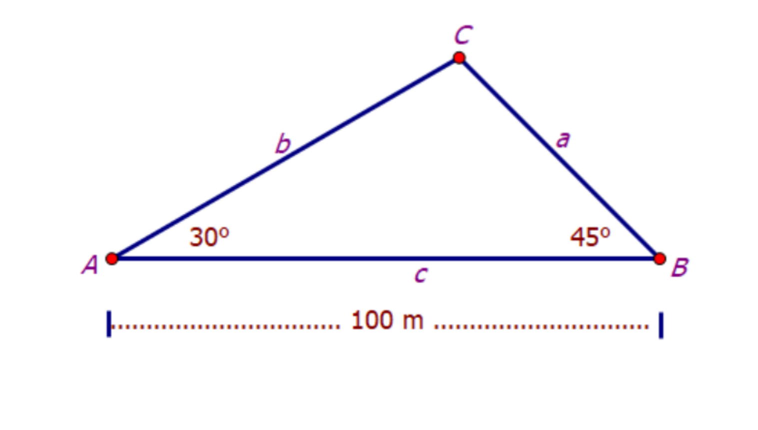
$$a^2 = b^2 + c^2 - 2bc\cos A$$

The *Tangent Rule* (also known as *the Law of Tangents*) is:

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{1}{2}(A+B)\right)}{\tan\left(\frac{1}{2}(A-B)\right)}$$

# Applications of Triangle Identities

Example $\Delta ABC$ has two angles and a side with $\angle A = 30°, \angle B = 45°$, and the length of the opposite side of $\angle C$ is 100 m. Find the length of the opposite side of $\angle A$ and $\angle B$.




Solution: Let $m\angle A = 30°, m\angle B = 45°$ and $c = 100\ m$.

$$m\angle C = 180° - (30° + 45°) = 105°.$$

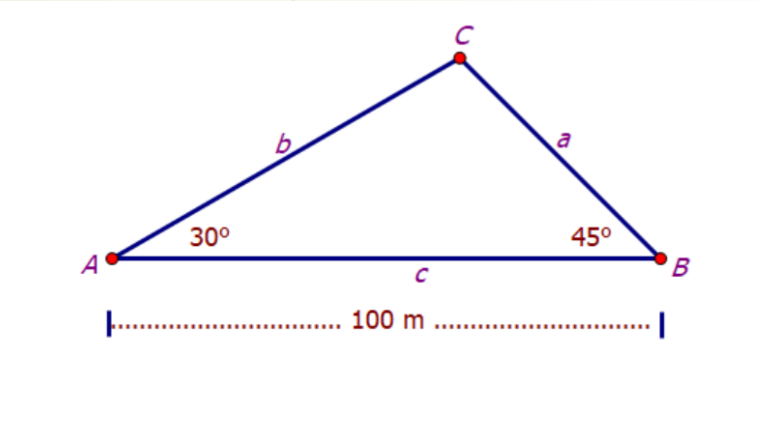
Using the Law of Sines as follows:

$$\frac{\sin A}{a} = \frac{\sin C}{c} \text{ and } \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 30°}{a} = \frac{\sin 105°}{100} \text{ and } \frac{\sin 45°}{b} = \frac{\sin 105°}{100}$$

Note: Using for AAS and ASA

# Applications of Triangle Identities (Cont.)




Note: Using for AAS and ASA

$$a = \frac{100 \sin 30°}{\sin 105°} \quad \text{and} \quad b = \frac{100 \sin 45°}{\sin 105°}$$

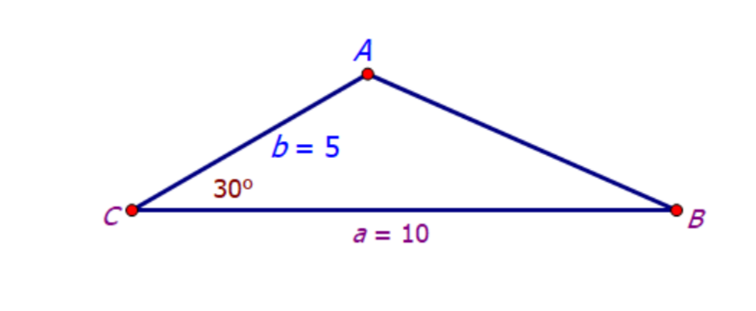
Since, $\sin 30° = 0.50, \sin 45° = 0.71,$ and $\sin 105° = 0.97.$

Therefore, $a = \frac{100(0.50)}{0.97} = 51.55\ m$

$$b = \frac{100(0.71)}{0.97} = 73.20\ m$$ #

# Applications of Triangle Identities (Cont.)

Example Solve $\Delta\, ABC$ with $a = 10, b = 5$, and $\angle C = 30°$.




Using the Law of Cosines to find the length of the opposite side of $\angle C$ as follows:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

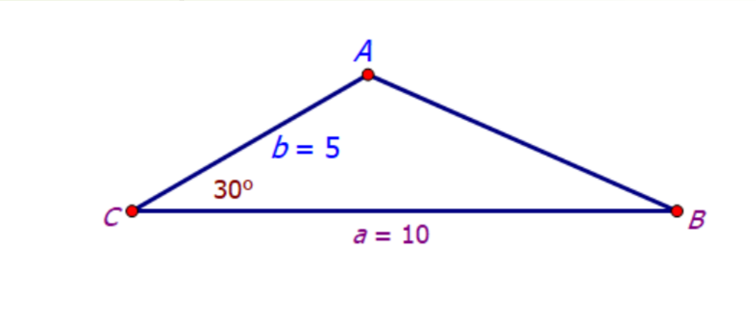
$$= 10^2 + 5^2 - 2(10)(5)\cos 30°$$

$$= 100 + 25 - 100(0.87) = 38$$

$$c = \sqrt{38} \approx 6.16$$

Note: Using for SAS and SSS

# Applications of Triangle Identities (Cont.)




Note: Using for SAS and SSS

Using the Law of Cosines to find one of the two unknown angles as follows: $a^2 = b^2 + c^2 - 2bc\cos A$

$$10^2 = 5^2 + 38 - 2(5)(6.16)\cos A$$

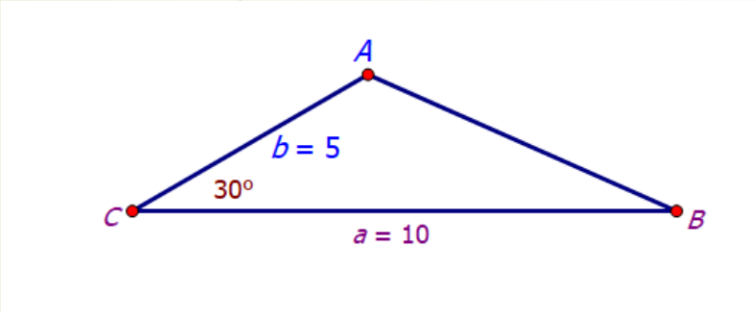
$$\cos A = \frac{25 + 38 - 100}{61.6} = -0.60$$

$$cos^{-1}(-0.60) = 126.87°$$

Then, $\angle A = 126.87°$

$\angle B = 180° - 30° - 126.87° = 23.13°$

# Applications of Triangle Identities (Cont.)




So, the six parts of the triangle are:

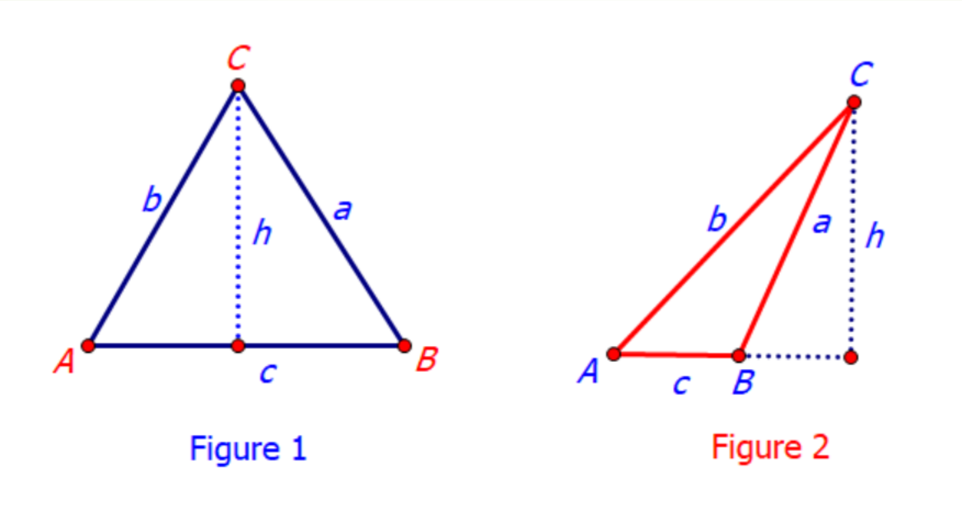
$\angle A = 126.87°, \qquad a = 10$

$\angle B = 23.13°, \qquad b = 5$

$\angle C = 30°, \qquad c = 6.16$ #

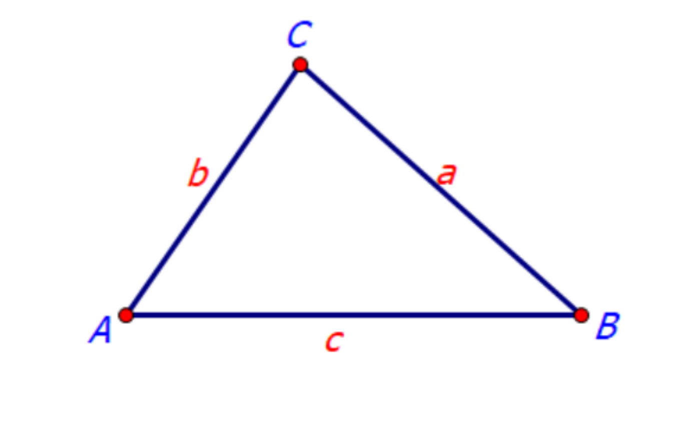
Note: Using for SAS and SSS

# Triangle Area




$$Area\ \Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C.$$

# Heron's Formula




In above figure, if $a, b,$ and $c$ are the lengths of $\Delta\, ABC$:

$$Area\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where, $s$ is half the perimeter or $s = \frac{a+b+c}{2}$.

# Example

Find the area of the triangle whose sides measure 4 cm, 13 cm, and 15 cm.

Solution: Half perimeter, $s = \frac{4+13+15}{2} = \frac{32}{2} = 16$

Using Heron's formula:

$$Area\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{16(16-4)(16-13)(16-15)}$$

$$= \sqrt{16 \times 12 \times 3 \times 1} = \sqrt{576} = 24 \text{ sq.cm}$$ #

# Assignment

Practice 5.2: Problem 4 and 5.

Practice 5.3: Problem 1, 3, and 4.

# Q & A