Chapter 5 Trigonometric Identities

Trigonometric identities are the equalities involving trigonometric functions and are true for every value of the variables involved. They are used for solving numerous mathematics problems and make problems easy to solve.

Reciprocal Trigonometric Identities

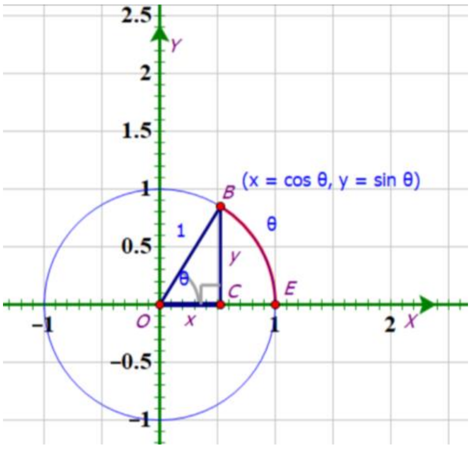


Figure 5.1 Right triangle in the unit circle

In Figure 5.1, ΔOCB is a right triangle with the measure of $\angle OCB$ is 90° and $\angle BOC$ is θ. Let (x, y) be the coordinate of the point B on the unit circle, then $OC = x$ (adjacent side), $BC = y$ (opposite side), and $OB = 1$ (hypothenuse). By the definition of six trigonometry functions, the relation of the sides of ΔOCB are as follows:

• $\sin\theta = \frac{BC}{OB} = \frac{y}{1} = y$	• $\csc\theta = \frac{OB}{BC} = \frac{1}{y}$
• $\cos\theta = \frac{OC}{OB} = \frac{x}{1} = x$	• $\sec\theta = \frac{OB}{OC} = \frac{1}{x}$
• $\tan\theta = \frac{BC}{OC} = \frac{y}{x}$	• $\cot\theta = \frac{OC}{BC} = \frac{x}{y}$

The reciprocal identities are given as follows:

• $\sin\theta = \frac{1}{\csc\theta}$	• $\csc\theta = \frac{1}{\sin\theta}$, provided $\sin\theta \neq 0$, if $\sin\theta = 0$ then $\csc\theta$ is undefined.
• $\cos\theta = \frac{1}{\sec\theta}$	• $\sec\theta = \frac{1}{\cos\theta}$, provided $\sin\theta \neq 0$, if $\sin\theta = 0$ then $\csc\theta$ is undefined.
• $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$, provided $\cos\theta \neq 0$, if $\cos\theta = 0$ then $\tan\theta$ is undefined.	• $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$, provided $\sin\theta \neq 0$, if $\sin\theta = 0$ then $\cot\theta$ is undefined.

Pythagorean Trigonometric Identities

The Pythagorean trigonometric identities are derived from the Pythagorean theorem.

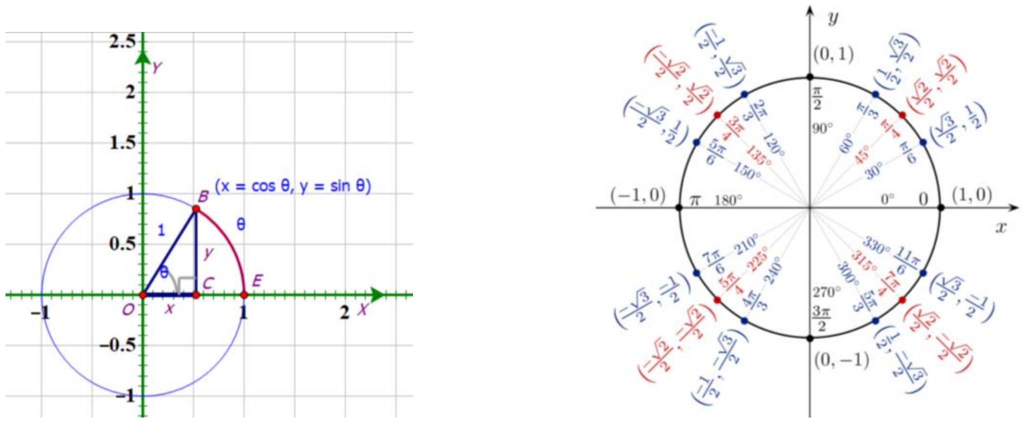


Figure 5.2 Pythagorean theorem in the unit circle

In Figure 5.2, let (x, y) be the point on the unit circle with center (0, 0) that determines the angle θ radians. Describe a right triangle with hypothenuse 1 and side of the length *x* and *y*, it will get $x^2 + y^2 = 1$ (Pythagorean Theorem). Since, the trigonometric ratios of the right triangle are $x = \cos\theta$ and $y = \sin\theta$. Replacing *x* and *y* by $\cos\theta$ and $\sin\theta$ respectively, this is the first of Pythagorean trigonometric identities as follows:

- $\cos^2\theta + \sin^2\theta = 1$ or $\sin^2\theta + \cos^2\theta = 1$

Common alternate forms:

$$1 - \sin^2\theta = \cos^2\theta \quad \text{and} \quad 1 - \cos^2\theta = \sin^2\theta$$

From this Pythagorean trigonometric identity, we can derive two other Pythagorean identities as the following:

- $1 + \tan^2\theta = \sec^2\theta$

Common alternate forms:

$$\sec^2\theta - \tan^2\theta = 1 \quad \text{and} \quad \sec^2\theta - 1 = \tan^2\theta$$

- $1 + \cot^2\theta = \csc^2\theta$

Common alternate forms:

- $\csc^2\theta - \cot^2\theta = 1$ and $\csc^2\theta - 1 = \cot^2\theta$

The identity $1 + \tan^2\theta = \sec^2\theta$ can be proved as follows:

Proof:

Method 1:

Since $\sin^2\theta + \cos^2\theta = 1$ (Pythagorean identity)

Divide both sides by $\cos^2\theta$, it follows that:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \#$$

Method 2:

L.H.S $= 1 + \tan^2\theta$

$= 1 + \frac{\sin^2\theta}{\cos^2\theta}$ (Definition of $\tan\theta$)

$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$ (Addition fractions)

$= \frac{1}{\cos^2\theta}$ (Pythagorean identity)

$= sec^2\theta$ (Reciprocal identity)

$=$ R.H.S #

The proof of $1 + \cot^2\theta = \csc^2\theta$ is left to the reader (See Practice 5.1).

Example 5.1 Prove that $(1 - sin^2\theta)sec^2\theta = 1$

Proof:

LHS = $(1 - sin^2\theta)sec^2\theta$

= $(cos^2\theta)\ sec^2\theta$ (Pythagorean identity: $cos^2\theta = 1 - sin^2\theta$)

= $\frac{1}{sec^2\theta} \times sec^2\theta$ (Reciprocal: $cos^2\theta = \frac{1}{sec^2\theta}$)

= 1 = RHS #

Example 5.2 Prove that $tan^4A + tan^2A = sec^4A - sec^2A$

Proof:

LHS = $tan^4A + tan^2A$

= $tan^2A(tan^2A + 1)$ (Common factor)

= $(sec^2A - 1)(sec^2A)$ (Pythagorean identity)

= $sec^4A - sec^2A$ (Distributive Law)

= RHS #

Example 5.3 Use Trigonometric Identities to write the expressions in terms of a single trigonometric identity or a constant.

(1) $\cot\theta \sin\theta$ (2) $\frac{\cos\theta \csc\theta}{\cot\theta}$

Solution:

(1) $\cot\theta \sin\theta = \frac{\cos\theta}{\sin\theta} \times \sin\theta = \cos\theta$ #

(2) $\frac{\cos\theta \csc\theta}{\cot\theta} = \frac{\cos\theta \times \frac{1}{\sin\theta}}{\cot\theta} = \frac{\cot\theta}{\cot\theta} = 1$ #

Complementary Trigonometric Identities

In a right triangle, the acute angles are ***complementary*** such that their sum is equal to $90°$. If one acute angle of a right triangle is θ, the other angle is $90° - \theta$.

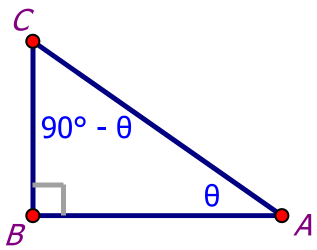


Figure 5.3 Complementary angles

In Figure 5.3, $\Delta\, ABC$ is a right triangle with the measure of $\angle ABC = 90°$. Since the sum of interior angles of a triangle is $180°$, then the sum of $\angle\, BAC$ and $\angle\, BCA$ is equal to $90°$. Let the measure of $\angle\, BAC = \theta$, then the measure of $\angle\, BCA = 90° - \theta$.

The trigonometric identities of complementary angles are:

• $\sin\,(90° - \theta) = \frac{AB}{AC} = \cos\theta$	• $\cos\,(90° - \theta) = \frac{BC}{AC} = \sin\theta$
• $\csc\,(90° - \theta) = \frac{AC}{AB} = \sec\theta$	• $\sec\,(90° - \theta) = \frac{AC}{BC} = \csc\theta$
• $\tan\,(90° - \theta) = \frac{AB}{BC} = \cot\theta$	• $\cot\,(90° - \theta) = \frac{BC}{AB} = \tan\theta$

Note that the complementary trigonometric identities are also called ***cofunction identities*** and the measure of angles can be the radians.

• $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	• $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
• $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$	• $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$
• $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$	• $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

Supplementary Trigonometric Identities

The supplementary angles are a pair of two angles such that their sum is equal to 180° or π radians. The ***supplement*** of an angle θ is $(180° - \theta)$ or $(\pi - \theta)$.

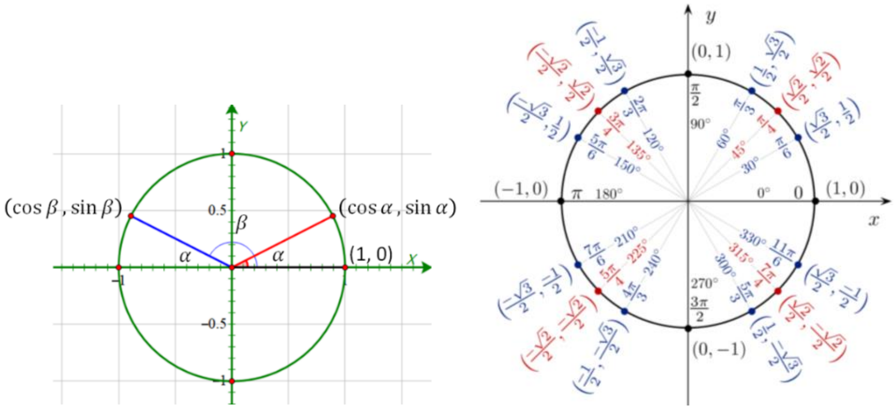


Figure 5.4 Supplementary angles on unit circle

In Figure 5.4, if α and β are two supplementary angles the following supplementary trigonometric identities are true.

• $\sin(\alpha) = \sin(\beta)$	• $\cos(\alpha) = -\cos(\beta)$	• $\tan(\alpha) = -\tan(\beta)$

The other supplementary trigonometric identities can be determined by using geometric transformations on the unit circle. For example, if the angle θ reflects in the y-axis, the coordinate of the point in Quadrant I will be changed in Quadrant II for the value of $\cos(\theta)$ from $\cos(\theta)$ to $-\cos(\theta)$, but the value of $\sin(\theta)$ remains the same (see Figure 5.5).

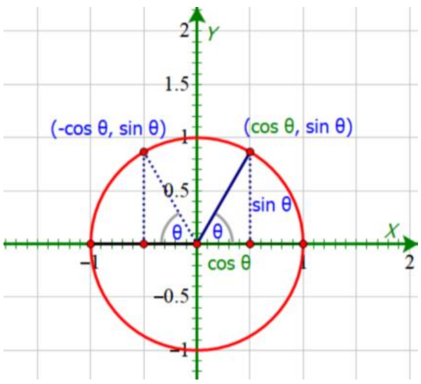


Figure 5.5 Reflection of angle θ

Determine the coordinate of the point $(-\cos(\theta), \sin(\theta))$ is the terminal side of angle $180° - \theta$ in standard position (see Figure 5.6).

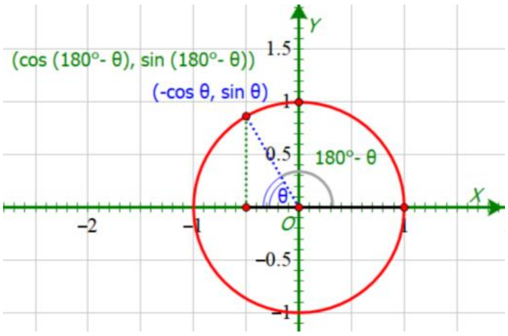


Figure 5.6 Angle $180° - \theta$ in standard position

Then by using the definition of sine and cosine of an angle in standard position, the following supplementary trigonometric identities are true for any angle θ measured in degrees and in radians:

- $\sin(180° - \theta) = \sin(\pi - \theta) = \sin\theta$
- $\cos(180° - \theta) = \cos(\pi - \theta) = -\cos\theta$
- $\csc(180° - \theta) = \csc(\pi - \theta) = \csc\theta$
- $\sec(180° - \theta) = \sec(\pi - \theta) = -\sec\theta$
- $\tan(180° - \theta) = \tan(\pi - \theta) = -\tan\theta$
- $\cot(180° - \theta) = \cot(\pi - \theta) = -\cot\theta$

Example 5.4 Evaluate $\csc\left(\frac{5\pi}{6}\right)$ by using the cofunction identities.

Solution: $\csc\left(\frac{5\pi}{6}\right) = \frac{1}{\sin\left(\frac{5\pi}{6}\right)} = \frac{1}{\sin\left(\frac{\pi}{2}+\frac{\pi}{3}\right)} = \frac{1}{\sin\left(\frac{\pi}{2}-\left(-\frac{\pi}{3}\right)\right)}$

$$= \frac{1}{\cos\left(-\frac{\pi}{3}\right)} = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{1}{2}} = 2$$

Therefore, $\csc\left(\frac{5\pi}{6}\right) = 2$ #

Example 5.5 Find the value of $\sin(135°)$.

Solution: $\sin(135°) = \sin(180° - 45°)$

$$= \sin(45°)$$

$$= \frac{\sqrt{2}}{2} \qquad \#$$

Practice 5.1

1. Prove that $\frac{1-sin^2\theta}{cot^2\theta} = sin^2\theta$.
2. Prove that $\frac{\tan\theta+\cot\theta}{\tan\theta} = csc^2\theta$.
3. Use Trigonometric Identities to write each expression in terms of a single trigonometric identity or a constant.

   (1) $\sin\theta\sec\theta$ (2) $\frac{\sec\theta}{\tan\theta}$ (3) $\frac{\sin\theta\csc\theta}{\cot\theta}$
4. Evaluate $\sec\left(\frac{3\pi}{4}\right)$ by using the cofunction identities.
5. Find the value of $\cos(120°)$.

Angle Addition and Subtraction Trigonometric Identities

The angle addition and subtraction trigonometric identities are useful to find the trigonometric functions at some angles where it is expressed the angle as the sum or difference of unique angles ($0°, 30°, 45°, 60°, 90°, and\ 180°$). In this section, the proof of the angle addition and subtraction identities for sine, cosine, and tangent will be used the techniques in analytic geometry. The list of sum and difference identities is as follows:

Angle Addition and Subtraction Identities for Cosine

- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Angle Addition and Subtraction Identities for Sine

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Angle Addition and Subtraction Identities for Tangent

- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$
- $\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

Proof of Angle Addition and Subtraction Identities for Cosine

Consider a unit circle in a coordinate plane. In the unit circle given below, point P makes an angle α with the positive x-axis and has coordinate $(\cos\alpha,\ \sin\alpha)$. Given point Q makes an angle β with the positive x-axis and has coordinate $(\cos\beta,\ \sin\beta)$.

First, Proof of $\cos(\alpha-\beta)$ by describing the measure of $\angle POQ$ which is equal to $\alpha-\beta$.

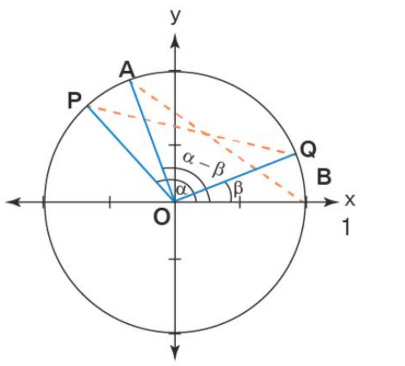


Figure 5.7 Measuring of angle $\alpha-\beta$

In Figure 5.7, label two points A and B on the unit circle such that the coordinate of point B is (1, 0) on the x-axis and the coordinate of point A is $(\cos(\alpha-\beta), \sin(\alpha-\beta))$. Draw $\overline{PQ}$ and $\overline{AB}$, note that $\Delta\ POQ$ and $\Delta\ AOB$ are rotations of one another and $PQ = AB$. Using the distance formula in analytic geometry to find PQ and AB as follows:

$$PQ = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \text{ .... (eq. 1)}$$

$$AB = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2} \text{ ..... (eq. 2)}$$

$$= \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta))^2} \text{ ..}$$

Since, $PQ = AB$ and $PQ^2 = AB^2$

$PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$

$= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$

$= 1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$

$= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$ …………………. (eq.3)

$AB^2 = (\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta))^2$

$= \cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)$

$= \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1$

$= 1 + 1 - 2\cos(\alpha - \beta)$

$= 2 - 2\cos(\alpha - \beta)$ ……………………………. (eq. 4)

$\text{eq.}\,3 = \text{eq.}\,4;\ 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$

Using algebraic properties, $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Therefore, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ #

Next, the proof of $\cos(\alpha + \beta)$ can derive from the difference identity of cosine by replacing $\beta = -(-\beta)$ as follows:

$\cos(\alpha + \beta) = \cos\big(\alpha - (-\beta)\big) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$

Since, $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$

Therefore, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ #

Proof of Angle Addition and Subtraction Identities for Sine

The proof of the angle addition identity for sine will apply the cofunction identities of trigonometry: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ *and* $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

Proof Angle Addition Identity for Sine:

$\sin(\alpha + \beta) = \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right)$

$$= \cos\left(\frac{\pi}{2} - \alpha - \beta\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right)$$

$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Therefore, $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ #

Proof Angle Subtraction Identity for Sine:

The proof of $\sin(\alpha-\beta)$ can derive from the angle addition identity of sine by writing $\alpha - \beta = \alpha + (-\beta)$ as follows:

$\sin(\alpha-\beta) = \sin(\alpha + (-\beta))$

$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$

Since, $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$

Therefore, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ #

Proof of Angle Addition and Subtraction Identities for Tangent

The proof of angle addition and subtraction identities for tangent can be written as the ratio of sine and cosine, that is, $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Proof Angle Addition Identity for Tangent:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

Dividing the numerator and denominator by $\cos\alpha\cos\beta$,

$$\tan(\alpha+\beta) = \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Therefore, $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$ #

Proof Angle Subtraction Identity for Tangent:

The proof of $\tan(\alpha-\beta)$ can derive from the angle addition identity of tangent by writing $\alpha - \beta = \alpha + (-\beta)$ as follows:

Proof:

$$\tan(\alpha-\beta)=\tan\big(\alpha+(-\beta)\big)=\frac{\tan(\alpha)+\tan(-\beta)}{1-\tan(\alpha)\tan(-\beta)}$$

Since, $\tan(-\beta)=-\tan\beta$

Therefore, $\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$ #

Example 5.6 Prove the following complementary and supplementary trigonometric identities by using sum and difference trigonometric identities.

(a) $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ (b) $\cos(180°-\theta)=-\cos\theta$

(c) $\tan(2\pi-\theta)=-\tan\theta$

Proof

(a) $\sin\left(\frac{\pi}{2}-\theta\right)=\sin\left(\frac{\pi}{2}\right)\cos\theta-\cos\left(\frac{\pi}{2}\right)\sin\theta$

$=(1)(\cos\theta)-(0)(\sin\theta)$

$=\cos\theta-0$

$=\cos\theta$ #

(b) $\cos(180°-\theta)=\cos(180°)\cos\theta+\sin(180°)\sin\theta$

$=(-1)\cos\theta+(0)\sin\theta$

$=-\cos\theta+0$

$=-\cos\theta$ #

(c) $\tan(2\pi-\theta)=\frac{\tan(2\pi)-\tan\theta}{1+\tan(2\pi)\tan\theta}$

$$=\frac{\frac{\sin(2\pi)}{\cos(2\pi)}-\tan\theta}{1+\frac{\sin(2\pi)}{\cos(2\pi)}\tan\theta}$$

$$=\frac{0-\tan\theta}{1+(0)\tan\theta}=\frac{-\tan\theta}{1+0}$$

$=-\tan\theta$ #

Double-Angle and Half-Angle Identities

Double-angle and half-angle identities are special case of the angle addition and subtraction identities for sine and cosine as shown in the following proof.

Double-Angle Identities of Sine

- $\sin 2A = 2\sin A\cos A$

Proof: $\sin 2A = \sin(A + A)$

$$= \sin A\cos A + \cos A\sin A$$

$$= \sin A\cos A + \sin A\cos A$$

$$= 2\sin A\cos A \qquad \#$$

Double-Angle Identities of Cosine

- $\cos 2A = \cos^2 A - \sin^2 A$

Proof: $\cos 2A = \cos(A + A)$

$$= \cos A\cos A - \sin A\sin A$$

$$= \cos^2 A - \sin^2 A \qquad \#$$

Note: There are ***common alternate forms*** for double-angle identities of cosine as follows:

- $\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$

  $= 2\cos^2 A - 1 \qquad \#$
- $\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A \qquad \#$

Double-Angle Identities of Tangent

- $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

Proof: $\tan 2A = \tan(A + A)$

$$= \frac{\tan A+\tan A}{1-\tan A\tan A}$$

$$= \frac{2\tan A}{1-\tan^2 A} \qquad \#$$

Half-Angle Identities of Sine

- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1-\cos\theta)}$

Proof: Since $\cos 2A = 1 - 2\sin^2 A$

Then, $2\sin^2 A = 1 - \cos 2A$ …………………(eq. 1)

Let $2A = \theta$, it follows that $A = \frac{\theta}{2}$

Replace $A = \frac{\theta}{2}$ in equation (1)

Then, $2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos\theta$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

Therefore, $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1-\cos\theta)}$ #

Half-Angle Identities of Cosine

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1+\cos\theta)}$

Proof: Since, $\cos 2A = 2\cos^2 A - 1$

Then, $2\cos^2 A = 1 + \cos 2A$ ………………(eq. 2)

Let $2A = \theta$, it follows that $A = \frac{\theta}{2}$

Replace $A = \frac{\theta}{2}$ in equation (2)

Then, $2\cos^2\left(\frac{\theta}{2}\right) = 1 + \cos\theta$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$$

Therefore, $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1+\cos\theta)}$ #

Half-Angle Identities of Tangent

- $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}$

Proof: The proof is left to the reader (See Practice 5.2).

Product Identities

The products of sine and cosine functions can be expressed as sums by using the angle addition and subtraction identities.

Product to Sum Identities

- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

Proof: Since, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ ……...(eq.1)

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ ………(eq.2)

(eq.1) + (eq.2); It follows that

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

Therefore, $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ #

Note: $\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$

Likewise, we can find:

- $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

Proof: The proof is left to the reader (See Practice 5.2).

- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

Proof: The proof is left to the reader (See Practice 5.2).

- $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

Proof: The proof is left to the reader (See Practice 5.2).

Sum to Product Identities

- $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
- $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Example 5.7 Write the following difference of sines expression as a product: $\sin(4\theta) - \sin(2\theta)$.

Solution: Formula $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

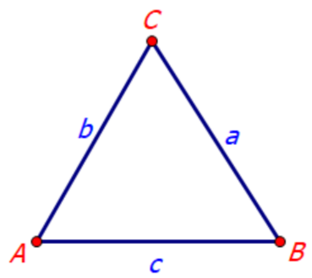
Substitute the values into the formula:

$$\sin(4\theta) - \sin(2\theta) = 2\cos\left(\frac{4\theta+2\theta}{2}\right)\sin\left(\frac{4\theta-2\theta}{2}\right)$$

$$= 2\cos(3\theta)\sin(\theta) \qquad \#$$

Triangle Identities (Sine, Cosine, Tangent Rules)

For any triangle *ABC*, if *A*, *B* and *C* are vertices of a triangle and a, b, and c are the respective sides, then ***sine rule***, ***cosine rule***, and ***tangent rule*** are derived as follows:

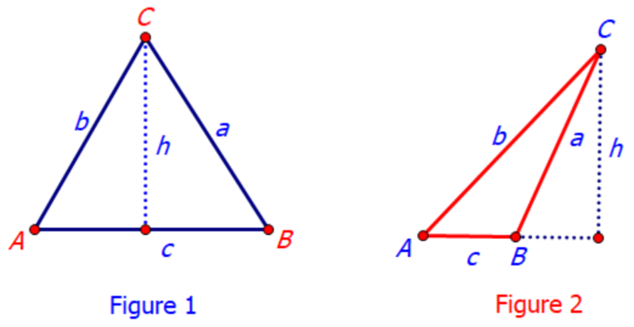


Sine Rule or Law of Sines

The Sine Rule (also known as the Law of Sines) is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof: Draw an altitude h to the side of length c as shown in Figure 1 and Figure 2



In each of Figure 1 and Figure 2, $\sin A = \frac{h}{b}$ ....(1)

In Figure 1, $\sin B = \frac{h}{a}$

In Figure 2, $\sin(\pi - B) = \frac{h}{a}$

So in either case $\sin B = \frac{h}{a}$ .... (2)

Solving for h in both equations yields $h = b \sin A$ and $h = a \sin B$

It follows that, $b \sin A = a \sin B$

Therefore, $\frac{\sin A}{a} = \frac{\sin B}{b}$ .... (3)

If we draw an altitude h to the side of length a and repeat the same steps as above, we would reach the conclusion that

$\frac{\sin B}{b} = \frac{\sin C}{c}$ .... (4)

From (3) and (4), it follows that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ #

Cosine Rule or Law of Cosines

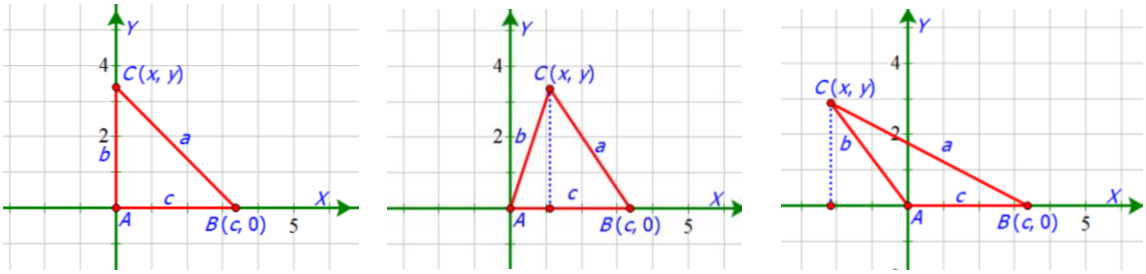
The Cosine Rule (also known as the Law of Cosines) is an extension of the *Pythagorean Theorem* to any triangle:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Or

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Proof:



(a) (b) (c)

In each of these cases, point C is on the terminal side of angle A with distance b from the origin A. The coordinate of C is (x, y). By the definition for trigonometric functions:

$$\frac{x}{b} = \cos A \quad \text{and} \quad \frac{y}{b} = \sin A$$

and therefore

$$x = b\cos A \quad \text{and} \quad y = b\sin A.$$

Since, a is the distance from C to B using the distance formula:

$a = \sqrt{(x-c)^2 + (y-0)^2}$ (Distance formula)

$a^2 = (x-c)^2 + y^2$ (Square both sides)

$= \left((b\cos A) - c\right)^2 + (b\sin A)^2$ (Substitute x and y)

$= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$

$= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A$

$= b^2 + c^2 - 2bc\cos A$ (Pythagorean identity) #

Tangent Rule or Law of Tangents

The Tangent Rule (also known as the Law of Tangents) is:

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{1}{2}(A+B)\right)}{\tan\left(\frac{1}{2}(A-B)\right)}$$

Proof: The proof is left to the reader (See Practice 5.2).

Practice 5.2

1. Prove the following identities:

   (a) $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}$

   (b) $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

   (c) $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

   (d) $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

2. Let A, B and C are vertices of a triangle $\Delta\, ABC$ and a, b, and c are the respective sides. Prove the Law of Tangent

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{1}{2}(A+B)\right)}{\tan\left(\frac{1}{2}(A-B)\right)}$$

3. Write the following expression as a product:

   (a) $\sin(3\theta) + \sin(\theta)$ (b) $\cos(3x) + \cos(9x)$

4. Evaluate the following difference of cosine expression using the sum to product identities: $\cos(15°) - \cos(75°)$.

5. Write each the following expressions as the sine or cosine of an angle.

   (a) $\sin(22°)\cos(13°) + \cos(22°)\sin(30°)$

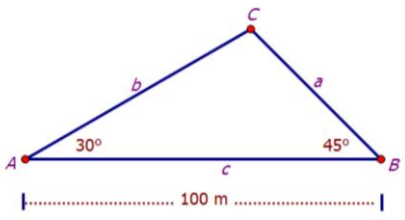
   (b) $\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

Applications of Triangle Identities

Solving Triangle Using the Law of Sines

Two angles and a side of a triangle, in any order, determine the size and shape of a triangle completely.

Example 5.8 ΔABC has two angles and a side with $\angle A = 30°, \angle B = 45°$, and the length of the opposite side of $\angle C$ is 100 m. Find the length of the opposite side of $\angle A$ and $\angle B$.



Solution: Let $m\angle A = 30°, m\angle B = 45°$ and $c = 100\ m$.

$$m\angle C = 180° - (30° + 45°) = 105°.$$

Using the Law of Sines as follows:

$$\frac{\sin A}{a} = \frac{\sin C}{c} \text{ and } \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 30°}{a} = \frac{\sin 105°}{100} \text{ and } \frac{\sin 45°}{b} = \frac{\sin 105°}{100}$$

$$a = \frac{100 \sin 30°}{\sin 105°} \text{ and } b = \frac{100 \sin 45°}{\sin 105°}$$

Since, $\sin 30° = 0.50, \sin 45° = 0.71$, and $\sin 105° = 0.97$.

Therefore, $$a = \frac{100(0.50)}{0.97} = 51.55\ m$$

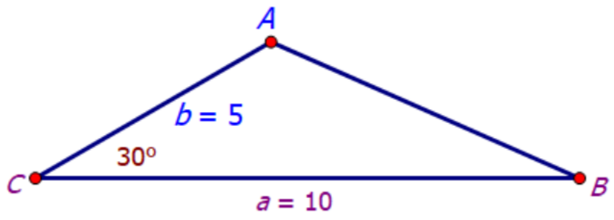
$$b = \frac{100(0.71)}{0.97} = 73.20\ m \qquad \#$$

Note: The Law of Sines is the tool for solving triangles in the AAS (Angle-Angle-Side) and ASA (Angle-Side-Angle) cases.

Solving Triangle Using the Law of Cosines

The Law of Cosines is the tool for solving triangles in the SAS and SSS cases.

Example 5.9 Solve ΔABC with $a = 10, b = 5$, and $\angle C = 30°$.



Solution: Let $a = 10, b = 5$, and $\angle C = 30°$.

Using the Law of Cosines to find the length of the opposite side of $\angle C$ as follows:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 10^2 + 5^2 - 2(10)(5)\cos 30°$$

$$= 100 + 25 - 100(0.87) = 38$$

$$c = \sqrt{38} \approx 6.16$$

Using the Law of Cosines to find one of the two unknown angles as follows:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$10^2 = 5^2 + 38 - 2(5)(6.16)\cos A$$

$$\cos A = \frac{25 + 38 - 100}{61.6} = -0.60$$

$$cos^{-1}(-0.60) = 126.87°$$

Then, $\angle A = 126.87°$

$$\angle B = 180° - 30° - 126.87° = 23.13°$$

So, the six parts of the triangle are:

$\angle A = 126.87°$, $a = 10$

$\angle B = 23.13°$, $b = 5$

$\angle C = 30°$, $c = 6.16$ #

Triangle Area

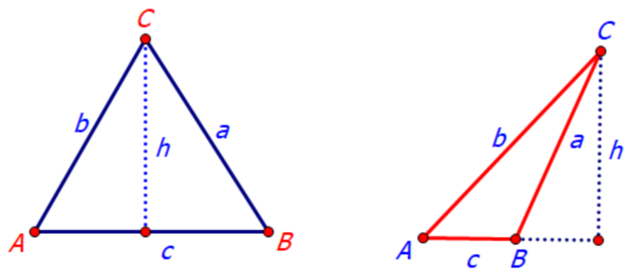


Figure 1 Figure 2

In Figure 1 and Figure 2, each triangle has base c and altitude h. Since, $\sin A = \frac{h}{b}$ $or\ h = b \sin A$, then applying the standard area formula:

$$Area\ \Delta = \frac{1}{2}(base)(height) = \frac{1}{2}(c)(b \sin A) = \frac{1}{2} bc \sin A$$

The following three formulas can be used for other bases:

$$Area\ \Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C.$$

Example 5.10 Find the area of a regular octagon (8 equal sides and 8 equal angles) inscribed inside a circle of radius 6 centimeters.

Solution: Split the regular octagon into 8 congruent triangles.

Each triangle has two 6-cm sides with an included angle of

$$\theta = \frac{360°}{8} = 45°.$$

The area of each triangle is

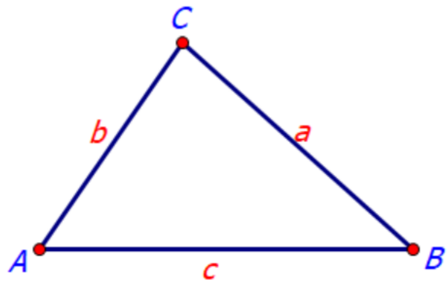
$$Area\ \Delta = \frac{1}{2}(6)(6) \sin 45° = 18\left(\frac{1}{\sqrt{2}}\right)$$

Therefore, the area of the octagon is

$$Area = 8\ \times \Delta\ Area = 8 \times 18\left(\frac{1}{\sqrt{2}}\right) \approx 101.82 \text{ sq.cm}$$ #

Heron's Formula

Heron's formula is used to find the area of a triangle when all its sides are given. This formula was written in 60 CE by Heron of Alexandria. He was an engineer and mathematician.



In above figure, if $a, b,$ and c are the lengths of $\Delta\ ABC$:

$$Area\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where, s is half the perimeter or $s = \frac{a+b+c}{2}$.

There are difference methods by which we can derive Heron's formula by using trigonometric identities or the Pythagorean Theorem. (Watch the following VDO: https://www.youtube.com/watch?v=LyI2BiyftvY)

Example 5.11 Find the area of the triangle whose sides measure 4 cm, 13 cm, and 15 cm.

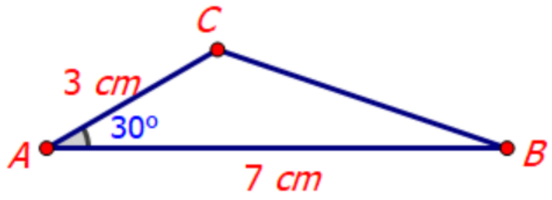
Solution: Half perimeter, $s = \frac{4+13+15}{2} = \frac{32}{2} = 16$

Using Heron's formula:

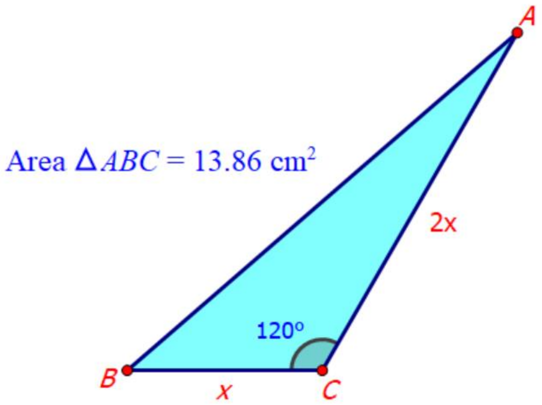
$$Area\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{16(16-4)(16-13)(16-15)}$$
$$= \sqrt{16 \times 12 \times 3 \times 1} = \sqrt{576} = 24 \text{ sq.cm} \qquad \#$$

Practice 5.3

1. Solve $\Delta\, ABC$ with $a = 8, b = 7$, and $\angle\, C = 45°$. Give the answer correct to 1 decimal place.
2. Find the area of triangle *ABC*. Give the answer correct to 2 decimal places.



3. In triangle *ABC* if $AC = 2BC$ and $\angle\, C = 120°$. The area of triangle *ABC* is 13.86 sq.cm. Find the length of $\overline{BC}$. Give the answer correct to 1 decimal place.



4. Find the area of the triangle whose sides measure 7 cm, 8 cm, and 12 cm.
5. A triangular field has three fences. One is 40 m long; another is 50 m and the other is 60 m. Find the area of the field.