Chapter 2 Unit Circle Trigonometry

Since trigonometric functions can be used to determine the dimensions of any triangle given any degrees of angles, then the definitions of trigonometric functions were defined as ***points on a unit circle***. This allowed the development of graphs of functions related to the angles they represent which were periodic. Today, using the periodic nature of trigonometric functions, mathematicians and scientists have developed mathematical models to predict many natural periodic phenomena.

The Unit Circle

The unit circle is a circle with a radius of 1, the center is at the origin of *xy*-plane, and the equation for the unit circle is $x^2 + y^2 = 1$. The Figure 2.1 shows the trigonometric ratios on the unit circle.

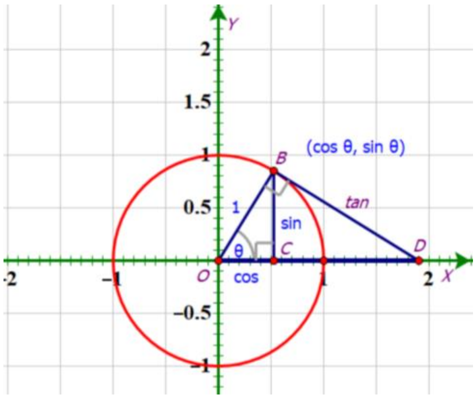


Figure 2.1 Trigonometric Ratios on a Unit Circle

In Figure 2.1, circle O is a unit circle. Let point B is on the unit circle $(OB = 1)$, $\overline{BC}$ perpendicular to x-axis, $\overline{OB} \perp \overline{BD}$ (tangent line of the unit circle), and the point D is on x-axis.

In the right triangle BOC with $\angle BCO = 90°$, $\angle BOC = \theta$, $\overline{BC}$ is side opposite θ, $\overline{OC}$ is side adjacent to θ, and $\overline{OB}$ is the hypotenuse $(OB = 1)$. Let (x, y) be the coordinate of the point B, then $OC = x$ and $BC = y$. Therefore, the trigonometric ratios of the right triangle BOC are as follows:

$$\sin\theta = \frac{BC}{OB} = \frac{y}{1} = y, \qquad \cos\theta = \frac{OC}{OB} = \frac{x}{1} = x$$

So, $B(x, y) = B(\cos\theta\,, \sin\theta)$. The trigonometric identities on xy-plane are as follows:

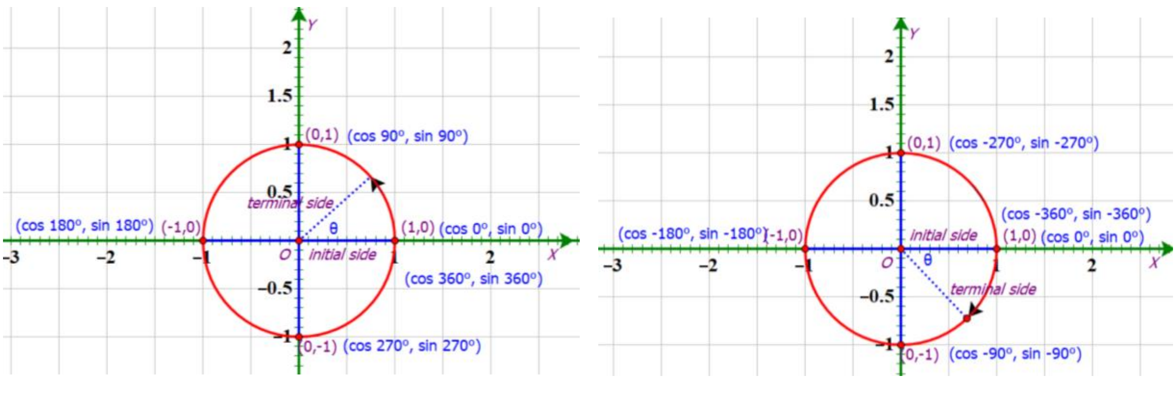
$$x = \cos\theta \text{ and } y = \sin\theta.$$

Since, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then $\tan\theta = \frac{y}{x}$.

Value of Sin $\boldsymbol{\theta}$, Cos $\boldsymbol{\theta}$, and Tan $\boldsymbol{\theta}$ on Unit Circle

Any point on the unit circle has coordinate (x, y), which are equal to the trigonometric identities of $(\cos\theta\,, \sin\theta)$. For any value of θ made by the radius of unit circle, the value of $(\cos\theta\,, \sin\theta)$ will be the positive or negative depending on the value of coordinate (x, y) in the four quadrants.

The entire circle represents a complete angle of $360°$, and the four quadrant lines of the circle make angles of $0°, 90°, 180°, 270°$, and $360°$. In trigonometry, the vertex is always placed at the origin and one ray is always place on the positive x-axis. This ray is called ***initial side*** of the angle. The other ray is called the ***terminal side*** of the angle. This positioning of an angle is called ***standard position***. The angle goes anticlockwise is defined to be the positive angle (see Figure 2.2(a) and the angle goes clockwise is defined to be the negative angle (see Figure 2.2 (b).



(a) (b)

Figure 2.2 Positioning of Angles: Anticlockwise and Clockwise

The angles in Figure 2.2 (a) go anticlockwise from the initial side of $\theta = 0°$ to the terminal side of $90°, 180°, 270°$, and $360°$, but the angles in Figure 2.2 (b) go clockwise from the initial side of $\theta = 0°$ to the terminal side of $-90°, -180°, -270°$, and $-360°$. The values of $(\cos\theta, \sin\theta)$ on x-axis and y-axis are listed in the following table.

θ	$0°$	$90°$	$180°$	$270°$	$360°$	$0°$	-90°	$-180°$	$-270°$	$-360°$
cos	1	0	-1	0	1	1	0	-1	0	1
sin	0	1	0	-1	0	0	-1	0	1	0

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ then the values of $\tan\theta$ in the above table are as follows:

θ	$0°$	$90°$	$180°$	$270°$	$360°$	$0°$	-90°	$-180°$	$-270°$	$-360°$
tan	0	undefined	0	undefined	0	0	undefined	0	undefined	0

Trigonometric Functions on the Unit Circle

In the previous section, a unit circle can be divided into 360 parts based on a complete angle of $360°$ of the entire circle. For this section, a unit circle can be divided by wrapping up a number line to measure the circumference of the circle (see Figure 2.3).

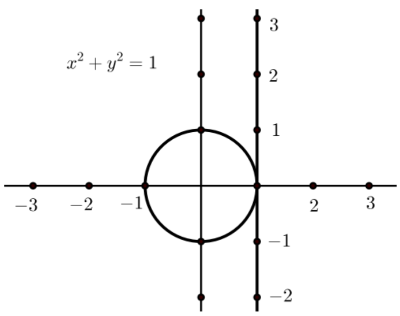


Figure 2.3 Setting up to wrap a number line around the unit circle

The formula for the circumference of a circle is $2\pi r$, where r is the radius of the circle. Therefore, the circumference of a unit circle is 2π. Since the formula to measure the arc length of a circle is $s = r\theta$ where r is the radius. In a unit circle, $r = 1$ then the arc length of a unit circle is $s = \theta$. This arc length is defined as 1 radian (see Figure 2.4).

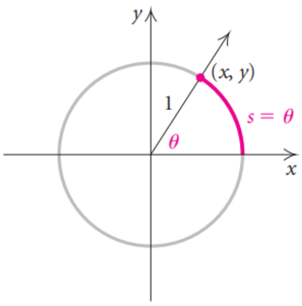


Figure 2.4

From Figure 2.4, the length of the arc around a unit circle is 2π is the same as the radian measure of the angle $\theta = 360°$. Therefore,

$$2\pi \text{ radians} = 360° \text{ or } \pi \text{ radians} = 180°$$

Example 2.1 Convert each degree measure into radians.

(a) $45°$ (b) $-60°$

Solution:

(a) Since $180° = \pi$ radians

Therefore, $45° = \frac{45\pi}{180} = \frac{\pi}{4}$ radians #

(b) Since $180° = \pi$ radians

Therefore, $-60° = -\frac{60\pi}{180} = -\frac{\pi}{3}$ radians #

Example 2.2 Convert each radian measure into degrees.

(a) $\frac{2\pi}{3}$ (b) $-\frac{\pi}{6}$

Solution:

(a) Since π radians $= 180°$

Therefore, $\frac{2\pi}{3}$ radians $= \frac{2}{3} \times 180° = 120°$ #

(b) Since π radians $= 180°$

Therefore, $-\frac{\pi}{6}\ radians = -\frac{180°}{6} = -30°$ #

The Unit Circle with Radian Measures

In Figure 2.4, the point (x, y) is the point on a unit circle where the terminal side of the angle with radian measure s intersects the unit circle. Since, wrapping a number line around a unit circle then s is a real number. To each real number s, there corresponds an arc length s on the unit circle is determined the trigonometric functions (see Figure 2.5).

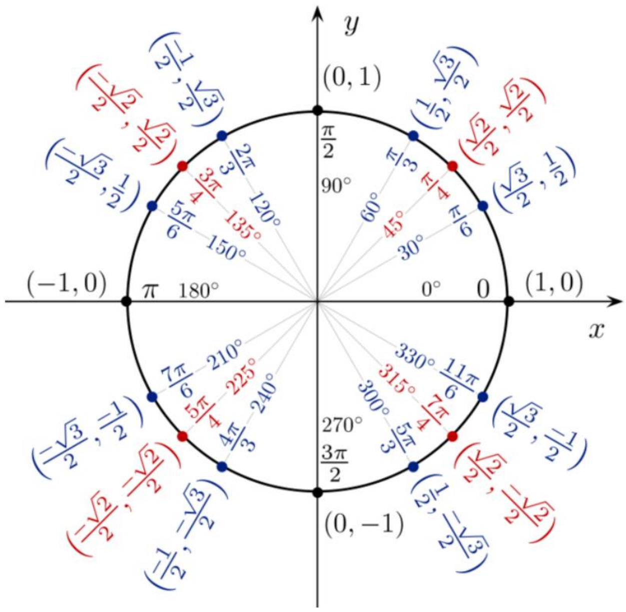


Figure 2.5 Trigonometric Functions with Radian Measures

The values of trigonometric functions with radian measures on x-axis and y-axis are listed in the following table.

θ	$0°$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$0°$	$-\frac{\pi}{2}$	$-\pi$	$-\frac{3\pi}{2}$	-2π
cos	1	0	-1	0	1	1	0	-1	0	1
sin	0	1	0	-1	0	0	-1	0	1	0
tan	0	undefined	0	undefined	0	0	undefined	0	undefined	0

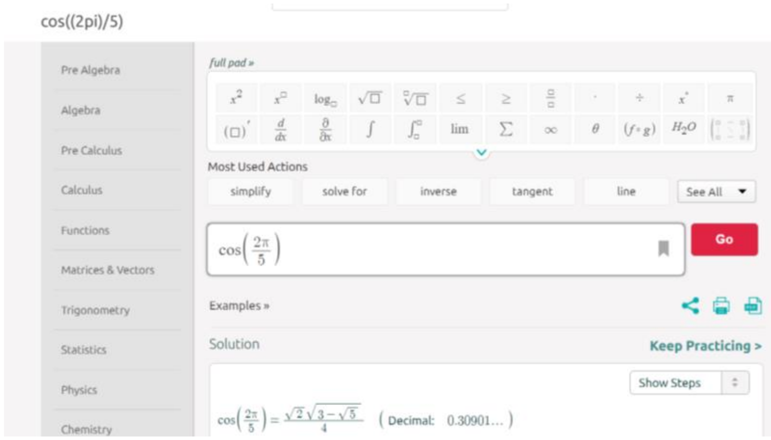
The trigonometric function values can be calculated by using online calculators such as “Symbolab Math Solver”, “Microsoft Math Solver”, and etc.

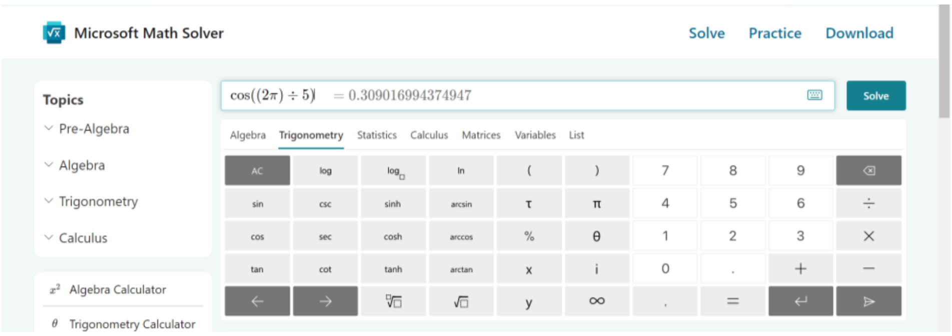
Example 2.3 Find each of the following function values of radian measures using a calculator, correct to 4 decimal places.

(a) $\cos\frac{2\pi}{5}$ (b) $\sin\frac{\pi}{7}$ (c) $\tan(-3)$ (d) $\sec(2.5)$

Solution:

(a)





The solution from the above calculators show that $\cos\frac{2\pi}{5} = 0.3090$ #

(b) $\sin\frac{\pi}{7} = 0.4339$ #

(c) $\tan(-3) = 0.1425$ #

(d) $\sec(2.5) = -1.2482$ #

Practice 2.1

1. Using a calculator to find the value of each trigonometric function (3 decimal points) or state that it is undefined.

1) $\sin\frac{\pi}{3}$ 2) $\sin\left(-\frac{\pi}{3}\right)$ 3) $\tan\frac{3\pi}{2}$

4) $\cos\frac{\pi}{6}$ 5) $\cos\left(-\frac{\pi}{6}\right)$ 6) $\cot\frac{2\pi}{6}$

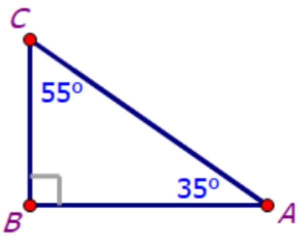
7) $\tan(-5\pi)$ 8) $\cot 5\pi$ 9) $\csc\frac{5\pi}{6}$

10) $\sec\left(-\frac{5\pi}{6}\right)$ 11) $\sin\frac{7\pi}{3}$ 12) $\cos\frac{11\pi}{6}$

Cofunctions Identities

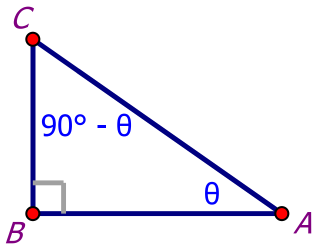
In a right triangle, the acute angles are ***complementary*** since the sum of all three angle measures is $180°$ and the right-angle measure is $90°$. Therefore, if one acute angle of a right triangle is θ, the other is $90° - \theta$.

The six trigonometric functions of each of acute angle in the right triangle below are listed in the table.



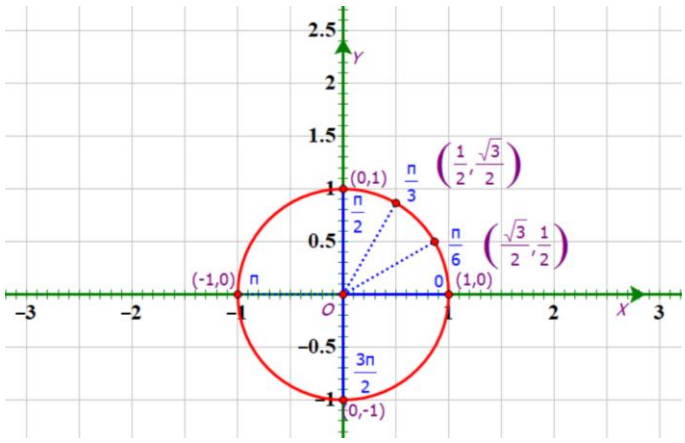
$\boldsymbol{\theta}$	$\sin\boldsymbol{\theta}$	$\cos\boldsymbol{\theta}$	$\tan\boldsymbol{\theta}$	$\csc\boldsymbol{\theta}$	$\sec\boldsymbol{\theta}$	$\cot\boldsymbol{\theta}$
$35°$	0.5736	0.1892	0.7002	1.7434	1.2208	1.4281
$55°$	0.8192	0.5736	1.4281	1.2208	1.7434	0.7002

In the above table, note that the sine of an angle is also the cosine of the complement. Similarly, the tangent of an angle is the cotangent of the complement, and the cosecant of an angle is the secant of the complement. These pairs of functions are called ***cofunctions***. A list of ***cofunction identities*** in terms of degrees and radians are as follows:



$\sin\theta = \cos(90° - \theta)$	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
$\cos\theta = \sin(90° - \theta)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot(90° - \theta)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$
$\cot\theta = \tan(90° - \theta)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\csc\theta = \sec(90° - \theta)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc(90° - \theta)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$

Example 2.4 Find the value of x, if $\sin x = \cos\frac{\pi}{6}$.



Solution:

Method 1 Using cofunction definition:

Given $\sin x = \cos\frac{\pi}{6}$.

It means that x and $\frac{\pi}{6}$ are complementary angles

Then, $x + \frac{\pi}{6} = \frac{\pi}{2}$

Solve for x, thus $x = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

Therefore, the value of angle x is $\frac{\pi}{3}$ #

Method 2 Using cofunction identity, $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

Thus, $\cos\frac{\pi}{6} = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sin\frac{\pi}{3}$

So, $\sin x = \sin\frac{\pi}{3}$

Therefore, the value of angle x is $\frac{\pi}{3}$ #

From the above figure, note that the value of $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ is the same as $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Example 2.5 Find the value of angle x, if $\sin x = \cos 35°$.

Solution:

Method 1 Using the meaning of cofunction:

Given $\sin x = \cos 35°$

It means that x and $35°$ are complementary angles

Then, $x + 35° = 90°$

Solve for x, thus $x = 90 - 35 = 55°$

Therefore, the value of angle x is $55°$. #

Method 2 Using cofunction identity, $\cos\theta = \sin(90° - \theta)$

Thus, $\cos 35° = \sin(90° - 35°) = \sin 55°$

So, $\sin x = \sin 55°$

Therefore, the value of angle x is $55°$. #

Note that the values of $\cos 35° \approx 0.819$ and $\sin 55° \approx 0.819$ on a calculator are equal.

Example 2.6 Given that $\sin 20° \approx 0.3420$, $\cos 20° \approx 0.9397$, and $\tan 20° \approx 0.3639$, find the six trigonometric functions value of $70°$.

Solution: Using reciprocal relations, we know that

$$\csc 20^\circ = \frac{1}{\sin 20^\circ} \approx 2.9239$$

$$\sec 20^\circ = \frac{1}{\cos 20^\circ} \approx 1.0642$$

and $$\cot 20^\circ = \frac{1}{\tan 20^\circ} \approx 2.7475$$

Since 70° and 20° are complementary, therefore

$\sin 70^\circ = \cos 20^\circ \approx 0.9397,\quad \cos 70^\circ = \sin 20^\circ \approx 0.3420$

$\tan 70^\circ = \cot 20^\circ \approx 2.7475,\quad \cot 70^\circ = \tan 20^\circ \approx 0.3639$

$\csc 70^\circ = \sec 20^\circ \approx 1.0642,\quad \sec 70^\circ = \csc 20^\circ \approx 2.9239$ #

Practice 2.2

1. Find $m\angle A$ from the given cofunction.

1) $\sin 6A = \cos 9A$ 2) $\tan \frac{7\pi}{6} = \cot A$ 3) $\csc A = \sec \frac{5\pi}{6}$

2. Find the value of each expression.

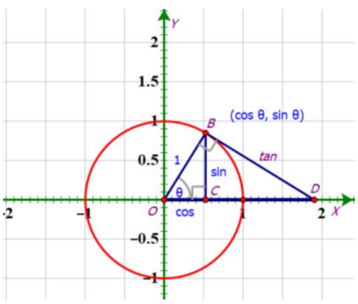
1) If $\cos\theta = \frac{1}{2}$, find $\sin\left(\frac{\pi}{2} - \theta\right)$.

2) If $\tan\theta = 1$, find $\cot\left(\frac{\pi}{2} - \theta\right)$.

3) If $\sec\theta = \frac{2}{\sqrt{3}}$, find $\csc\left(\frac{\pi}{2} - \theta\right)$.

Inverse Trigonometric Functions

The trigonometric functions $\sin x$, $\cos x$, and $\tan x$ can be used to find an unknown side length of a right triangle, if one side length and an angle measure are known.

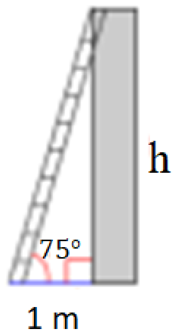


In $\Delta\, BOC$, if one side length and an angle measure are known, the trigonometric functions or ratios can be found by using the following properties.

$\sin\theta = \frac{BC}{OB}$ $\qquad\cos\theta = \frac{OC}{OB}$ $\qquad\tan\theta = \frac{BC}{OC}$

The application of trigonometric ratios can be used to solve the following example.

Example 2.7 The base of a ladder is placed 1 meter away from the wall so that the top of the ladder hits the top of the wall. If the measure of the angle formed by the ladder at the ground is $75°$. What is the height of the wall?



Solution: From the right triangle, $\tan 75° = \frac{h}{1}$

Using a calculator, $\tan 75° \approx 3.73$

Thus, $h \approx 3.73$ m

Therefore, the height of the wall is about 3.73 meters #

In Figure 2.11, the length of arc BE on the unit circle is θ and it is equal to the measure of $\angle EOB$ (θ radians) at the center of the unit circle. Note that the coordinate of point B is (x, y) which is the same as $(\cos\theta, \sin\theta)$. This means

$$x = \cos\theta \text{ and } y = \sin\theta.$$

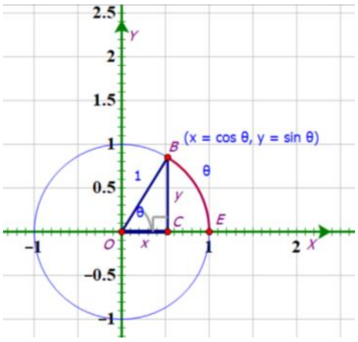
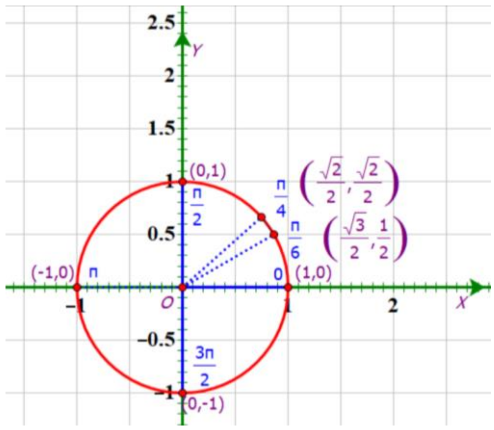


Figure 2.11 Trigonometric functions on unit circle

If the two side lengths of a right triangle or x and y are known, the following inverse trigonometric functions can be used to find the unknown measure of the angle as shown in the following figure.



- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then $arcsine\ \left(\frac{1}{2}\right) = \frac{\pi}{6}$.
- Since $\cos(\pi) = -1$, then $arccosine\ (-1) = \pi$.
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then $arctangent\ (1) = \frac{\pi}{4}$.

Finding Angles by Using Inverse Trigonometric Functions

Inverse of sine function is used evaluate the angle whose sine value is equal to the ratio of its opposite side and hypotenuse. For example, if $f(x) = \sin x$, then the inverse of sine function is written as $f^{-1}(x) = \sin^{-1} x$. The inverse of $\sin x$ is called '$\boldsymbol{arcsin\ x}$' denoted by '$\mathbf{sin^{-1}}\boldsymbol{x}$'(read as sine inverse *x*). Be aware that $\sin^{-1} x \neq \frac{1}{\sin x}$.

Example 2.8 Given $f(x) = \sin x$ and $\sin x = \frac{1}{2}$, find the value of angle *x*.

Solution: For $\sin x = \frac{1}{2}$, using a calculator to evaluate $\sin^{-1}\left(\frac{1}{2}\right)$.

Since $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}\ or\ 30°$

Therefore, $x = \frac{\pi}{6} \approx 0.5236$ radians or $30°$. #

Inverse of cosine function is used evaluate the angle whose cosine value is equal to the ratio of its adjacent side and hypotenuse. For example, if $f(x) = \cos x$, then the inverse of cosine function is written as $f^{-1}(x) = \cos^{-1} x$. The inverse of $\cos x$ is called '$\boldsymbol{arccos\, x}$' denoted by '$\mathbf{cos^{-1}}\, \boldsymbol{x}$'(read as cosine inverse *x*). Again, be aware that $\cos^{-1} x \neq \frac{1}{\cos x}$.

Example 2.9 Given $f(x) = \cos x$ and $\cos x = \frac{1}{2}$, find the value of angle *x*.

Solution: For $\cos x = \frac{1}{2}$, using a calculator to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$.

Since $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}\ or\ 60°$

Therefore, $x = \frac{\pi}{3} \approx 1.0472$ radians or $60°$. #

Inverse of tangent function is used evaluate the angle whose tangent value is equal to the ratio of its opposite side and adjacent side. For example, if $f(x) = \tan x$, then the inverse of tangent function is written as $f^{-1}(x) = \tan^{-1} x$. The inverse of $\tan x$ is called '$\boldsymbol{arctan\, x}$' denoted by '$\mathbf{tan^{-1}}\, \boldsymbol{x}$'(read as tangent inverse *x*). Also, be aware that $\tan^{-1} x \neq \frac{1}{\tan x}$.

Example 2.10 Given $f(x) = \tan x$ and $\tan x = 1$, find the value of angle *x*.

Solution: For $\tan x = 1$, using a calculator to evaluate $\tan^{-1}(1)$.

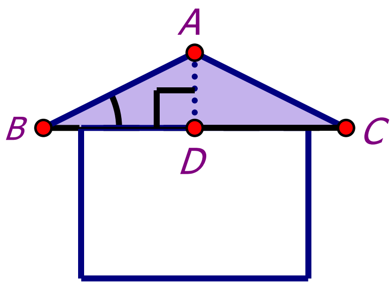
Since $\tan^{-1}(1) = \frac{\pi}{4}$ *or* $45°$

Therefore, $x = \frac{\pi}{4} \approx 0.7854$ radians or $45°$. #

Applications of Inverse Trigonometric Functions

The inverse trigonometric functions are found in applications where the measure of an angle is required as demonstrated in the following example.

Example 2.11 The roof on the house has a rise of 2 meters over the bottom line 8 meters. Find the angle of inclination from the bottom to the top of the roof. Express the answer in decimal degrees, rounded to two decimal places.



Solution: $\Delta\, ABC$ represents the roof of the house and $\overline{AD} \perp \overline{BC}$.

$AD = 2\ m$ and $BC = 8\ m$

Let $m\angle ABD = \theta°$

In $\Delta\, ABD$, $\angle ADB = 90°, AD = 2\ m,$ and $BD = \frac{1}{2} \times 8 = 4\ m$

Thus, $\tan\theta = \frac{AD}{BD} = \frac{2}{4} = \frac{1}{2}$

So, $\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.57°$

Therefore, the angle of inclination from the bottom to the top of the roof is $26.57°$ #

Practice 2.3

1. Find the values of the following inverse trigonometric functions.

1) $\sin^{-1}(-1)$ 2) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ 3) $\cos^{-1}\left(-\frac{1}{2}\right)$ 4) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

5) $\tan^{-1}(\sqrt{3})$ 6) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ 7) $\cot^{-1}(-\sqrt{3})$ 8) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

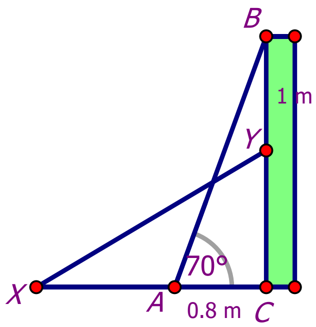
2. Using a calculator to approximate the following value to four decimal places.

1) $\sec^{-1}(1)$ 2) $\sec^{-1}(-\sqrt{2})$ 3) $\csc^{-1}(-2)$ 4) $\csc^{-1}(-\sqrt{2})$

3. A ladder *AB* rests against a vertical building *BC*, making an angle of $70°$ with the horizontal ground. Its lower end *A* is 0.8 m away from the foot *C* of the building.

(1) Find the length of the ladder.

(2) When move the ladder *B* down 1 m to *Y*, the new position of the ladder is *XY* as shown in the diagram. Find $\angle CXY$.



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