Chapter 3 Trigonometric Functions in Coordinate Plane

The trigonometric functions can be graphed in the coordinate plane or the Cartesian Plane. The unit circle is a circle with a radius of 1. The center point of the unit circle is at the origin of *xy*-plane or at the coordinate (0, 0) and the equation for the unit circle is $x^2 + y^2 = 1$.

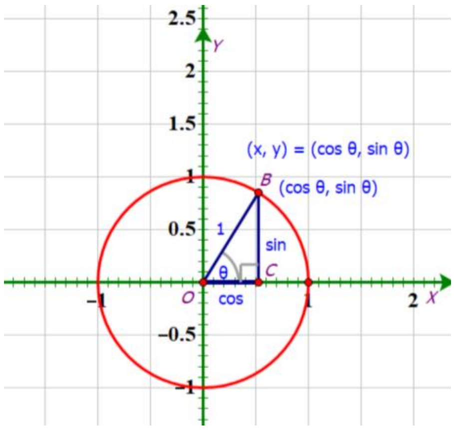


Figure 3.1 Coordinates on the unit circle $(x, y) = (\cos\theta, \sin\theta)$

In Figure 3.1, any point on the unit circle has coordinates $(x, y) = (\cos\theta, \sin\theta)$, where θ is the angle. For any value of θ made by the radius of unit circle, the value of $(\cos\theta, \sin\theta)$ will be the positive or negative depending on the value of coordinate (x, y) in the four quadrants.

Graph of Trigonometric Functions

The trigonometric ratios on a right triangle have input values (θ) in degrees and output values in real numbers, for example, $\sin 30° = \frac{1}{2}$. The trigonometric functions defined on a unit circle have input values (θ) in degrees or radians, and output values in real numbers for example, $f(x) = \sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$. Some of the principal values of trigonometric functions at specific angles are presented below in the following table.

$\boldsymbol{\theta}$	$\mathbf{0}$	$\boldsymbol{\frac{\pi}{6}}$	$\boldsymbol{\frac{\pi}{4}}$	$\boldsymbol{\frac{\pi}{3}}$	$\boldsymbol{\frac{\pi}{2}}$	$\boldsymbol{\pi}$	$\boldsymbol{\frac{3\pi}{2}}$	$\boldsymbol{2\pi}$
$\mathbf{sin}\,\boldsymbol{\theta}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\mathbf{cos}\,\boldsymbol{\theta}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\mathbf{tan}\,\boldsymbol{\theta}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined	0	Undefined	0
$\mathbf{csc}\,\boldsymbol{\theta}$	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	Undefined	-1	Undefined
$\mathbf{sec}\,\boldsymbol{\theta}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined	-1	Undefined	1
$\mathbf{cot}\,\boldsymbol{\theta}$	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	Undefined	0	Undefined

The graphs of trigonometric functions have the domain value of θ represented on the horizontal x-axis and the range value represented along the vertical y-axis.

Graphs of Sine Function

The matching a correspondence between the y-coordinates of points on the unit circle and the values of $f(\theta) = \sin\theta$ on *xy*-plane are shown in Figure 3.2 for the angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$.

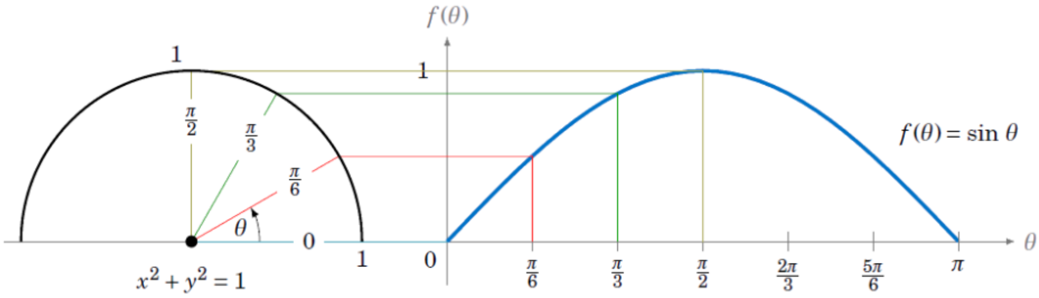


Figure 3.2 Corresponding points on the unit circle and *xy*-plane

If the terminal side of the angle θ rotate around the origin including 0 to 2π radians, as shown in Figure 3.3. This illustrates what is called the period of sine function.

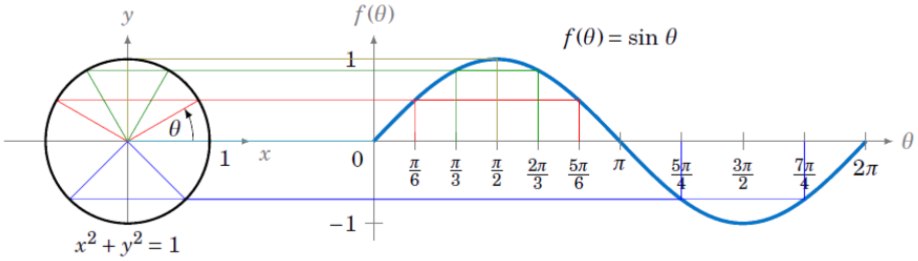


Figure 3.3 The period of sine function

Since the trigonometric functions repeat every 2π radians or $360°$, then the graph of function $f(x) = \sin x$ or $y = \sin x$ for x in the interval $[-2\pi, 2\pi]$ is the curve shown in Figure 3.4.

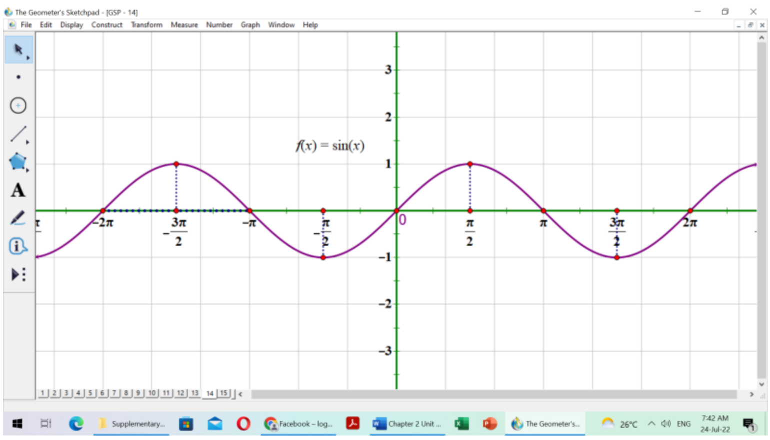


Figure 3.4 Graphs of Sine Functions

In Figure3.4, the ***sine function*** is a smooth curve that maximum value of $\sin x$ is 1, where $x = \frac{\pi}{2}$ radians $(90°)$ and minimum value of $\sin x$ is -1, where $x = \frac{3\pi}{2}$ radians $(270°)$. From the figure shows that for all the real numbers x, the values of $\sin x$ always be between -1 and 1. So, the domain of $f(x) = \sin x$ contains all real numbers and the range of values of $\sin x$ is $[-1,1]$. The shape of ***sine curve*** starts at 0 and is the same for each full rotation, so the function repeats itself over and over in both directions called '***periodic function***'.

Graph of Cosine Function

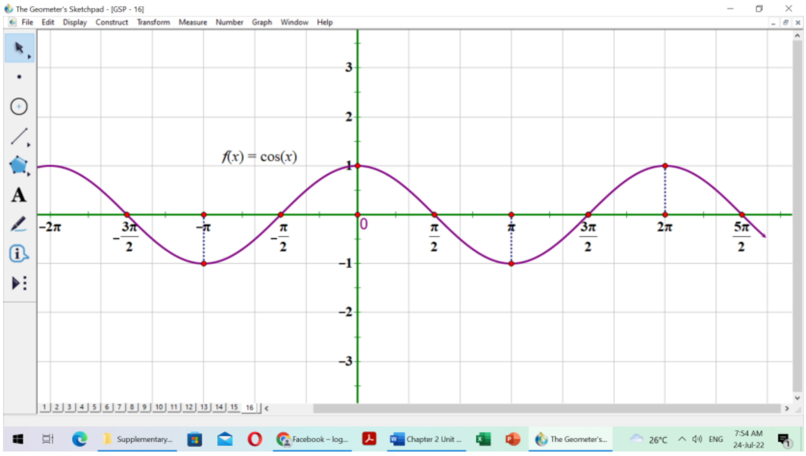


Figure 3.5 Graphs of Cosine Functions

In Figure 3.5, the cosine function is also a smooth curve that varies in the range $[-1,1]$ but translated to the left by $\frac{\pi}{2}$ radians along the x-axis. Therefore, $y = \cos x$ crosses the y-axis at 1. The shape of ***cosine curve*** starts at 1 and is the same for each full rotation, so the function is also called '***periodic function***'.

Connecting Sine and Cosine Function

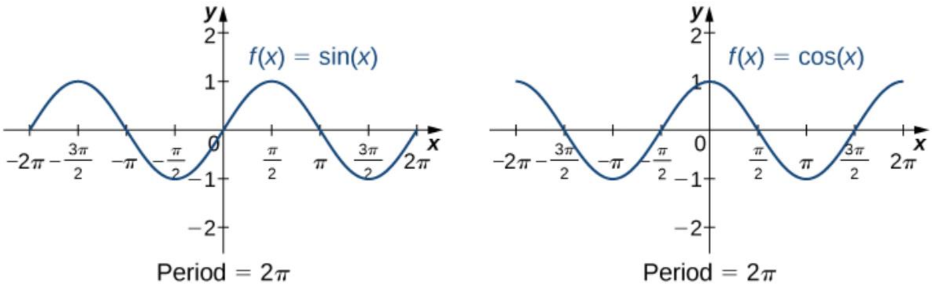


Figure 3.6 Graphs of $\sin x$ and $\cos x$

In Figure 3.6, the period of sine function is 2π radians or $360°$. That is $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 2k\pi)$ for any integer k. For example, $\sin 0 = 0, \sin(0 + 2\pi) = \sin(2\pi) = 0, \sin(4\pi) = 0$. The sine function is symmetric about the origin because it is an odd function and $\sin(-x) = -\sin x$. The ***amplitude*** of $\sin x$ is 1. This refers to the distance from the peak and the baseline (the horizontal line). The period of the cosine function is 2π, therefore the value of $\cos x = \cos(x + 2\pi) = \cos(x + 4\pi) = \cos(x + 2k\pi)$ for any integer k. For example, $\cos 0 = 1, \cos(0 + 2\pi) = \cos 2\pi =$

$1, \cos(4\pi) = 1$. The cosine function is symmetric about y-axis because it is an even function and $\cos(-x) = \cos x$. The distance from the peak of the graph of $\cos x$ and the baseline (the horizontal line) is 1, so the ***amplitude*** of the $\cos x$ is 1.

Comparing the sine and cosine functions are listed in the following table:

Concepts	$\sin x$	$\cos x$
Pattern	Continuous	Continuous
Period	2π	2π
Domain	All real numbers	All real numbers
Range	$[-1,1]$	$[-1,1]$
Amplitude	1	1

Graph of Tangent Function

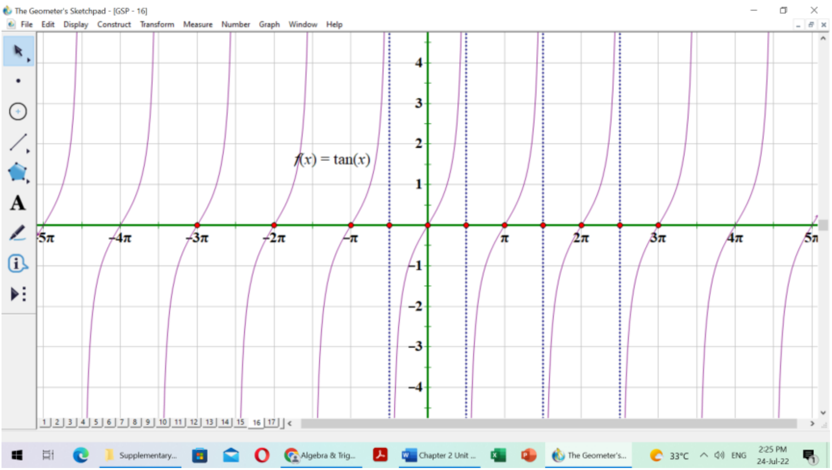


Figure 3.7 Graph of $\tan x$

Since $\tan x = \dfrac{\sin x}{\cos x}$, then the tangent function is not defined when $\cos x = 0$. That is, it is not defined for any number $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. Therefore, we draw vertical asymptotes at these locations (see Figure 3.7). Note that the function value is 0 when $x = 0$, and the values increase without bound as x increases toward $\frac{\pi}{2}$. The graph gets closer and closer to an asymptote as x gets closer to $\frac{\pi}{2}$, but it never touches to the line. As x decreases from 0 to $-\frac{\pi}{2}$, the values decrease without bound. Again, the graph gets closer and closer to an asymptote, but it never touches it. Now, the drawing tangent graph is complete for the period is π. Note that there is no amplitude because there are no maximum and minimum values. Thus, the domain of tangent function is the set of all real numbers $except\,\frac{\pi}{2} + k\pi$, $where\ k\ is\ an\ integer$. The range of function is the set of all real numbers.

Graphs of the Cosecant, Secant, and Cotangent Functions

The cosecant, secant, and cotangent functions are the reciprocal functions of sine, cosine, and tangent functions, respectively. The graphs in Figure 3.8 (a), (b), and (c) are shown by thin curves for reference.

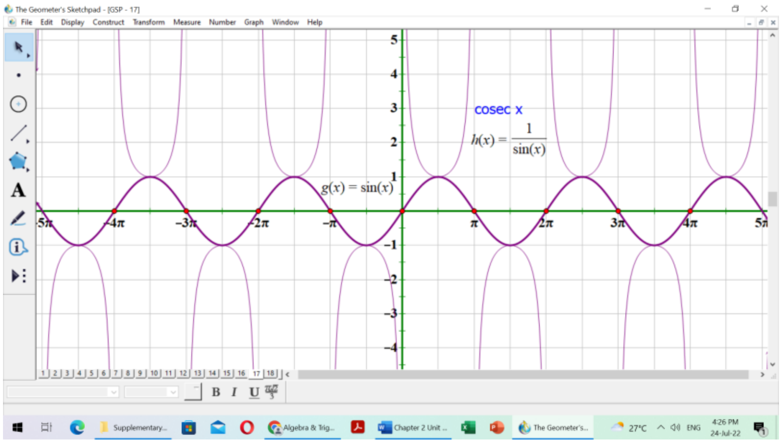


Figure 3.8 (a) Graph of $\csc x$

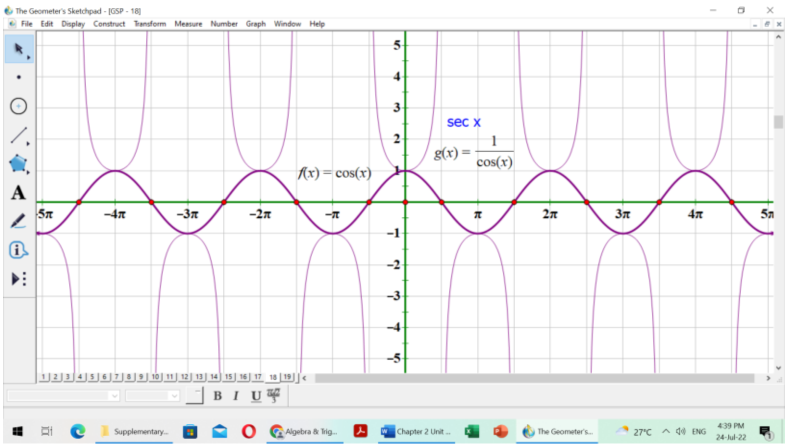
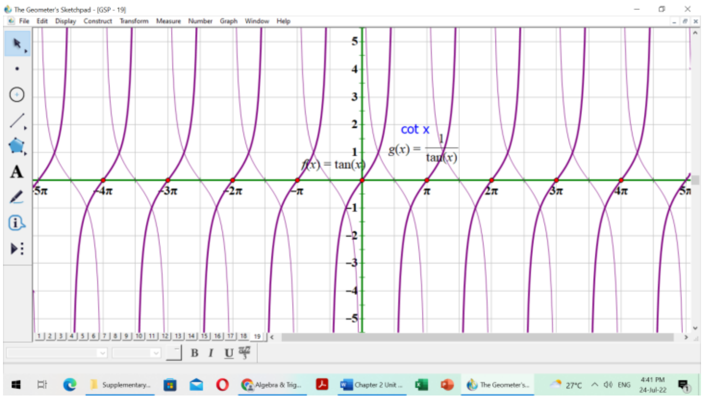


Figure 3.8 (b) Graph of $\sec x$

Comparing the cosecant and secant functions are listed in the following table:

Concepts	**csc *x***	**sec *x***
Period	2π	2π
Domain	All real numbers *except* $k\pi$, *where k is an integer*	All real numbers *except* $\frac{\pi}{2} + k\pi$, *where k is an integer*
Range	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$



Graph 3.8 (c) Graph of $\cot x$

Comparing the tangent and cotangent functions are listed in the following table:

Concepts	**tan *x***	**cot *x***
Period	π	π
Domain	All real numbers *except* $\frac{\pi}{2} + k\pi$, *where k is an integer*	All real numbers *except* $k\pi$, *where k is an integer*
Range	All real numbers	All real numbers

Sinusoidal Graphs

A ***sinusoid*** or ***sine wave*** is the name given to any curve that can be written in the form

$$y = A\sin(B(x - C)) + D$$

where A, B, C, and D are constants such that:

$|A|$ is the ***amplitude***.

B is the cycle from 0 *to* 2π (frequency) and $\frac{2\pi}{B}$ is the ***period***.

C is the $\boldsymbol{horizontal\ shift}$ (or phase shift). If C is positive, the graph shifts right. If C is negative, the graph shifts left.

D is the ***vertical shift*** (or displacement). If D is positive, the graph shifts up. If it is negative the graph shifts down.

The general form of cosine function can be $y = A\cos(B(x - C)) + D$.

The simple graphs of $y = A\sin(Bx)$ and $y = A\cos(Bx)$ are illustrated in Figure 3.9.

Investigation for changing the value A:

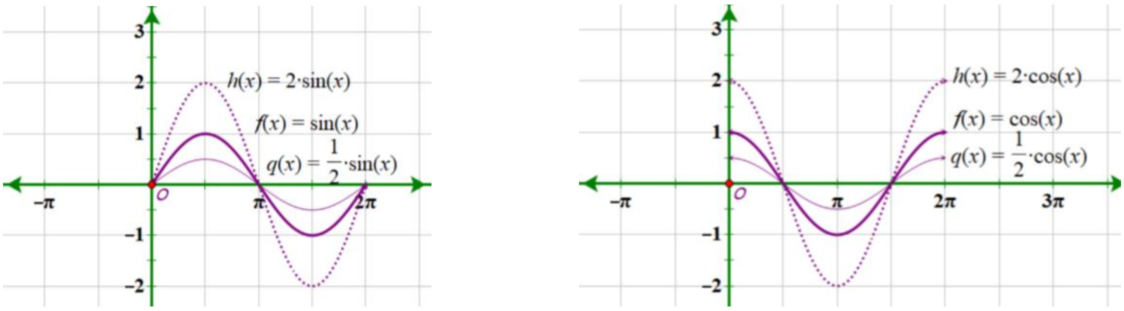


Figure 3.9 Sinusoidal graphs in different amplitudes

In Figure 3.9, the value ***A*** (in front of $\sin x$ or $\cos x$) affects the ***amplitude*** (height). The amplitude is half the distance between the maximum and minimum values of the function evaluated by $|\boldsymbol{A}|$**.** Increasing or decreasing the value of ***A*** will ***vertically stretch*** or ***shrink*** the graph.

Investigation for changing the value B:

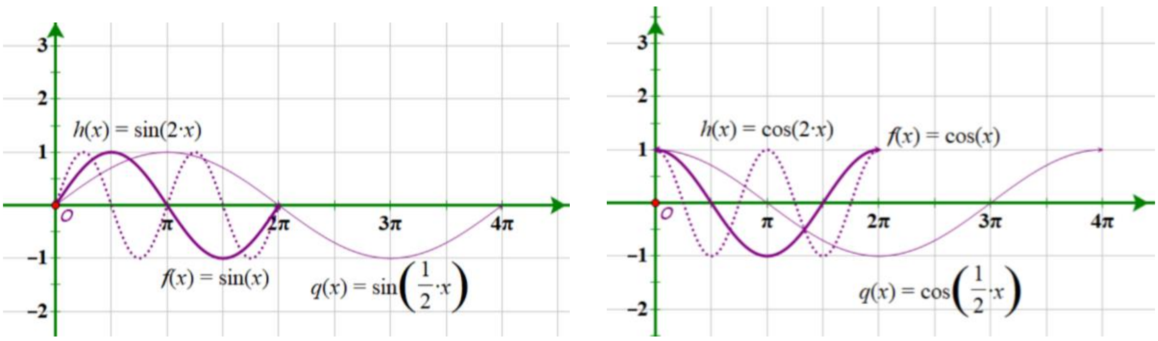


Figure 3.10 Sinusoidal graphs in different periods

In Figure 3.10, the value $\boldsymbol{B}$ (in front of x) is the number of cycles the graph completes in an interval 0 to 2π or $360°$. The value $\boldsymbol{B}$ affects the ***period***, the period is evaluated by $\left|\frac{2\pi}{B}\right|$. When $\mathbf{0 < B < 1}$, the period of function will be greater than 2π and will ***horizontally stretch*** the graph. When $\mathbf{B > 1,}$ the period of the function will be less than 2π and the graph will be a ***horizontal shrinking***. The periods are calculated as the following examples:

Sine Functions	Cosine Functions
$y = \sin x\,;\text{period} = \dfrac{2\pi}{1} = 2\pi$	$y = \cos x\,;\text{period} = \dfrac{2\pi}{1} = 2\pi$
$y = \sin\left(\frac{1}{2}x\right);\ \text{period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$	$y = \cos\left(\frac{1}{2}x\right);\ \text{period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$
$y = \sin(2x);\ \text{period} = \frac{2\pi}{2} = \pi$	$y = \cos(2x);\ \text{period} = \frac{2\pi}{2} = \pi$

Example 3.1 Graph $3\sin(2x)$

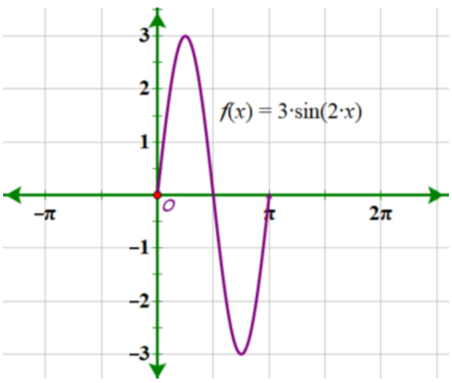
Solution: Amplitude: $|A| = |3| = 3$

Period: $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

$C = 0$, so there is no horizontal shift

$D = 0$, so there is no vertical shift

The graph of $y = 3\sin(2x)$ is shown below:



Example 3.2 Graph $y = 2\sin\left(x - \frac{\pi}{4}\right) + 3$.

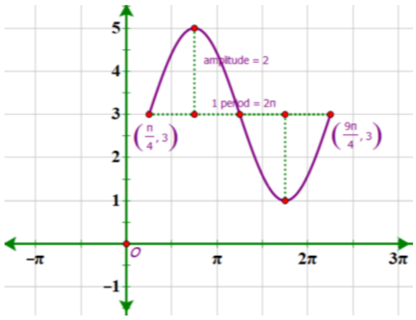
Solution: Amplitude: $|A| = |2| = 2$

Period: $\frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$

$C = \frac{\pi}{4}$, so the graph shifts right $\frac{\pi}{4}$

$D = 3$, so the graph shifts up 3

The graph of $y = 2\sin\left(x - \frac{\pi}{4}\right) + 3$ is shown below:



Practice 3.1

1. Draw the graphs of the following trigonometric functions in *xy*-plane.

   1) $y = 2\sin 3x$ 2) $y = 3\cos 2x$ 3) $y = \tan 2x$

2. Find the amplitude, period, horizontal shift, and vertical shift of each function.

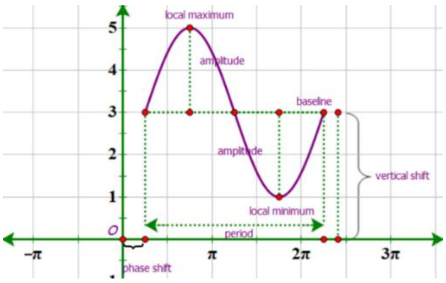
   1) $y = 3\sin\left(x - \frac{\pi}{4}\right) + 2$. 2) $y = 2\sin\left(x + \frac{\pi}{4}\right) - 3$.

3. Draw the graphs:

   1) $y = 3\sin\left(x - \frac{\pi}{4}\right) + 2$. 2) $y = 2\sin\left(x + \frac{\pi}{4}\right) - 3$.

Equation of a Sinusoidal Curve

An equation of the form $y = A\sin(B(x - C)) + D$ can be characterized by four properties: period, amplitude, phase shift, and vertical shift.

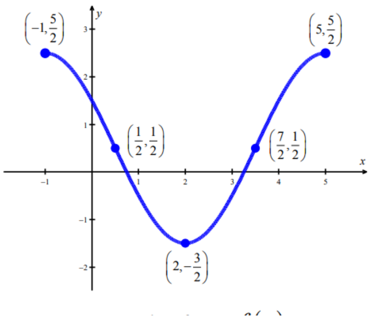


- ***Period*** is the number of cycles the graph completes in an interval 0 to 2π or $360°$
- ***Amplitude*** is a measure of how 'tall' the wave curve is. The amplitude of the standard sine and cosine functions is 1, but vertical scale can alter this.
- ***Phase shift*** of the sinusoid is the horizontal shift to the right of the origin along x-axis.
- ***Vertical shift*** of the sinusoid is assumed to shift up along y-axis.

Example 3.3 Below is the graph of one complete cycle of a sinusoid $y = f(x)$.

(a) Find a sine function whose graph matches the graph $y = f(x)$.

(b) Find a cosine function whose graph matches the graph $y = f(x)$.



One cycle of graph $y = f(x)$.

Solution:

(a) Determine an equation of the form $y = A\sin(B(x - C)) + D$ or $y = A\sin(Bx - BC) + D$.

- Find the value of B:

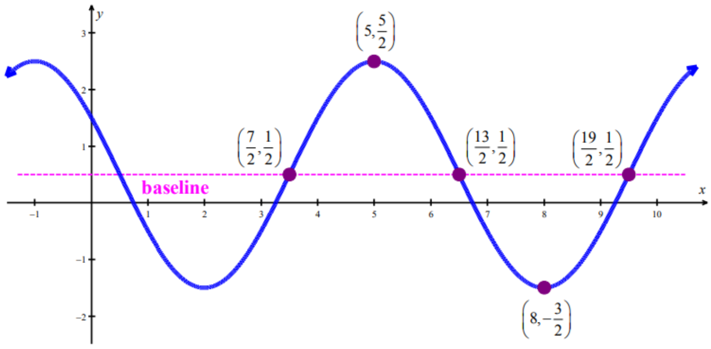
Since one cycle is graphed over the interval $[-1, 5]$, its period is $5 - (-1) = 6$. So that the value of $B = \frac{2\pi}{6} = \frac{\pi}{3}$.

- Find the value of A:

  Notice that the range of the sinusoid is $\left[-\frac{3}{2}, \frac{5}{2}\right]$. The midpoint of the range is $\frac{1}{2}\left(\frac{5}{2} - \frac{3}{2}\right) = \frac{1}{2}$. This is the baseline of $y = \frac{1}{2}$. Therefore, the amplitude $A = \frac{5}{2} - \frac{1}{2} = 2$.

- Find the phase shift:

  Extend the graph of the given sinusoid as in the figure below so that we can identify a cycle beginning at $\left(\frac{7}{2}, \frac{1}{2}\right)$. Taking the phase shift to be $\frac{7}{2}$, we get $BC = \frac{\pi}{3} \times \frac{7}{2} = \frac{7\pi}{6}$.



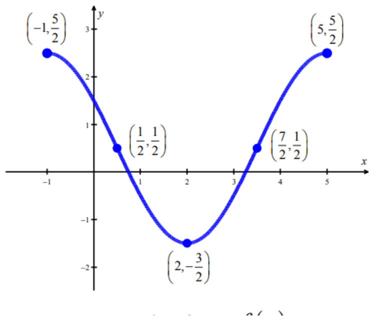
- Find the vertical shift:

  The baseline is used to determine a vertical shift of $D = \frac{1}{2}$.

Therefore, the sine function whose graph matches the graph $y = f(x)$.is

$y = 2\sin\left(\frac{\pi}{3}x - \frac{7\pi}{6}\right) + \frac{1}{2}$ #

(b) Determine an equation of the form $y = A\cos(B(x - C)) + D$. or $y = A\cos(Bx - BC) + D$.



- Find the value of ***B***

Since one cycle is graphed over the interval $[-1, 5]$, its period is $5 - (-1) = 6$. So that the value of $B = \frac{2\pi}{6} = \frac{\pi}{3}$.

- Find the value of A:

Notice that the range of the sinusoid is $\left[-\frac{3}{2}, \frac{5}{2}\right]$. The midpoint of the range is $\frac{1}{2}\left(\frac{5}{2} - \frac{3}{2}\right) = \frac{1}{2}$. This is the baseline of $y = \frac{1}{2}$. Therefore, the amplitude $A = \frac{5}{2} - \frac{1}{2} = 2$.

- Find the phase shift:

Extend the graph of the given sinusoid as in the figure above so that we can identify a cycle beginning at $\left(-1, \frac{5}{2}\right)$. Taking the phase shift to be -1, we get $BC = \frac{\pi}{3} \times (-1) = -\frac{\pi}{3}$.

- Find the vertical shift:

The baseline is used to determine a vertical shift of $D = \frac{1}{2}$.

Therefore, the cosine function whose graph matches the graph $y = f(x)$.is $y = 2\cos\left(\frac{\pi}{3}x + \frac{\pi}{3}\right) + \frac{1}{2}$ #

Applications of Sinusoidal Functions

Sine and cosine functions can be used to model many real-life situations such as sunrise and sunset, tides, musical tones, radio waves, electrical current.

In the following example contains the number of hours of daylight at the sun rises and sets on the fifteenth of every month in a year.

Let x =1 represent January 15

x = 2 represent February 15.

x = 3 represent March 15.

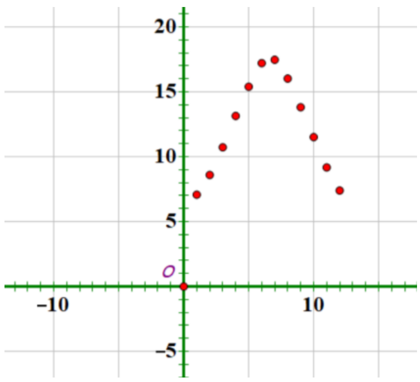
…

x = 12 represent December 15.

The number of daylight hours for x = 1 to x = 12 list in the following table.

Month	Jan.	Feb.	March	April	May	June
x	1	2	3	4	5	6
Hours of Daylight	7.08	8.60	10.73	13.15	15.40	17.22

Month	July	August	Sept.	Oct.	Nov.	Dec.
x	7	8	9	10	11	12
Hours of Daylight	17.48	16.03	13.82	11.52	9.18	7.40



The above graph resembles a type of sine curve. Since there are 12 months in a year, month 13 is the same as month 1, month 14 is the same as moth 2, and so on. The function is periodic. We can write a sinusoidal function to represent the data and can be function of the form

$$y = A\sin(B(x - C)) + D$$

Given the graph of a sinusoidal function, we can write its equation in the above form using the following steps.

- To find the equation of the midline between the maximum value (17.48 h) and the minimum value (7.08 h), so the equation of the midline is,

$$y = \frac{17.48+7.08}{2} = 12.28.$$

This midline indicates the vertical shift $D = 12.28$.

- To find the amplitude ***A***, find the perpendicular distance between the midline and either a local maximum or minimum using half the difference between the local maximum and the local minimum.

In the above table, the value of ***A*** (amplitude) is half the difference between the maximum value (17.48 h) and the minimum value (7.08 h). Therefore,

$$A = \frac{17.48 - 7.08}{2} = 5.2$$

So, the amplitude $A = 5.2$.

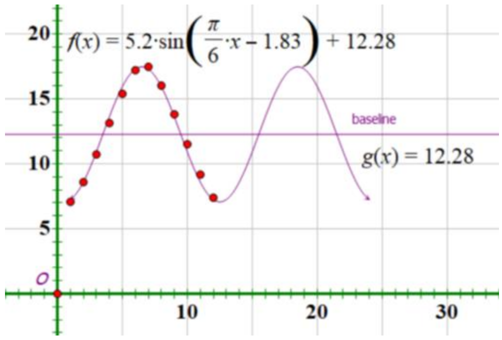
- To find the value ***B***, examine the graph to determine its period or the horizontal distance before the graph repeats itself by calculating $B = \frac{2\pi}{period}$ or period = $\frac{2\pi}{B}$.

There are 12 months (periods) for collecting the hour daylight data, then the value of B is calculated by $\frac{2\pi}{B} = 12$ or $B = \frac{2\pi}{12} = \frac{\pi}{6}$

- To find phase shift, identify a cycle beginning at the first point at which the sinusoid intersects the midline $y = 12.28$ that precedes the local maximum. It is about a quarter of a year. Taking the phase shift between March and April $\left(\frac{3+4}{2} = 3.5\right)$ and the number of hour daylight between March and April $\left(\frac{10.73+13.15}{2} = 11.94\right)$ or at $(3.5, 11.94)$, we get $BC = \frac{\pi}{6} \times (3.5) \approx 1.83$. Therefore, the sinusoidal function represents the data is

$$y = 5.2 \sin\left(\frac{\pi}{6}x - 1.83\right) + 12.28. \qquad \#$$

Note that the scatter plot with a graph of this function fits the graph of the final answer as shown in the following diagram.



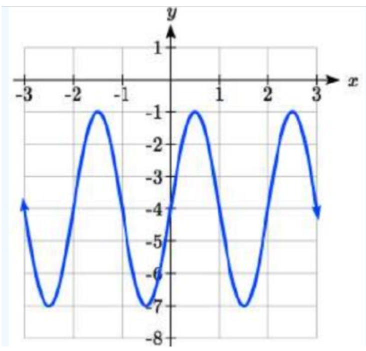
Practice 3.2

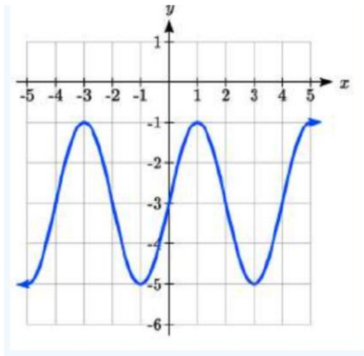
1. Each of the following equations, find the amplitude, period, phase shift, and midline.

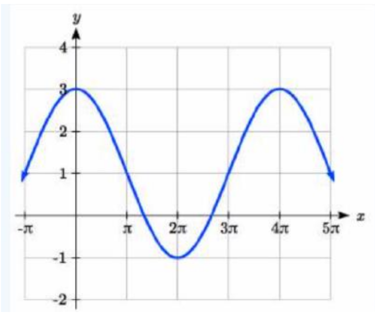
   1) $y = 4\sin\left(\frac{\pi}{2}(x+4)\right) + 5$ 2) $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$

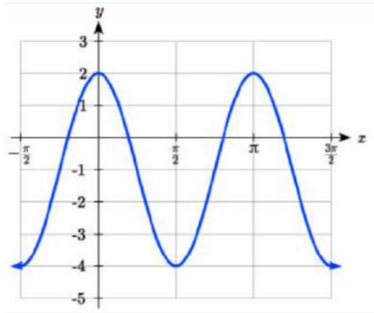
   3) $y = 2\sin(3x - 21) + 4$ 4) $y = 5\sin(5x + 20) - 2$

2. For the graph below, determine the amplitude, midline, and period, then find the equation for the function.

1) 

2) 

3) 

4) 

Graphs of Inverse Trigonometric Functions

In Figure 3.9, trigonometric functions are not one to one because their graphs fail the horizontal line test (intersect more than one point).

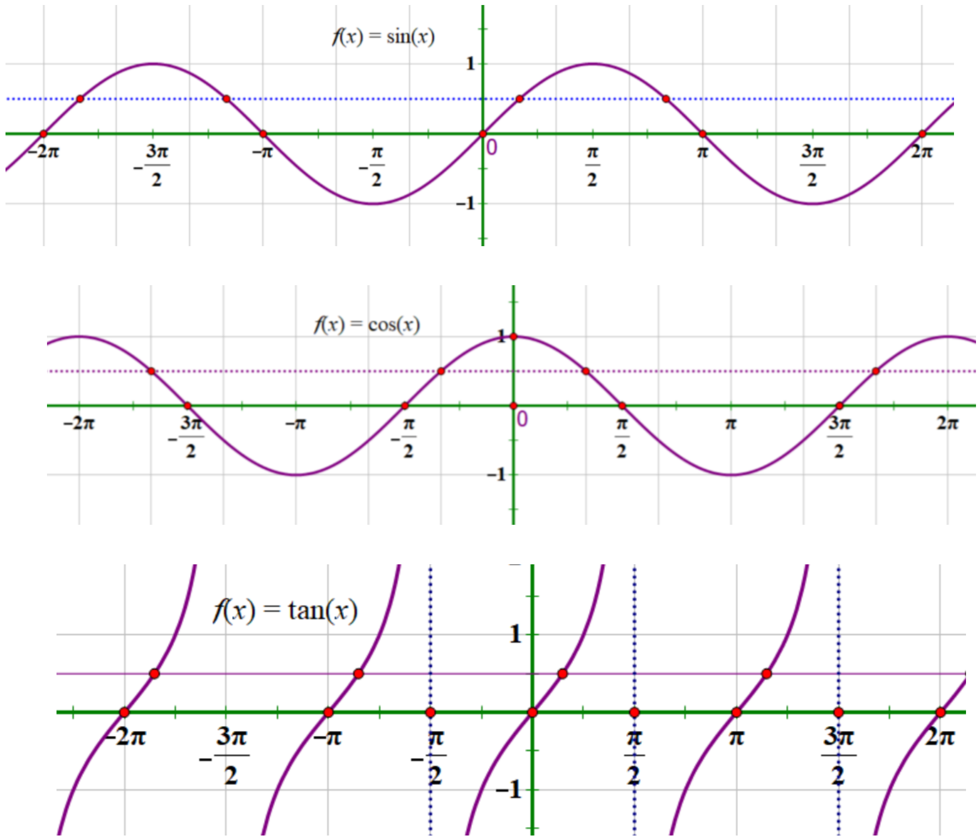
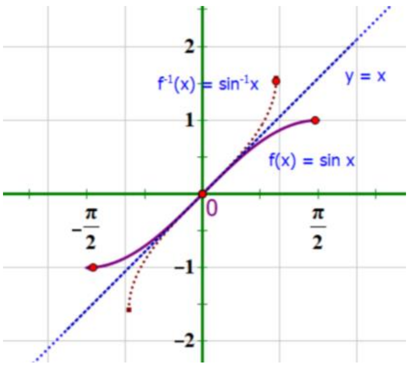


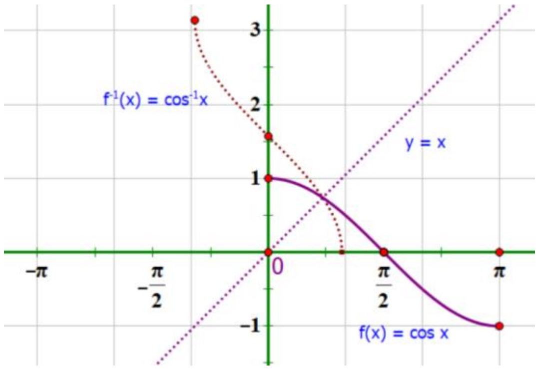
Figure 3.9 Horizontal line intersects each trigonometric graph

However, there is a part of the graph that satisfies the horizontal line test. With appropriate restriction on θ or x, we define the domain and range of inverse trigonometric functions as follows:

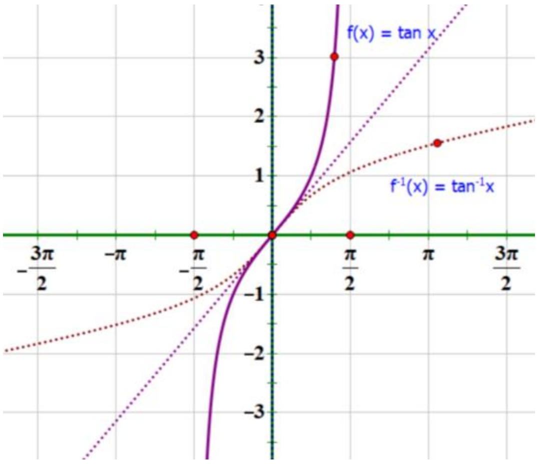
- For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $-1 \leq k \leq 1$, we have $\theta = \sin^{-1} k$ if and only if $k = \sin\theta$.
- For $0 \leq \theta \leq \pi$ and $-1 \leq k \leq 1$, we have $\theta = \cos^{-1}\theta$ if and only if $k = \cos\theta$.
- For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $k \in R$, we have $\theta = \tan^{-1}\theta$ if and only if $k = \tan\theta$.



Graphs of $\sin x$ and $\sin^{-1} x$



Graphs of $\cos x$ and $\cos^{-1} x$



Graphs of $\tan x$ and $\tan^{-1} x$

Practice 3.3

1. Evaluate the given expression (in degrees and radians) without the aid of a calculator.

1) $\sin^{-1}\left(\frac{1}{2}\right)$ 2) $\cos^{-1}\left(\frac{1}{2}\right)$ 3) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

4) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ 5) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ 6) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

2. Using a calculator to find the approximate value in degrees and radians.

1) $\sin^{-1}(0.8621)$ 2) $\cos^{-1}(-0.3218)$ 3) $\tan^{-1}(0.5893)$

4) $\sin^{-1}(-0.6821)$ 5) $\cos^{-1}(0.2814)$ 6) $\tan^{-1}(-1.7321)$

3. Evaluate, answer the questions and graph:

1) (a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ 2) $\cos^{-1}(0)$ 3) $\cos^{-1}\left(-\frac{1}{2}\right)$

2) What is the domain of $\cos^{-1}(x)$?

3) What is the range of $\cos^{-1}(x)$?

4) Graph $\cos^{-1}(x)$.

5) (a) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (b) $\tan^{-1}(0)$

6) What is the domain of $\tan^{-1}(x)$?

7) What is the range of $\tan^{-1}(x)$?

8) Graph $\tan^{-1}(x)$.

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