Chapter 1 Concepts of Functions

Calculus is a branch of mathematics that describing how things change. The fundamental idea of calculus is to study changes over tiny interval of functions. For example, changes the speed function of objects which could be modeled by simple laws. All the functions necessary to study calculus are polynomial, rational, trigonometric, exponential, and logarithmic functions.

Functions

Functions are used many areas in mathematics to describe relationships between two sets. Given two sets *A* and *B*, a set with elements that are ordered pairs (x, y), where x is an element of *A* and y is an element of B, is defined as a relation from *A* to *B*.

A ***function*** is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the *input*; the element of the second set is called the *output*. The set of all first elements or *input* of the function *f* is called the ***domain*** of f (denoted by D_f). In symbol

$$D_f = \{x|(x, y) \in f\}$$

The set of all second elements or output of the function *f* is called the ***range*** of f (denoted by R_f). In symbol

$$R_f = \{y|(x, y) \in f\}$$

Definition of Functions

Definition 1.1: Let *A* and *B* be sets. *f* be a relation from *A* to *B*, denoted by $f: A \rightarrow B$ (read *f is a function from A to B*), if and only if

a) $D_f = A$, and

b) $\forall x \in A, \forall y, z \in B, [(x, y) \in f \land (x, z) \in f \rightarrow y = z]$

In words, this says that if *f* is a relation from *A* to *B* such that for every $x \in A$ there exist exactly one $y \in B$ such that $(x, y) \in f$ then *f* is a function.

For a general function *f* with *domain* $D_f = \{x|(x, y) \in f\}$, we use *x* to denote the ***input*** and refer to *x* as the ***independent variable***. Then, the function f with $R_f = \{y|(x, y) \in f\}$, we use *y* to denote the ***output*** associate with *x* and refer to *y* as the ***dependent variable***, because it depends on *x*. Using function notation, $f: A \rightarrow B$ we write $y = f(x)$, and we read this equation as "*y* equals *f* of *x*".

A function can be visualized as the mapping every element in the domain to exactly one element in the range as shown in Figure 1.1.

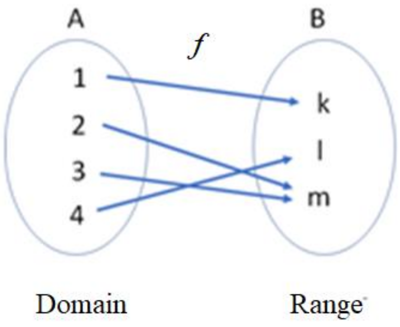


Figure 1.1

From Figure 1.1, $f = \{(1,k),(2,m),(3,m),(4,l)\}$

$D_f = \{1,2,3,4\}$, $R_f = \{k,l,m\}$

$f(1) = k, f(2) = m, f(3) = m,$ and $f(4) = l.$

Examples: (a) Let $f = \{(x,y) | y = 3 - x \text{ and } x \in \{1,2,3\}\}$.

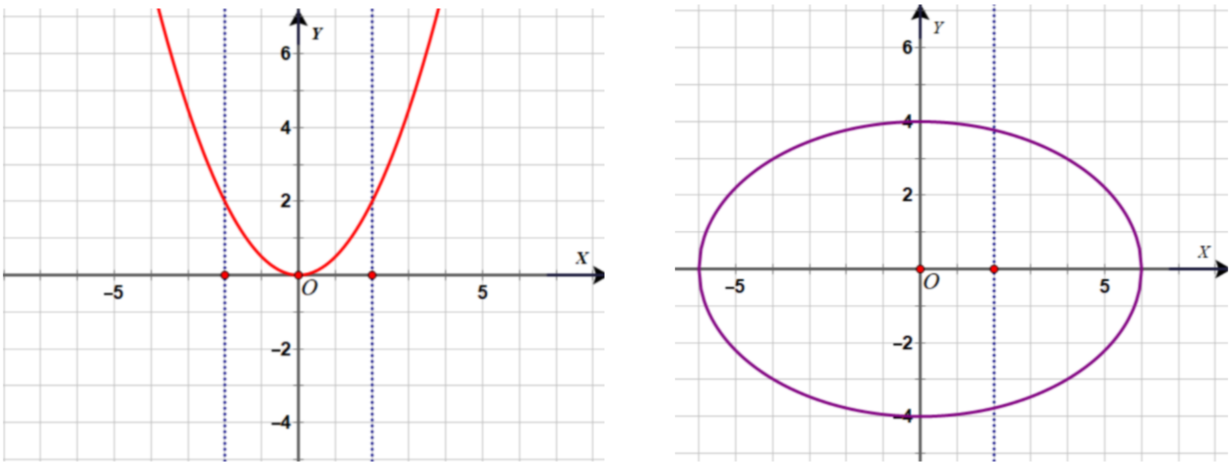
Then, $f = \{(1,2),(2,1),(3,0)\}$ #

(b) Let $f = \{(x,y) | y = x^2 \text{ and } x \in \{4,5,6\}\}$.

Then, $f = \{(4,16),(5,25),(6,36)\}$ #

Identify Function: Vertical Line Test

The vertical line test is a graphical method to determine whether a curve in the coordinate plane represents the graph of a function. Any vertical line in the plane can intersect the graph of a function at most once (Figure 1.2 a). If any vertical line intersects a graph more than once, then the graph does not represent a function (Figure 1.2b).



(a) Function (b) Not Function

Figure 1.2

Names and Formulas of Functions

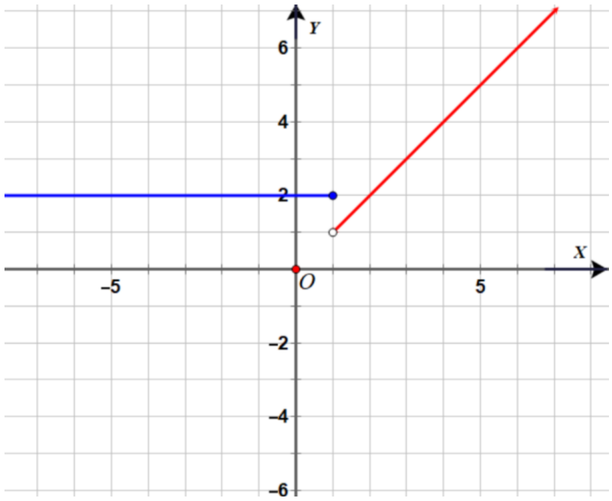
The most common name of function is "f", but we can have other names like "g" or "h" or name by using subscript such as $f_1, f_2, \dots$.

The formula or rule which specifies a function can be many different forms. For examples:

- A function is defined as $f(x) = 2x + 1$, it is a linear function.
- A function is defined as $g(x) = 3x^2 - 2x + 1$, it is a quadratic function.
- A function is defined as $h(x) = \cos(x^2 + 1)$, it is a trigonometric function.

Sometimes functions are defined by different rules on different intervals called ***piecewise functions*** as shown in the following examples:

$$f(x) = \begin{cases} 2 \; if \; x \le 1 \\ x \; if \; x > 1 \end{cases}$$



Note that the graph does pass the vertical line test at $x = 1$ because the point (1,1) is not the part of the piecewise function.

Evaluate of Functions

When we have a function in formula form, it is a simple way to replace the input variable in the formula with the value given and calculate the result.

Example 1: Given a function $f(x) = 8x^2 + 7x - 2$. Find the value of $f(x)$ that corresponds to a given value of x.

(a) $f(0)$, (b) $f(2)$, (c) $f(-2)$.

Solution:

Given: $f(x) = 8x^2 + 7x - 2$.

(a) $f(0) = 8(0)^2 + 7(0) - 2 = 0 + 0 - 2 = -2$. #

(b) $f(2) = 8(2)^2 + 7(2) - 2 = 32 + 14 - 2 = 44$. #

(c) $f(-2) = 8(-2)^2 + 7(-2) - 2 = 32 - 14 - 2 = 16$. #

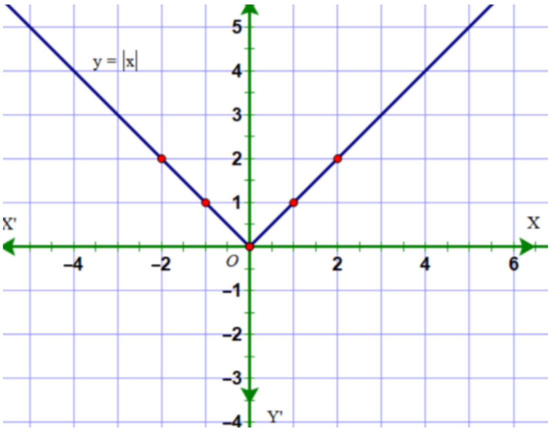
Example 2: Given an absolute function $f(x) = |x|$ is defined as

$$f(x) = \begin{cases} x & if\ x > 0 \\ 0 & if\ x = 0 \\ -x & if\ x < 0 \end{cases}$$

Evaluate the given absolute function as shown in the following table:

x	-2	-1	0	1	2
$f(x) = \|x\|$	$-(-2) = 2$	$-(-1) = 1$	0	1	2

From the value of x and $f(x)$ in the table, we can plot the graph on a coordinate plane at (-2,2), (-1,1), (0,0), (1,1), (2,2) and connect them as the following figure.



- The vertex of the graph is $(0,0)$.
- The axis of symmetry (x = 0 or y-axes) is the line that divides the graph into two congruent halves.
- The domain is the set of all real numbers.
- The range is the set of all real numbers greater than or equal to 0. That is $y \geq 0$.
- The x-intercept and the y-intercept are both 0.

Practice 1.1

1. For the function, $f(x) = x^2 + 3x - 4$, evaluate each of the following.

(a) $f(2)$ (b) $f(a)$ (c) $f(a+h)$ (d) $\frac{f(a+h)-f(a)}{h}$

2. Determine the domain of the functions

(a) $f(x) = \sqrt{4 - x^2}$ (b) $f(x) = \frac{x^2+3}{(x-1)(x+3)}$ (c) $f(x) = \frac{x}{x^2+4}$

Polynomial Functions and Graphing

A polynomial function is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for some integers $n \geq 0$ and constants $a_n, a_{n-1}, \dots , a_0$ are real numbers, where $a_n \neq 0$. The value n is called the ***degree*** of the polynomial and the *degree of a polynomial is the* ***highest power of x in its expression.*** The constants are called the ***coefficients*** of the polynomials.

There are different forms of polynomial functions. The general form of different types of polynomial functions are given below:

Degree	Type of Function	General Form
0	Constant	$f(x) = k$
1	Linear	$f(x) = mx + b$
2	Quadratic	$f(x) = ax^2 + bx + c$
3	Cubic	$f(x) = ax^3 + bx^2 + cx + d$
4	Quartic	$f(x) = ax^4 + bx^3 + cx^2 + dx + e$

Constant Function

A ***constant function*** is a function *having the same range for different values of the domain* Graphically a constant function is a straight line, which is parallel to the x-axis.

Example: Let D_f is the set of real numbers R, that is $R = \{x_i | x_i \text{ are real numbers}\}$.

Let $R_f = \{k\}$.

A constant function is defined as $f(x) = k$ for each $x \in R$ and k is a constant. It is a linear function where $f(x_1) = f(x_2) = \cdots = f(x_i)$ for all $x_i \in R$.

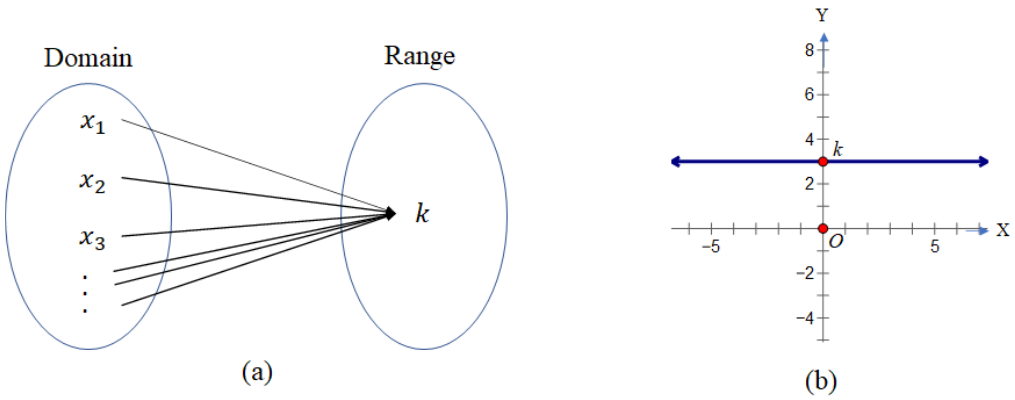


Figure 1.3

From Figure 1.3 (a), *different values of the domain map to the same value k*. Form Figure 1.3 (b), the of $(x) = k$ is a straight line, which is parallel to the x-axis.

Domain and Range of Constant Function

The *domain of a constant function is the set of all real numbers, and the range of a constant function is the set of the constant k.*

Linear Function

A ***Linear function*** has the form $f(x) = mx + b$, where m and b are constants. The constant *m* and *b* are the ***slope*** and ***y-intercept*** of the line.

Example:

- $f(x) = 5x + 25$; here $m = 5$ and $b = 25$.
- $g(x) = \frac{1}{2}x - 3$; here $m = \frac{1}{2}$ and $b =$ -3.
- $h(x) = -x + 5$; *here* $m =$ -1 and $b = 5$.

Notice that the *y-intercept* occurs where $f(x) = f(0)$.

Slope of the Line

The ***slope*** measures both the ***steepness and the direction of a line***. If the slope is positive, the line is increasing when x increases, y also increases (see Figure 1.4 (a)). If the slope is zero, the line is horizontal and parallel to the x-axis (see Figure 1.4 (b)). If the slope is negative, the line is decreasing when x increases, but y decreases (see Figure 1.4 (c)).

Example: Determine the *slope* and *y-intercept* of the line that graphically linear functions.

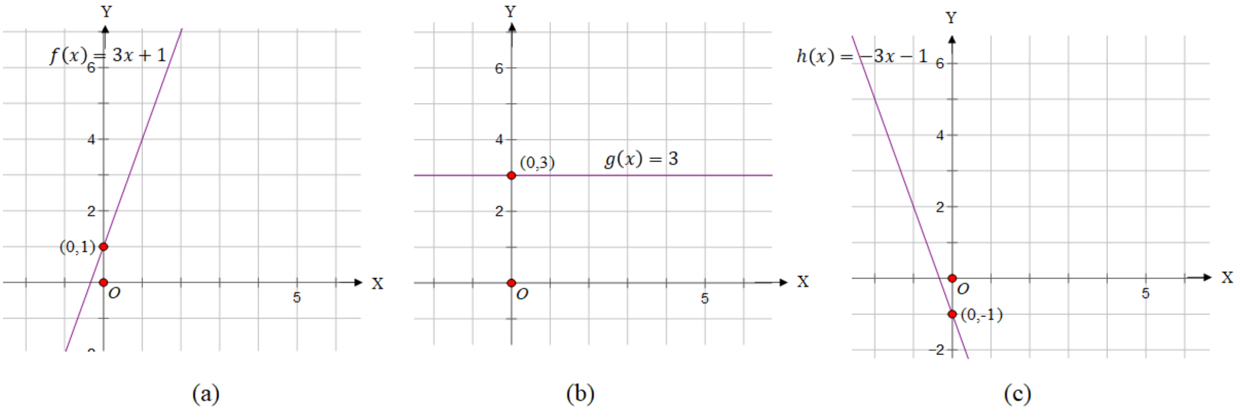


Figure 1.4

In Figure 1.4 (a), the linear function $f(x) = 3x + 1$ has $m = 3$, then the slope of the function is 3. The function has $b = 1$, then the y-intercept is (0, 1). The line moves upward from the left to the right or increases when x increases, y also increases.

In Figure 1.4 (b), the linear function $g(x) = 3x$ has $m = 3$, then the slope of the function is 3. The function has $b = 0$, then the graph of the line is horizontal and parallel to the x-axis.

In Figure 1.4 (c), the linear function $h(x) = -3x - 1$ has m = -3, then the slope of the function is -3. The function has b = -1, then the y-intercept is (0, -1). The line moves downward from the left to the right or decreases when x increases, but y decreases.

Calculating the Slope

The ***slope*** is the change in *y* for each unit change in *x*. There are three steps in calculating the slope of a straight line when we are not given its equation (see Figure 1.5).

Step 1: Identify two points on the line.

Step 2: Select one to be (x_1, y_1) and the other to be (x_2, y_2).

Step 3: Use the slope equation to calculate slope from two selected points as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

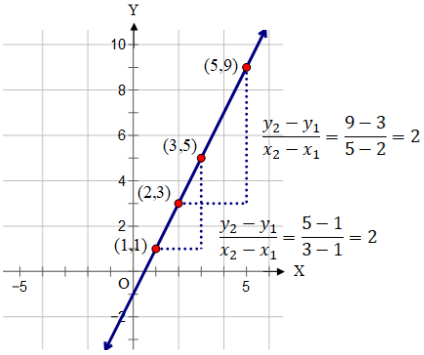


Figure 1.5

Standard Form of a Line

Consider a line passing through the point (x_1, y_1) with slope *m*, the equation

$$y - y_1 = m(x - x_1)$$

is the ***point-slope form*** for that line.

Consider a line with slope *m* and *y*-intercept (0, b). The equation

$$y = mx + b$$

is the ***slope-intercept form*** for a line.

The ***standard form*** of a line is given by the equation

$$ax + by = c,$$

where *a* and *b* are not both zero. This equation allows for vertical line, $x = k$.

Example: Consider the line passing through points $(-3, 2)$ and $(1,4)$.

(a) Find the slope of the line.

(b) Find an equation of that line in point-slope form.

(c) Find an equation of that line in slope-intercept form.

(d) Find an equation of that line in standard form.

Solution:

(a) Let m be the slope of the line passing through points $(-3, 2)$ and $(1,4)$.

Find the slope of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{1-(-3)} = \frac{2}{4} = \frac{1}{2} \qquad \#$$

(b) Point-slope form is:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \tfrac{1}{2}(x - (-3))$$

$$y - 2 = \tfrac{1}{2}(x + 3) \qquad \#$$

(c) Slope-intercept form is:

$$y - 2 = \frac{1}{2}(x + 3)$$

$$y = \tfrac{1}{2}x + \tfrac{3}{2} + 2 = \tfrac{1}{2}x + \tfrac{7}{2} \qquad \#$$

(d) Standard form is:

$$-x + 2y = 7 \qquad \#$$

Domain and Range of Linear Function

The *domain of a linear function is the set of all real numbers, and the range of a linear function is also the set of all real numbers*. The domain and the range of the given function written in two ways:

Notation	**Domain**	**Range**
Set Notation	$\{x \mid x \in R\}$	$\{y \mid y \in R\}$
Interval Notation	$(-\infty, +\infty)$	$(-\infty, +\infty)$

Applications of Linear Functions

Example 1: You are comparing two cell phone voice only plans. Customers need to pay additional fees for data usage. Company A offers a monthly fee of $10 plus $1 per GB of data. Company B offers a monthly fee of $15 plus $0.5 per GB of data.

1. Write linear function that model the cell phone plans.
2. If 4 GB of data is used each month, which company is cheaper and by how much?
3. If 12 GB of data is used each month, which company is cheaper and by how much?

4. How many GB per month would yield equal monthly statements from both companies? What would the monthly cost be?

Solution: Let *C* = the monthly cost, and *x*=the number of GB of data used per month

1. Company A: $C = 10 + x$, and Company B: $C = 15 + 0.5x$. #
2. If $x = 4$, then

   Company A: $C = 10 + 4 = \$14$.

   Company B: $C = 15 + (0.5 \times 4) = 15 + 2 = \$17$

   Therefore, Company A is cheaper than Company B = $\$17 - \$14 = \$3$. #
3. If $x = 12$, then

   Company A: $C = 10 + 12 = \$22$.

   Company B: $C = 15 + (0.5 \times 12) = \$21$.

   Therefore, Company B is cheaper than Company A = $\$22 - \$21 = \$1$. #
4. If equal monthly statements from both companies, then

$$10 + x = 15 + 0.5x$$
$$x - 0.5x = 15 - 10$$
$$0.5x = 5$$
$$x = 10$$

   Therefore, the monthly cost would be $10 + 10 = \$20$.

   Or $15 + (0.5 \times 10) = 15 + 5 = \$20$ #

Practice 1.2

1. For each pair of points, (i) find the slope of the line passing through the points y-intercept and (ii) indicate whether the line is increasing, decreasing, horizontal, or vertical.

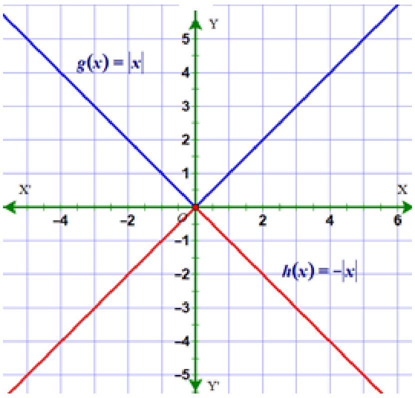
   (a) (-2, 4) and (1, 1) (b) (3, 5) and (-1, 2) (c) (2, 4) and (1, 4) (d) (1, 4) and (1, 0).
2. Write the equation of the line satisfying the given conditions in slope-intercept form.

   (a) Slope = 3, passes through (-3, 2) (b) x-intercept = 5 and y-intercept = -3.
3. The total cost *C* (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28{,}5000$, where *x* is the number of items produced. Determine the cost to produce 175 items.

Absolute Value Functions

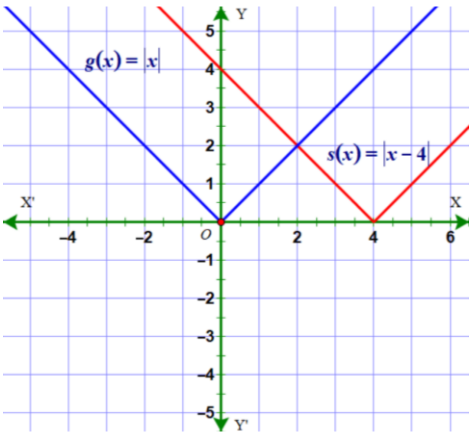
An absolute value function is a function that contains an algebraic expression within absolute value symbols. The form of the equation for an absolute function is $y = a|x - h| + k$, where:

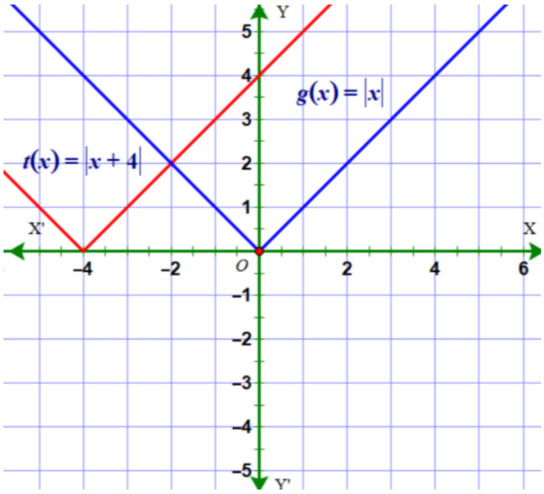
- The vertex of the graph is (h, k).
- The domain of the graph is set of all real numbers and the range is $y \geq k$ when $a > 0$.
- The domain of the graph is set of all real numbers and the range is $y \leq k$ when $a < 0$.
- The axis of symmetry is $x = h$.
- The graph of the absolute value function opens up if $a > 0$ and opens down if $a < 0$ as shown in the following figure.



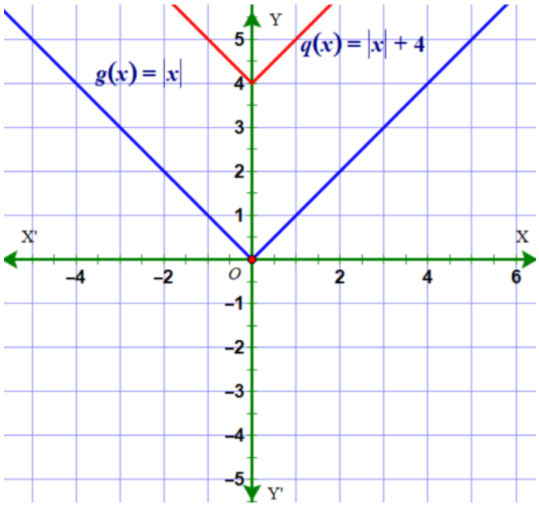
- The graph $y = |x|$ can be translated h units horizontally and k units vertically to get the graph $y = a|x - h| + k$.

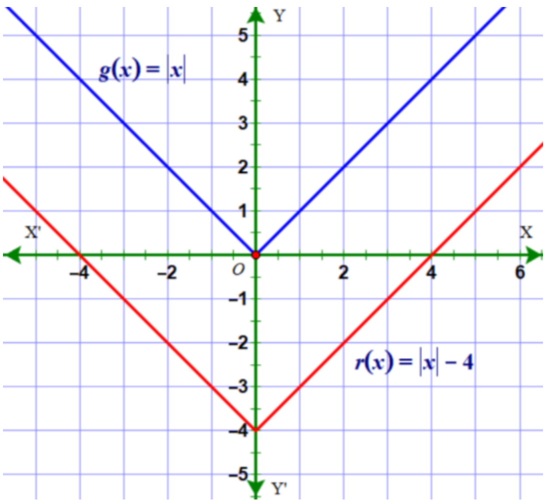
Horizontal Shift



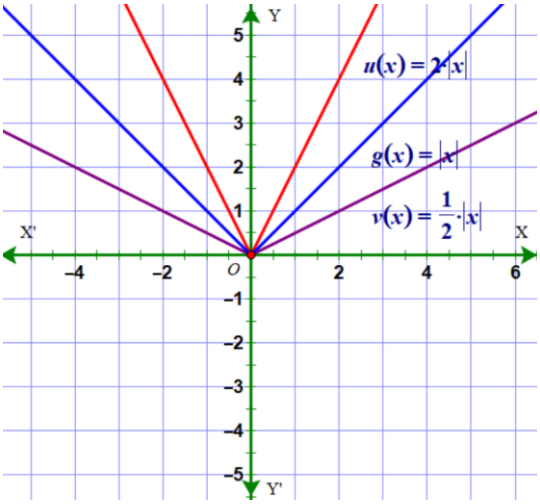


Vertical Shift





- The graph $y = a|x|$ is wider than the graph $y = |x|$ if $|a| < 1$ and narrower if $|a| > 1$.



Practice 1.3

Draw the graphs of the following functions, and find their domains and ranges.

(a) $f(x) = |x - 3|$ (b) $f(x) = \frac{|x|}{x}$ (c) $f(x) = x - |x|$

Quadratic Function

A ***quadratic function*** has the form $f(x) = ax^2 + bx + c$, where *a*, *b*, and *c* are real numbers with $a \neq 0$.

Example:

- $f(x) = 2x^2 - 5x + 4$; here $a = 2, b = -5,$ and $c = 4$.
- $g(x) = 4x^2 - 16$; here $a = 4, b = 0,$ and $c = -16$.
- $h(x) = x^2 - x$; here $a = 1, b = -1,$ and $c = 0$.

Vertex of Quadratic Function

The ***vertex*** of a quadratic function indicates ***a maximum value or minimum value of the function***. The graph of a quadratic function is a curve in ∪ shape called a ***parabola***; the ***vertex of the graph*** is at the point of *a maximum value or minimum value of the function*. If the graph of a quadratic function has two x-intercepts, then *the* ***line of symmetry*** *is the vertical line through the midpoint of the x-intercepts*. The axis of symmetry of the quadratic function intersects the function at the vertex (see Figure 1.6).

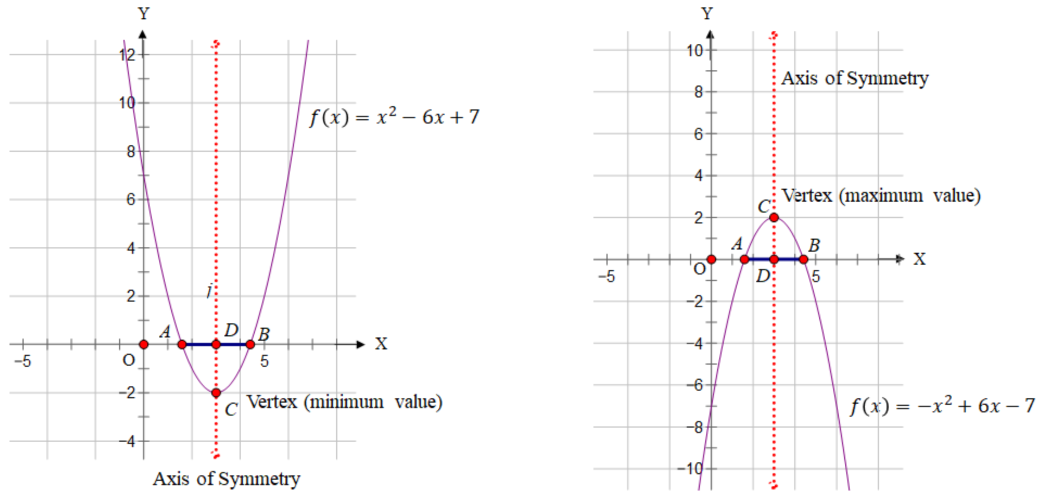


Figure 1.6

Standard Form of Quadratic Functions

A quadratic function can be in different forms: standard form, vertex form, and intercept form.

S***tandard form*** is given by the equation

$$f(x) = ax^2 + bx + c \ \text{ or } \ y = ax^2 + bx + c, \text{ where } a \neq 0.$$

Vertex form is given by the equation

$$f(x) = a(x - h)^2 + k \ \text{ or } \ y = a(x - h)^2 + k$$

where $a \neq 0$ and (h, k) is the ***vertex*** of the parabola representing the quadratic function.

Intercept form is given by the equation

$$f(x) = a(x - p)(x - q) \ \text{ or } \ y = a(x - p)(x - q)$$

where $a \neq 0$ and $(p, 0)$ and $(q, 0)$ are the ***x-intercepts*** of the parabola representing the quadratic function.

The ***x-intercepts*** are the points where the graph $y = f(x)$ cuts the x-axis. They also called the ***zeros of the function***, because they are the x-values where $y = f(x) = 0$. It simply finds the *x-intercepts* by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Shape of Quadratic Function

Graph of a parabola representing the quadratic function opens upward or downward according to the value of a (see Figure 1.5):

- If $a > 0$, then the parabola opens upward.
- If $a < 0$, then the parabola opens downward.

Domain and Range of Quadratic Function

Consider the vertex form of a quadratic function (parabola), $f(x) = a(x-h)^2 + k$, the domain will be all values of real numbers. The range must be considered the minimum value for the parabola opens upward and the maximum value for the parabola opens downwards. The domain and the range of the given function written in two ways:

Notation	Domain	Range
Set Notation (upward)	$\{x \mid x \in R\}$	$\{y \in R \mid y \geq k\}$
Interval Notation (upward)	$(-\infty, +\infty)$	$[k, +\infty)$
Set Notation (downward)	$\{x \mid x \in R\}$	$\{y \in R \mid y \leq k\}$
Interval Notation (downward)	$(-\infty, +\infty)$	$(-\infty, k]$

Applications of Quadratic Functions

There are many real-world situations that deal with quadratic functions, such as throwing a ball, hitting a golf ball, and diving from a platform. From these situations, we can draw a graph of parabola of the situation and find the highest point or lowest point which is known as the vertex.

The standard form of a parabola is $y = ax^2 + bx + c$, where $a \neq 0$. The vertex is the minimum or maximum point of a parabola.

- If $a > 0$, then the parabola opens upward.
- If $a < 0$, then the parabola opens downward.

To find the vertex, we need to find x- and y- coordinates.

The formula for the x-coordinate of the vertex is:

$$x = \frac{-b}{2a}$$

To find the y-coordinate of the vertex, substitute the value of *x* into the equation of parabola and solve for *y*.

$$y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c.$$

Example: A toy rocket is fired into the air from the top of a barn. Its height (h) above the ground in meters after *t* seconds is given by the function $h(t) = -5t^2 + 10t + 20$.

(a) What was the initial height of the rocket?

(b) When did the rocket reach its maximum height?

Solution:

(a) The initial height of the rocket is the height from which it was fired. The time is zero.

$h(t) = -5t^2 + 10t + 20 = -5(0)^2 + 10(0) + 20 = 20$ meters.

Therefore, the initial height of the rocket is 20 meters above the ground. #

(b) The time at which the rocket reaches its maximum height is x-coordinate of the vertex.

$$t = \frac{-b}{2a} = \frac{-10}{2(-5)} = 1 \text{ sec.}$$

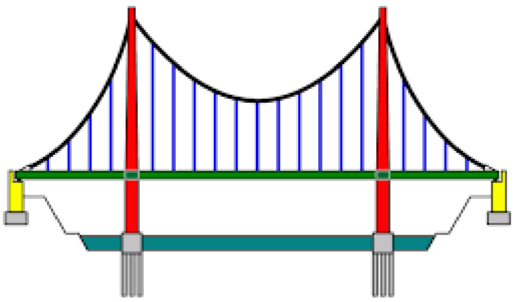
Therefore, it takes the toy rocket 1 second to reach its maximum height. #

Practice 1.4

1.Graph the following quadratic functions and find the domains and ranges.

(a) $f(x) = x^2 + 2x + 1$ (b) $f(x) = x^2 - 3x - 4$ (c) $f(x) = -2x^2 - 4x + 1$

2. A suspension bridge is 20 meters long and is supported by a suspension cable.



The height *h* meters of the cable over the bridge at a distance *x* meter from one end of the bridge is given by the formula:

$$h(x) = 0.03x^2 - 0.6x + 5, \text{ for } 0 \le x \le 20.$$

(a) Draw the graph of $h(x) = 0.03x^2 - 0.6x + 5$, for $0 \le x \le 20$.

(b) Find the height of the cable at one end.

(c) Find the minimum height of the cable above the bridge at 10 meters from one end.

(d) State the line of symmetry of the graph.

(**Hint**: To find the vertex of a parabola for maximum or minimum point using the formula for the x-coordinate of the vertex:

$$x = \frac{-b}{2a}$$

To find the y-coordinate of the vertex, substitute the value of *x* into the equation of parabola and solve for *y*.

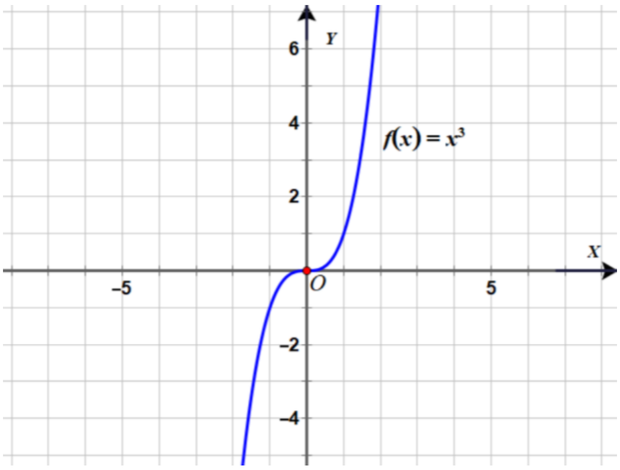
$$y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c).$$

Cubic Function

A cubic function is a polynomial of degree three. It is one in the form:

$f(x) = ax^3 + bx^2 + cx + d$ or $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$.

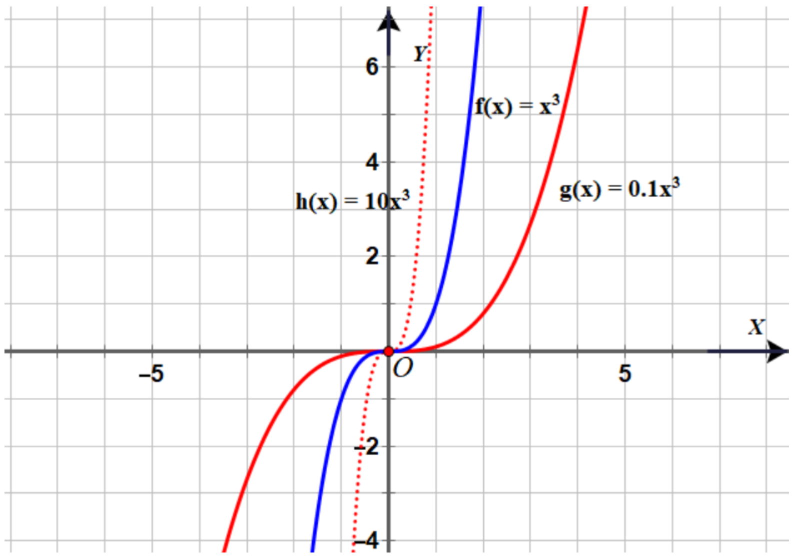
The graph of the basic cubic function, $f(x) = x^3$, is graphed below.



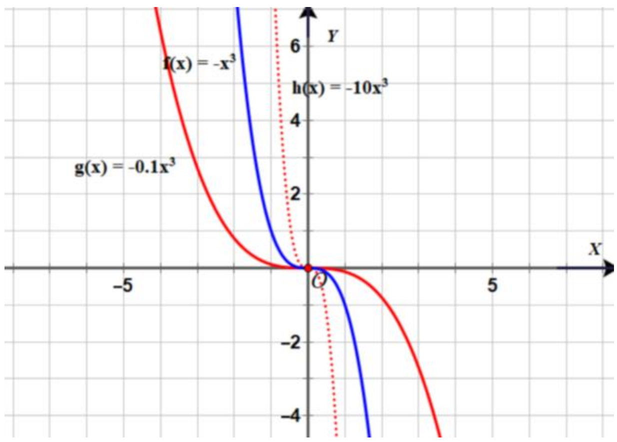
Standard Form of Cubic Function

The standard form of a cubic function is $f(x) = a(x-h)^3 + k$, where *a* indicates steep or flat, *h* indicates a horizontal shift in x-direction, and *k* indicates a vertical shift in y-direction.

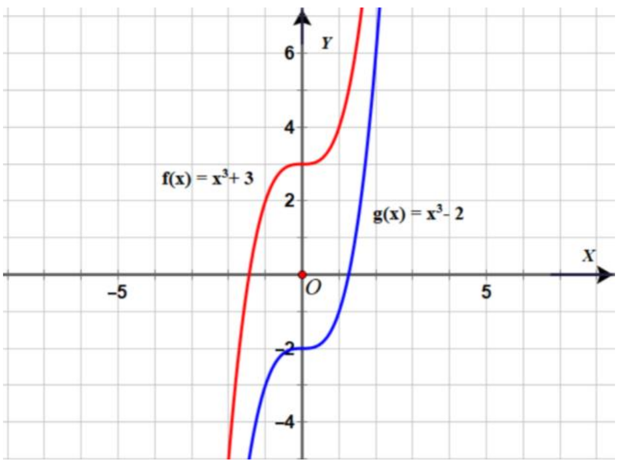
Case 1: If $a > 1$, then the graph of cubic function will be ***steeper*** compared to the graph $f(x) = x^3$. If $a < 1$ then the graph of function will be ***flatter*** compared to the graph $f(x) = x^3$.



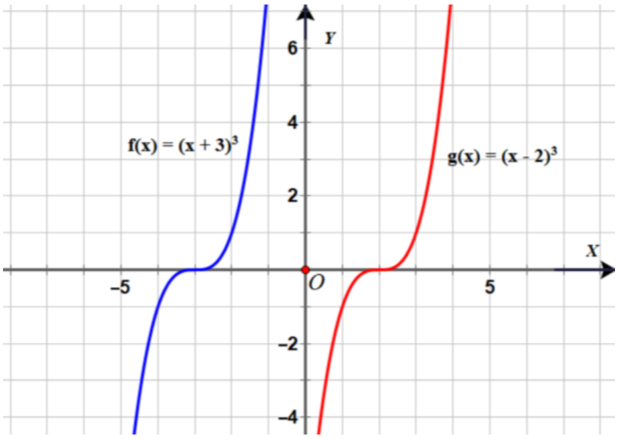
Case 2: If *a* is negative, the graph of cubic function is ***inverted***.



Case 3: The *k* has the effect of moving the graph of a cubic function up or down the y-axis by *k* units.



Case 4: The h has the effect to moving the graph of a cubic function along the x-axis left or right by h units.



Example: Graph the cubic function, $g(x) = 2(x + 1)^3 - 1$.

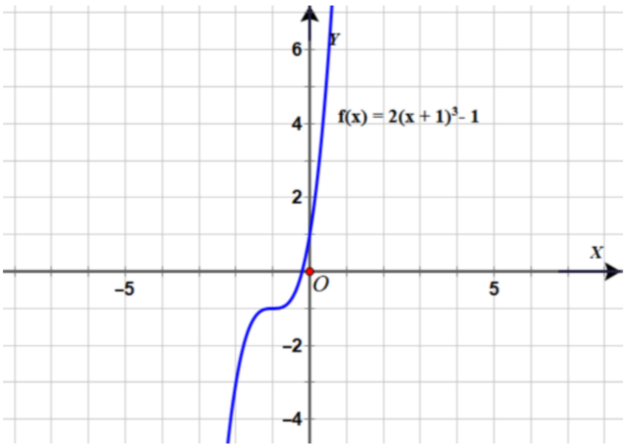
Solution: Since $g(x)$ is in standard form, where $a = 2, h = -1$, and $k = -1$.

Step 1: $a = 2$ which means $a > 1$. Therefore, the graph of $g(x)$ will be steeper compared to the graph of $f(x) = x^3$.

Step 2: $h = -1$ which means the graph of $g(x)$ is shifted one unit to the left from the location of the graph $f(x) = x^3$.

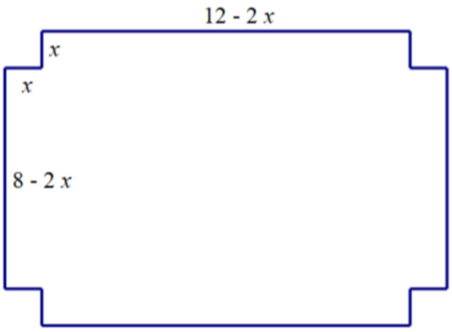
Step 3: $k = -1$ which means the graph of $g(x)$ is shifted down 1 unit from the location of the graph $(x) = x^3$.

When we combine the findings of step 1 to step 3, we can draw the graph of $g(x)$ as follows:



Applications of Cubic Functions

Example 1: Find the volume function of a box made from a sheet of cardboard with the length 12 cm. and the width 8 cm. by cutting square out of the four corners of the cardboard. Then draw the graph of volume function.

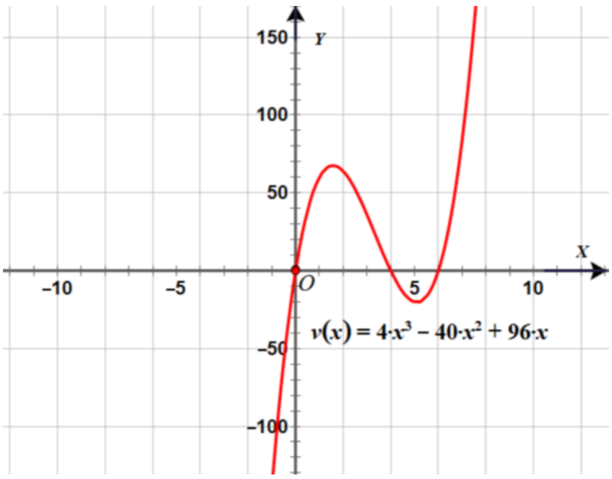


Solution: Let x be the side-length of the squares cut out of the corners.

Then the volume function is:

$v(x) = (12 - 2x)(8 - 2x)x = 4x^3 - 40x^2 + 96x$

Draw the graph of function $v(x)$



Analyze the graph of the function to find the maximum value of the box, focusing on the interval (0,4). We can see the volume of the box increase, and then decreases. To determine the maximum value will be studied in a later chapter. #

Practice 1.5

1.Draw the graphs of the following cubic function for $-3 \leq x \leq 3$ on separate diagrams using The Geometer's Sketchpad.

(a) $f(x) = 1.5x^3$ (b) $f(x) = 0.8x^3$ (c) $f(x) = -\frac{1}{4}x^3$ (d) $f(x) = -1.5x^3$

2.The function $y = -0.2t^3 + 5t$ for $0 \le t \le 5$ gives the approximate concentration y units of medicine found in a person's blood t hours after taking a pill containing the medicine.

(a) Draw the graph of $y = -0.2t^3 + 5t$ for $0 \le t \le 5$ using the horizontal axis as t-axis to represent the time in hours and the vertical axis as y-axis to represent the concentration in units.

(b) Estimate from the graph, the times at which the concentration is 6 units.

Power Functions

A ***power function*** is a *single-term function of the form* $f(x) = kx^a$, where k is a nonzero coefficient and a is a real number. All of the following functions are the examples of power functions.

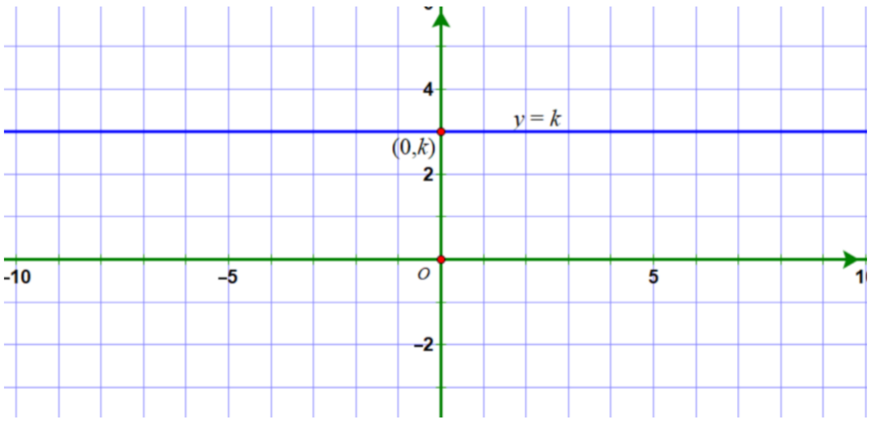
- $f(x) = 3$; here $k = 3$ and $a = 0$
- $g(x) = -5x^2$; here $k = -5$ and $a = 2$.
- $h(x) = \frac{3}{x^2}$; here $k = 3$ and $a = -2$.
- $2\sqrt{x}$; here $k = 2$ and $a = \frac{1}{2}$.

The power functions have a variety of forms based on the type of the power a is:

- $a = n$ where $n = 0$;
- $a = n$ where n is a positive integer;
- $a = \frac{1}{n}$ where n is a positive integer; and
- $a = -1$

Case 1: If $a = n$ where $n = 0$.

Notice that if $n = 0$ then the power function $f(x) = k$ is a *constant function*, since $x^0 = 1$.



Case 2: If $a = n$ where n is a positive integer.

Notice that if n is a positive integer, then the power function is a type of *polynomial function*. Explore the variety graphs in Figure 1.7:

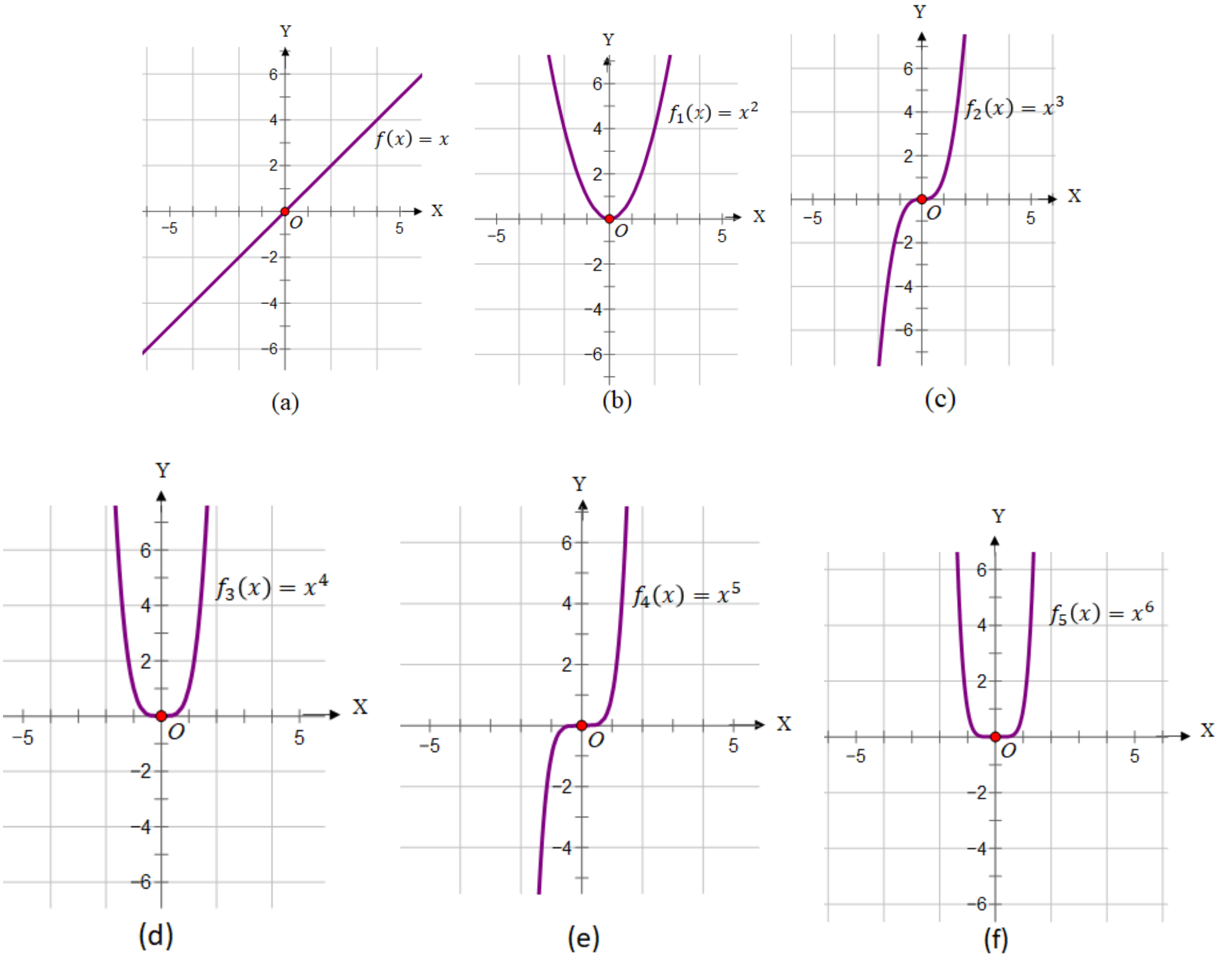


Figure 1.7

Figure 1.7 (b), (d), and (f) show the graphs of $f_1(x) = x^2, f_3(x) = x^2$, and $f_5(x) = x^6$, which all power of the functions is even. Notice that these graphs have similar shapes and look like graph of quadratic function. However, as the power increases, the graphs flatten near the origin and are symmetric about the y-axis. For the even-power function, as the value of *x* increases or decreases without bound, the values of function become very large with positive numbers. In symbolic form, we write

$$as\ x \to \pm\infty, f(x) \to \infty,$$

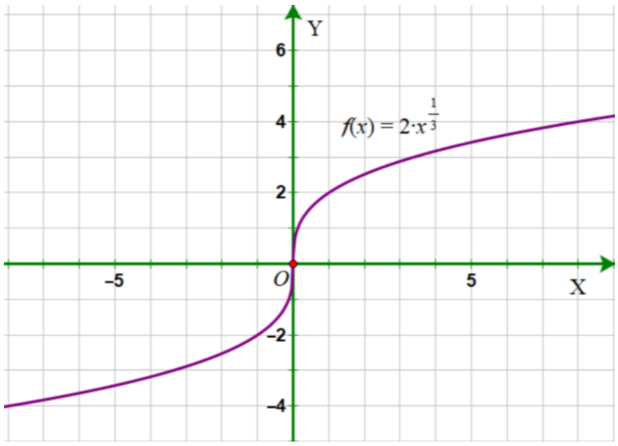
read "as *x* approaches positive or negative infinity, the $f(x)$ values increase without bound".

Figure 1.7 (c) and (d) show the graph of $f_2(x) = x^3$, and $f_4(x) = x^5$, which all power of the functions is odd. Notice that these graphs have similar shapes and look like graph of cubic function and the graphs flatten near the origin as the power increases. We see that odd-power function $f(x) = x^n, n$ odd, are symmetric about the origin. For the odd-power function, as *x* approaches negative infinity, the values of functions decrease without bound. As *x* approaches positive infinity, the values of functions increase without bound. In symbolic form, we write

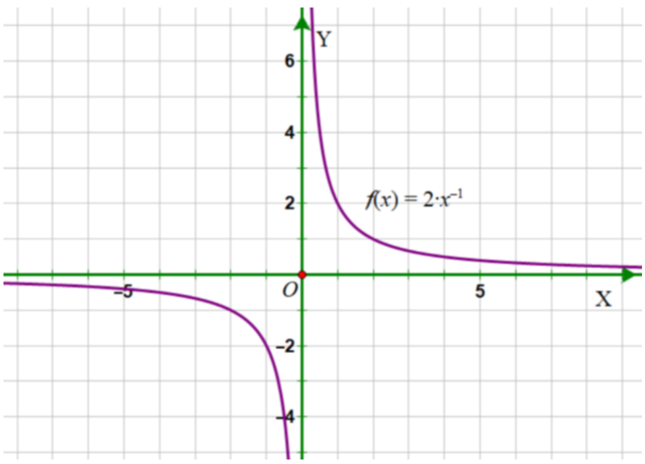
$$as\ x \to -\infty, f(x) \to -\infty, \text{ and } as\ x \to \infty, f(x) \to \infty.$$

The symbols $x \to -\infty$ and $x \to \infty$ are referred to the ***end behavior*** of the function. The end behavior of power functions in the form $f(x) = kx^a$ where *n* is a ***non-negative integer*** *depending on the power n and the constant k.*

Case 3: $a = \frac{1}{n}$ where n is a positive integer



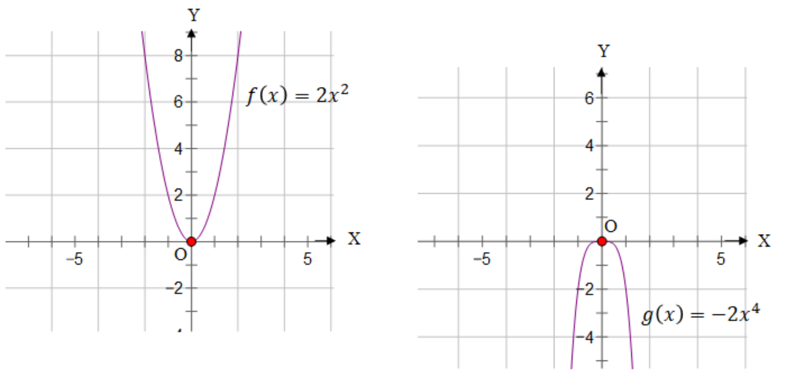
Case 4: $a = -1$



Power Function in Odd and Even

Some power functions are either even or odd, so their graphs are different shapes but they are either symmetric about the y-axis and origin.

Case 1: Even power function with $k > 0$ its graph is upward, if $k < 0$ its graph is downward and symmetric about y-axis (see Figure 1.8).

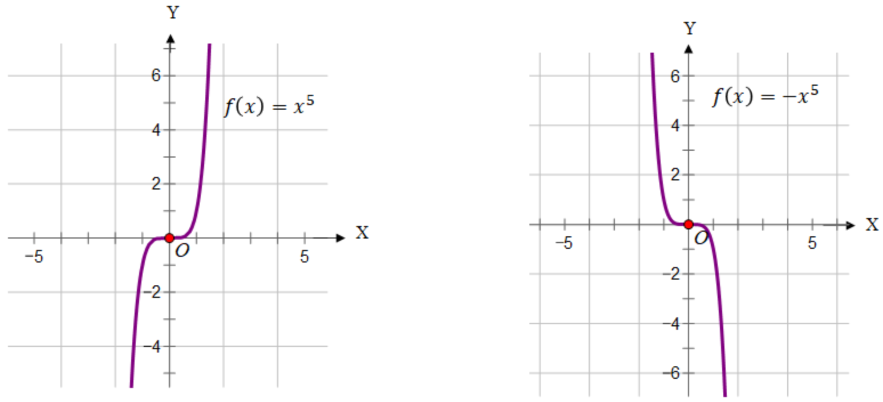


$x \to -\infty, f(x) \to \infty$
and $x \to \infty, f(x) \to \infty$

$x \to -\infty, f(x) \to -\infty$
and $x \to \infty, f(x) \to -\infty$

Figure 1.8

Case 2: Odd power function with $k > 0$ its graph is increasing, if $k < 0$ its graph is decreasing and symmetric about origin (see Figure 1.9).



$x \to -\infty, f(x) \to -\infty$
and $x \to \infty, f(x) \to \infty$

$x \to -\infty, f(x) \to \infty$
and $x \to \infty, f(x) \to -\infty$

Figure 1.9

Applications of Power Functions

A power function can be used to model the relationships of variables in direct and inverse variation as the following definitions.

Direct variation describes a simple relationship between two variables. We say *y* varies directly with *x* (symbol $y \propto x$) if:

$$y = kx \text{ , for some constant } k.$$

This means that as *x* increases, *y* increases and as *x* decreases, *y* decreases – and that the *ratio* between them always stays the same.

Inverse variation describes the inverse relationship between two variables. We say *y* varies inversely with *x* (symbol $y \propto \frac{1}{x}$) if:

$$y = \frac{k}{x} \text{ , for some constant } k.$$

The power function for *direct variation* is $f(x) = kx^a$, where *k* is a nonzero coefficient and *a* is a real number.

The power function for *indirect variation* is $f(x) = \frac{k}{x^a}$, where *k* is a nonzero number and *a* is a real number.

Example 1: The distance, *d* meters, that the rock falls varies directly with the square of the time taken, *t* second. If the rock falls 6 meters in 2 seconds, find the distance the rock has fallen after 5 seconds.

Solution: Let *d* be the distance in meters and *t* be the time in seconds.

Since $d \propto t^2$, so $d = kt^2$.

Given $d = 6$ meters and $t = 2$ seconds.

Then, $6 = k(2)^2$

Thus, $k = 1.5$

If $t = 5$ seconds, then $d = 1.5(5)^2 = 1.5(25) = 37.5$ meters. #

Example 2: The number of hours *h* taken to build a wall varies inversely with the number of workers *x* who are available to work on it. If three workers are available the wall takes two hours to build. Find the time it takes to build the wall when four people are available to work on it.

Solution: Let *h* be the number of hours and *x* be the number of the workers.

Since $h \propto \frac{1}{x}$, then $h = \frac{k}{x}$.

Given $h = 2$ hours and $x = 3$ workers.

Then, $2 = \frac{k}{3}$.

Thus, $k = 2(3) = 6$.

If $x = 4$, then $h = \frac{6}{4} = 1.5$ hours or 1 hour 30 minutes. #

Practice 1.6

1. Given function $f(x) = x^3 - x^2 - 2x$ for $-2 \leq x \leq 3$
   (a) Draw the given function by using The Geometer's Sketchpad.
   (b) Use your graph to solve the equation $x^3 - x^2 - 2x = 5$. (Hint: Draw a graph of line for $y = 5$, then find the intersection point of this line and the graph of $f(x)$.
2. The area of a rectangular poster is 4 m^2. The dimensions of the poster are *x* meters by *y* meters.
   (a) Express *y* as a function of *x*.
   (b) Draw the graph of the function for $0 < x \leq 5$.

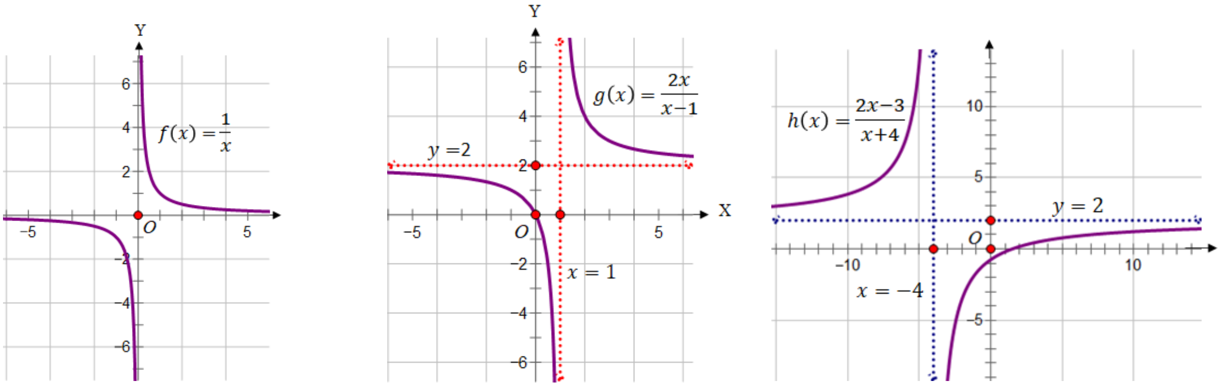
Rational Function

A rational function is a ratio of two polynomials, $p(x)$ and $q(x)$ and $q(x) \neq 0$.

$$f(x) = \frac{p(x)}{q(x)}$$

- $f(x) = \frac{1}{x}$, here $p(x) = 1$ and $q(x) = x$ and $x \neq 0$.
- $g(x) = \frac{2x}{x-1}$, here $p(x) = 2x$ and $q(x) = x - 1$ and $x \neq 1$.
- $h(x) = \frac{2x-3}{x+4}$, here $p(x) = 2x - 3$ and $q(x) = x + 4$ and $x \neq -4$.

All above rational functions can be graphed as follows:



The dot lines are the asymptote of each rational function which the function is not defined when the value of its denominator is zero.

Facts about Asymptote

The following facts about asymptotes are described as the following.

Let n be the degree of polynomial of the numerator, and m be the degree of polynomial of the denominator.

1. The graph will have a vertical asymptote at $x = a$ if the denominator is zero at $x = a$ and the numerator is not zero at $x = a$.
2. If $n < m$ then the x-axis is the horizontal asymptote. However, if $n = m + 1$, then there exists a slant asymptote.
3. If $n = m$, then a horizontal asymptote exists, and the equation is:

$$y = \frac{Coeficient\ of\ highest\ power\ term\ in\ numerator}{Coeficient\ of\ highest\ power\ term\ in\ denominator}$$

Example: Graph the rational function, $f(x) = \frac{x^2+x-1}{x^2-1}$

Solution: Find the vertical asymptotes of the rational function

Step 1: Calculate the zero function of the denominator using the linear factors in the denomination that two singularities exist at $x = 1$ and $x = -1$. However, the linear factors of both numerator and denominator are as follows:

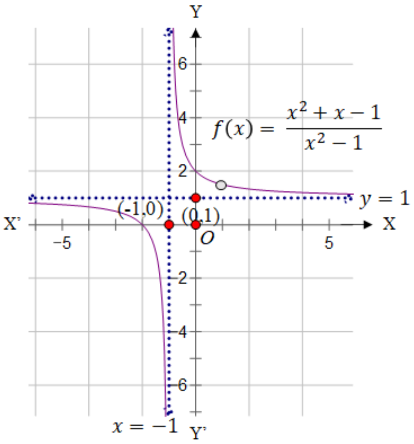
$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x+1)} = \frac{x+2}{x+1}$$

Therefore, the linear factor $(x-1)$ cancels with a factor in the denominator. Thus, the only vertical asymptote of this function is at $x=-1$. The domain of the function in the simplest form $\frac{x+2}{x+1}$ is all real numbers except $x=-1$, but at $x=1$ the graph will have a hole.

Step 2: Since $n=m$, then a horizontal asymptote exists, and the equation is:

$$y=\frac{Coeficient\ of\ highest\ power\ term\ in\ numerator}{Coeficient\ of\ highest\ power\ term\ in\ denominator}=\frac{1}{1}=1$$

Step 3: When we combine the findings of step 1 to step 2, we can draw the graph of $f(x)$ as follows:



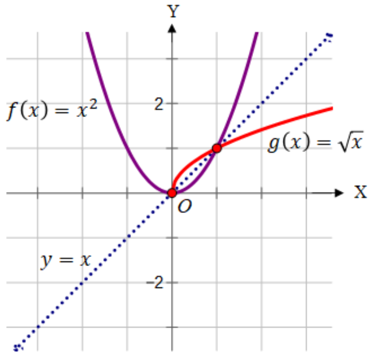
Radical Function

A radical function is a function that can be written with a variable under a root:

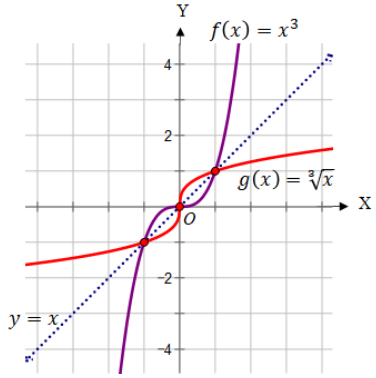
$$f(x)=\sqrt[n]{p(x)}$$

The simple radical function of the form $f(x)=\sqrt[n]{x}$ is an inverse relationship with power function of the form $f(x)=x^n$, where n is whole number or non-negative integer. It is a reflection of the graph across the line $y=x$ as shown in the following figures.

Even Power Functions and Radicals



Odd Power Functions and Radicals



Domain and Range of Radical Functions

Consider the properties of radicals $\sqrt[n]{a}$ as follows:

Case 1: The index n is an ***even*** number.

- If $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- If $a < 0$, then $\sqrt[n]{a}$ is not a real number.

Case 2: The index n is an ***odd*** number, $\sqrt[n]{a}$ is a real number for all values of a.

To find the domain and range of radical functions, we consider the above properties of radicals.

- When the ***index*** of the radical is ***even***, the radicand must be greater than or equal to zero.
- When the ***index*** of the radical is ***odd***, the radicand can be any real number.

Example 1: Find the domain and range of the function, $f(x) = \sqrt{3x - 5}$.

Solution: Since $f(x) = \sqrt{3x - 5}$ has a radical with an index of 2 (even number), then the radicand must be greater than or equal to zero.

Solve, $3x - 5 \geq 0$

$$3x \geq 5$$

$$x \geq \frac{5}{3}$$

Therefore, $D_f = [\frac{5}{3}, \infty)$.

Since, the range of a function is the set of all possible function values.

Therefore, $R_f = \{y \in R | y \geq 0\}$ or $[0, \infty)$. #

Example 2: Find the domain and range of the function, $f(x) = \sqrt{\frac{4}{x-1}}$.

Solution: Since $(x) = \sqrt{\frac{4}{x-1}}$ has a radical with an index of 2 (even number), then the radicand must be greater than or equal to zero. However, the radicand cannot be zero since the numerator is not zero.

Solve, $x - 1 > 0$

$x > 1$

Also, since the radicand is a fraction, so $x - 1 \neq 0$ and then $x \neq 1$.

Therefore, $d_f = \{x \in R | x > 1\}$ or $d_f = (1, \infty)$.

Since, the range of a function is the set of all possible function values.

Therefore, $R_f = \{y \in R | y > 0\}$ or $(0, \infty)$. #

Example 3: Find the domain and range of the function, $f(x) = \sqrt[3]{3x^2 - 1}$.

Solution: Since $f(x) = \sqrt[3]{3x^2 - 1}$ has a radical with an index of 3 (odd number), then the radicand can be any real number.

Therefore, $D_f = \{x | x \in R\}$ or $(-\infty, \infty)$. #

Since, the range of a function is the set of all possible function values.

Therefore, $R_f = \{y \in R | y \geq -1\}$ or $[-1, \infty)$. #

Practice 1.7

For each polynomial, (a) find the degree, (b) draw the graph, (c) find the y-intercept(s) (if any) (d) use the leading coefficient to determine the graph's end behavior.

(1) $f(x) = 2x^2 - 3x - 5$ (2) $f(x) = -3x^2 + 6x$

(3) $f(x) = \frac{1}{2}x^2 - 1$ (4) $f(x) = x^3 + 3x^2 - x - 3$

Exponential Functions

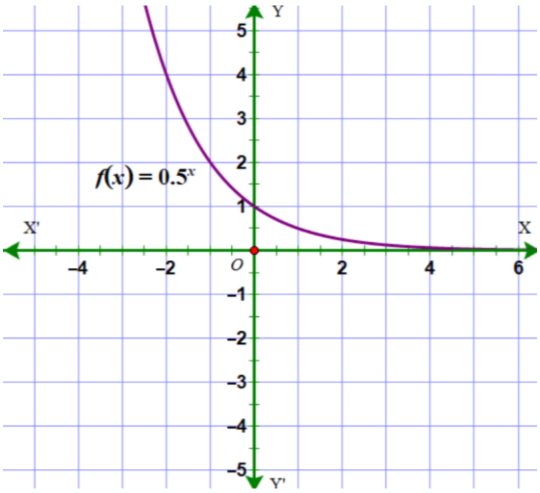
An exponential function is a function in form $f(x) = a^x$, where "x" is a ***variable*** and "a" is a ***constan***t which is called the ***base*** of the exponential function and it should be any value greater than 0.

Properties of Exponential Functions

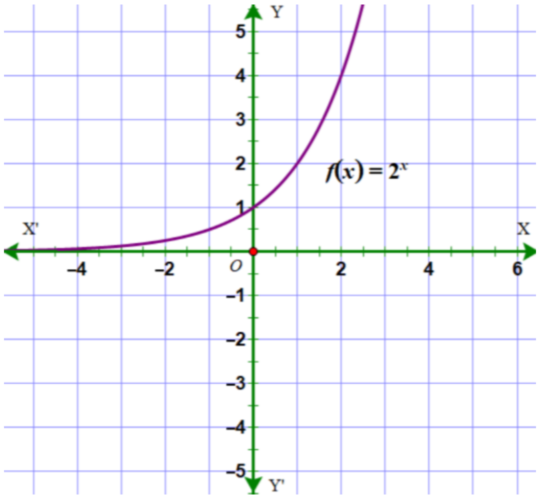
Properties of exponential functions depend on value of "a":

Case 1: When $a = 1$, then $f(x) = 1^x = 1$. Its graph is a horizontal line at $y = 1$.

Case 2: When $0 < a < 1$, then as x increases its graph heads to 0 (called exponential decay), and as x decreases its graph heads to infinity.



Case 3: When $1 < a$, then as x increases its graph heads to infinity (called exponential growth), and as x decreases its graph heads to 0.



In general, the properties of an exponential function are as follows:

- It is always greater than 0, and **never crosses the x-axis**.
- It always **passes through point (0,1)**.
- At $\boldsymbol{x = 1}, f(x) = a$, and **passes through point** $(\boldsymbol{1, a})$**.**
- It is an **injective (one-to-one) function**.
- Its **domain is the set of all real numbers**.
- Its **range is the positive real numbers**: $(0, +\infty)$.

Applications of Exponential Functions

Example: The number of yeast cells in a cup of water at time *t* hours after the start of observation is given by $y = 60(2^t)$

(a) Find the number of yeast cells in the cup 2 hours later.

(b) Draw the graph of $y = 60(2^t)$ for $0 \le t \le 4$

Solution: (a) Let y be the number of yeast cell and t be the time in hours.

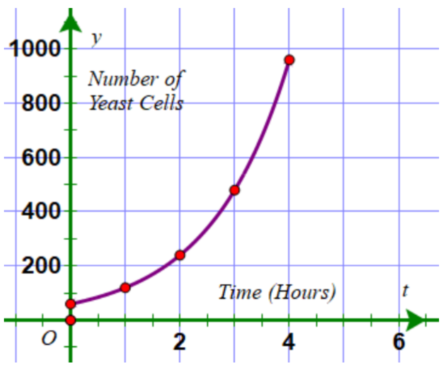
Since $y = 60(2^t)$, then at the start $t = 0$.

Thus, the number of yeast cells at the start is $y = 60(2^t) = 60(2^0) = 60(1) =$ 60 cells.

If $t = 2$ hours, then $y = 60(2^t) = 60(2^2) = 60(4) = 240$ cells.

(b) Draw the graph by evaluating the value of $y = 60(2^t)$, for $0 \le t \le 4$

t	0	1	2	3	4
y	60	120	240	480	960



Logarithmic Functions

In mathematics, the logarithmic function is an inverse of exponential function. The logarithmic function is a function of the form:

$y = \log_a x$ if and only if $x = a^y$, for $x > 0, a > 0,$ and $a \neq 1$.

The function $y = \log_a x$ can be read "y equals the log of *x* base *a*" or "y equals the log base a of *x*". The most two common bases used in logarithmic functions are base 10 and base *e*.

Common Logarithmic Function

The logarithmic function with base 10 is called the common logarithmic function and it is denoted by $y = \log_{10} x$ or $y = \log x$.

Natural Logarithmic Function

The logarithmic function with base *e* is called the natural logarithmic function and it is denoted by $y = \log_e x$ or $y = \ln x$ (note: $e \approx 2.718281828459$ ...).

Properties of Logarithms

There are four basic properties of logarithms as follows:

- $\log_a xy = \log_a x + \log_a y$.
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.
- $\log_a x^n = n \log_a x$.
- $\log_a x = \frac{\log_b x}{\log_b a}$

Additional properties are listed below for reference.

- $\log_a 1 = 0$.
- $\log_a a = 1$.
- $\log_a a^x = x$.
- $a^{\log_a x} = x$.
- $\log_a b = \frac{1}{\log_b a}$.

Graph of Logarithmic Functions

Since the logarithmic function is an inverse of exponential function. Then we can produce the table of values of exponential function and produce the table of logarithmic function by using inverse values of exponential function as shown in the following example.

Example: Draw the graph of $g(x) = \log_2 x$.

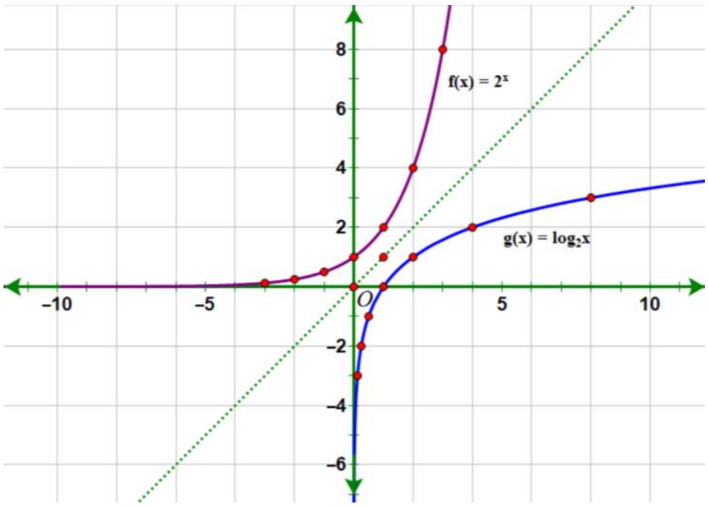
Solution: Let $f(x) = 2^x$, produce the table of values of $f(x) = 2^x$.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	0.125	0.25	0.5	1	2	4	8

Produce the table of logarithmic function by using inverse values of exponential function.

x	0.125	0.25	0.5	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

Draw the graph of $g(x) = \log_2 x$.



Note that the logarithmic function $g(x) = \log_2 x$ is an inverse of exponential function $f(x) = 2^x$.

Remark:

- The function $f(x) = 0^x$ is ***not*** an exponential function.
- The function $g(x) = 1^x$ is ***not*** an exponential function.
- The function $h(x) = (-2)^x$ is ***not*** an exponential function.
- The base of logarithm cannot be negative, means $a > 0$.
- The base of logarithm ***cannot be unity***, means $a \neq 1$.
- The logarithm of any ***negative number is not defined***.

Applications of Logarithmic Functions

Logarithm is used in the following application of real life.

- Earthquake intensity measurement

- Acidic measurement of solution (pH value)
- Sound intensity measurement
- Express larger values

Example: The Richter scale is a base 10 logarithmic scale. If there are the first earthquake with magnitude M_1 on the Richter scale and a second earthquake with magnitude M_2 on the Richter scale. Suppose $M_1 > M_2$, which means the earthquake $M_1$is stronger. A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$M_1 - M_2 = \log_{10}\left(\frac{A_1}{A_2}\right)$$

Consider an earthquake that measures 7 on the Richter scale and an earthquake that measures 6 on the Richter scale. Then

$$7 - 6 = \log_{10}\left(\frac{A_1}{A_2}\right)$$

Therefore,

$$\log_{10}\left(\frac{A_1}{A_2}\right) = 1$$

Which implies

$$\frac{A_1}{A_2} = 10 \text{ or } A_1 = 10A_2$$

Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as strong as the second earthquake.

In the following table, the comparison of the intense of earthquakes in India, US, and Japan which measured on the Richter scale. We can convert logarithm value into exponential value to compare the intense of the earthquake in India, US, and Japan as follows:

	India	US	Japan
$\log_{10} M$	6.0	7.0	9.0
M	10^6	10^7	10^9

$$\underbrace{10\ times\ stronger}_{} \qquad \underbrace{100\ times\ stronger}_{}$$

$$\underbrace{1000\ times\ stronger}_{}$$

Practice 1.8

1. Evaluate the given exponential function as indicated, accurate to two significant digits after the decimal.

   1.1 $f(x) = 5^x$ (a) $x = 3$ (b) $x = \frac{1}{2}$ (c) $x = \sqrt{2}$

   1.2 $f(x) = 10^x$ (a) $x = -2$ (b) $x = 4$ (c) $x = \frac{5}{3}$

2. Sketch the graph of the given exponential function. Determine the domain, range, and horizontal asymptote.

   (a) $f(x) = 4^{-x}$ (b) $f(x) = 2^{x+1}$ (c) $f(x) = \left(\frac{1}{2}\right)^x + 2$

3. Write the equation in equivalent exponential form.

   (a) $log_3 81 = 4$ (b) $log_5 25 = 2$ (c) $log 0.1 = -1$

4. Write the equation in equivalent logarithmic form.

   (a) $2^3 = 8$ (b) $4^{-2} = \frac{1}{16}$ (c) $e^x = y$

5. Sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

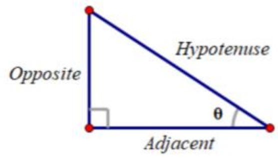
   (a) $f(x) = logx - 1$ (b) $f(x) = 1 - logx$ (c) $f(x) = \log(x+1)$

Trigonometric Functions

Trigonometric functions are functions of an angle. There are six trigonometric functions: sine function, cosine function, tangent function, cosecant function, secant function, and cotangent function.

Definition of Basic Trigonometric Functions

Trigonometric function relates between an angle of a right-angled triangle and the ratio of a pair of its sides. Three main basic trigonometry functions are defined as follows:



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Figure 1.10

In Figure 1.10, let a be the length of the adjacent side, b be the length of the opposite side, and c be the length of hypotenuse side. Therefore, the values of the trigonometric functions are:

$$\sin\theta = \frac{b}{c}, \cos\theta = \frac{a}{c}, \text{ and } \tan\theta = \frac{b}{a}$$

There are three other trigonometric functions that are not commonly used, they are equal to 1 divided by the three main basic trigonometric functions which are defined follows:

$$\text{Cosecant Function: } \csc\theta = \frac{1}{\sin\theta} = \frac{c}{b} \text{ or } \frac{Hypotenuse}{Opposite}$$

$$\text{Secant Function: } \sec\theta = \frac{1}{\cos\theta} = \frac{c}{a} \text{ or } \frac{Hypotenuse}{Adjacent}$$

$$\text{Cotangent Function: } \cot\theta = \frac{1}{\tan\theta} = \frac{a}{b} \text{ or } \frac{Adjacent}{Opposite}$$

Measure of Angles on Unit Circle

The trigonometric functions can be developed by using angle measurement which relates to the concept of the circular functions and relies on movement around the perimeter of a unit circle.

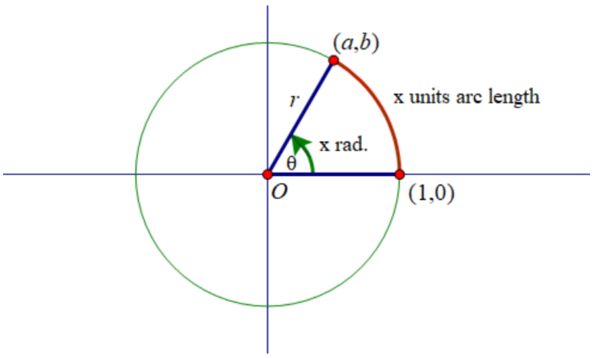


Figure 1.11

On unit circle with radius $r = 1$, the distance around the perimeter or circumference of the circle is equal to 2π. In Figure 1.1, it is determined that a complete rotation of a straight line around the vertex of an angle is equal to 360 degrees or 2π radians. Since one radian is the angle made at the center of a circle by an arc whose length is equal to the radius of the circle. Therefore, if θ is an angle measured in radians and x is the length of the corresponding arc on the unit circle, then $\theta = x$ as shown in Figure 1.11.

Example 1: What is the measure of a given angle in radians if its length is 4π and the radius has length 12?

Solution: Let the arc length $s = 4\pi$ and the radius $r = 12$.

Then substitute the values $s = 4\pi$ and the $r = 12$ into the angle formula:

$\theta = \frac{s}{r} = \frac{4\pi}{12} = \frac{\pi}{3}$ radians. #

Example 2: Convert an angle measuring 180 degrees to radians.

Solution: Since 360 degrees = 2π radians

Therefore, 180 degrees = $2\pi \times \frac{180}{360} = \pi$. #

Note that to convert an ***angle in radians to degrees*** can be used the following formula:

angle in degrees = angle in radians $\times \frac{180°}{\pi}$.

Trigonometric Functions on Circle

Let $P(x, y)$ be the coordinate of the end point of the arc on the circle corresponding to θ radian $(\theta = x)$. Then the trigonometric functions can be defined in terms of the horizontal component value of x and the vertical component value of y on a circle of radius r as shown in Figure 1.12.

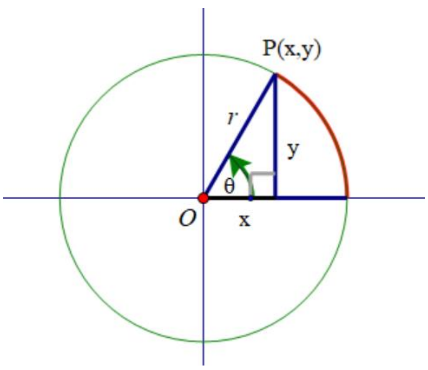


Figure 1.12

In Figure 1.12, the trigonometric functions for θ are defined in terms of the coordinates x and y on a circle of radius r as the following:

$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{r}{y}, y \neq 0$
$\cos\theta = \frac{x}{r}$	$\sec\theta = \frac{r}{x}, x \neq 0$
$\tan\theta = \frac{y}{x}, x \neq 0$	$\cot\theta = \frac{x}{y}, y \neq 0$

For a point $P(x, y)$ on a circle of radius r with a corresponding angle θ, we can use the definitions of sine and cosine to solve for the coordinates x and y on the terminal side of the angle θ as the following:

$$\cos\theta = \frac{x}{r}\text{, then } x = r\cos\theta \text{ and } \sin\theta = \frac{y}{r}\text{, then } y = r\sin\theta.$$

Since the radius of a unit circle is 1, then the values of x and y on a unit circle are as follows:

$$x = \cos\theta \text{ and } y = \sin\theta.$$

Sign of Trigonometric Functions

The sign of a trigonometric function is dependent on the signs of the coordinates of the points on the terminal side of the angle. The distance from a point to the origin is always positive, but the signs of the x and y coordinates may be positive or negative as follows:

Quadrant	Sign of x	Sign of y
1	+	+
2	-	+
3	-	-
4	+	-

The values of $\sin\theta$ and $\cos\theta$ in the first quadrant can be used to determine the values of $\sin\theta$ and $\cos\theta$ in the other quadrants as shown in Figure 1.13 and Figure 1.14.

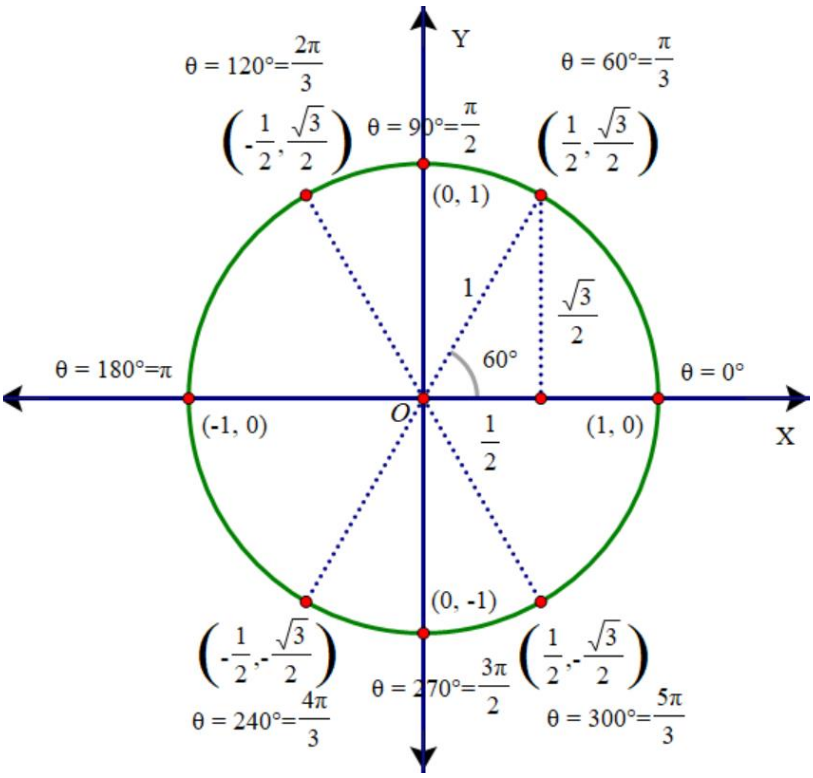


Figure 1.13

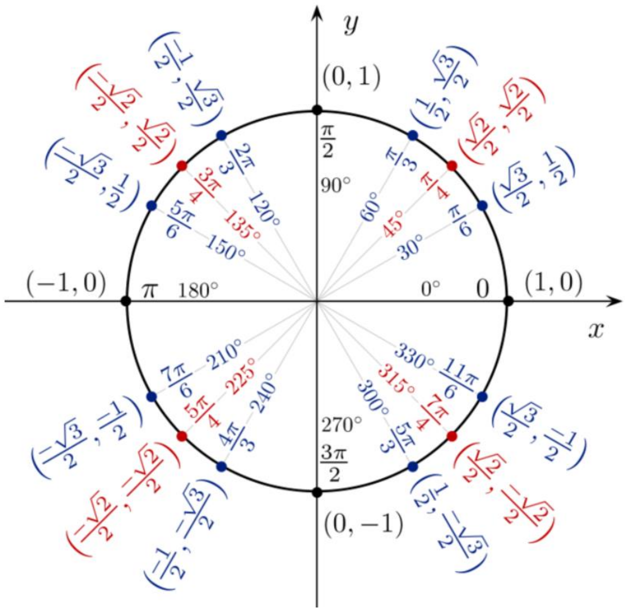


Figure 1.14

Source: https://courses.lumenlearning.com/boundless-algebra/chapter/trigonometric-functions-and-the-unit-circle/

Positive and Negative Angles

The measure of an angle for a trigonometric function describes the magnitude and the direction of the side of angle rotation from its initial position to its terminal position. If the rotation is counterclockwise or anticlockwise, the angle has a ***positive measure***.

In Figure 1.14, *all of angles in quadrant 1 to quadrant 4 have* ***positive measures*** *because the side of each angle rotates counterclockwise from its initial position at point (1,0) on positive x-axis around the center of a unit circle to the terminal position in four quadrants*. In the first quadrant, all angles are the acute angles formed by the terminal side and the positive x- axis which are called ***reference angles***. A reference angle is always between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. For any angle in quadrant 1, 2, 3, and 4, there is a reference angle in quadrant 1.

If the rotation is clockwise, the angle has a ***negative measure***, but the values of $x = \cos\theta$ and $y = \sin\theta$ may be positive or negative are based on the sign of $x \; and \; y$ on positive or negative axis of the quadrant as shown in Figure 1.15.

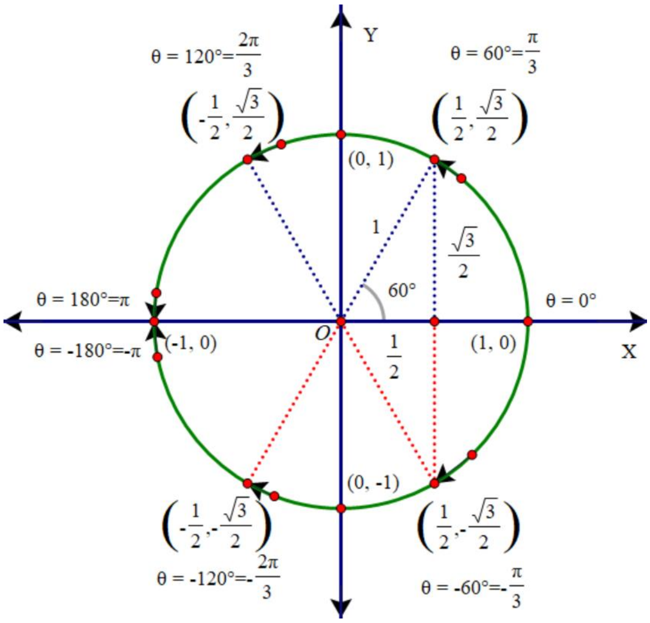


Figure 1.15

Since $x = \cos\theta$ and $y = \sin\theta$, then we can find $\tan\theta$ by determining $\frac{\sin\theta}{\cos\theta} = \frac{y}{x}$.

Example: Find $\tan 60°$ and $\tan(-60°)$.

Solution:

$$(1)\ \tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \times 2 = \sqrt{3}.$$

$$(2)\ \tan(-60°) = \frac{\sin(-60°)}{\cos(-60°)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2} \times 2 = -\sqrt{3}. \qquad \#$$

There are six negative angle identities as follows:

$\sin(-\theta) = -\sin\theta$	$\csc(-\theta) = -\csc\theta$
$\cos(-\theta) = \cos\theta$	$\sec(-\theta) = \sec\theta$
$\tan(-\theta) = -\tan\theta$	$\cot(-\theta) = -\cot\theta$

Domain, Range, and Graphs of Trigonometric Functions

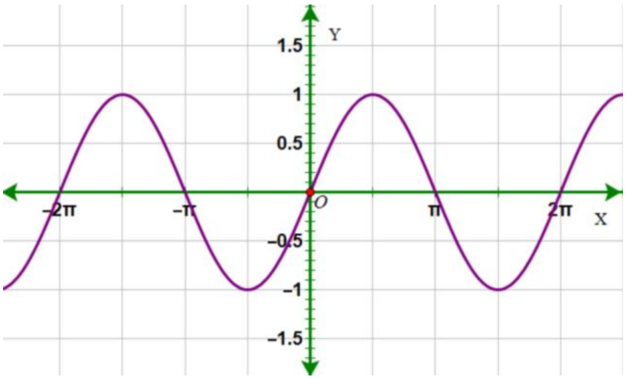
The following table shows the values of the basic trigonometric functions from 0 degree to 360 degrees or 0 radian to 2π radians.

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not Defined	0	Not Defined	0
cosec θ	Not Defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	Not Defined	-1	Not Defined
sec θ	1	$\frac{1}{2}$	$\sqrt{2}$	2	Not Defined	-1	Not Defined	1
$\cot\theta$	Not Defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	Not Defined	0	Not Defined

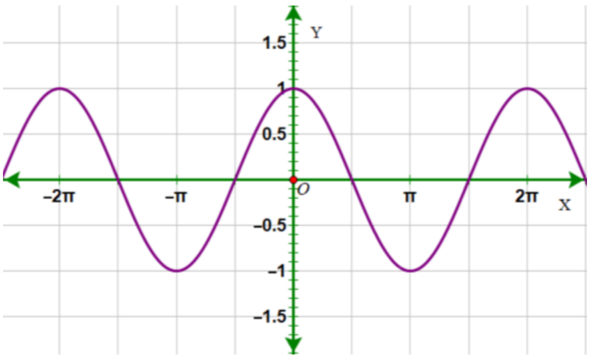
The domain of trigonometric function is defined as the particular set values that an independent variable contained in a function can accept the work. The range exists as resulting values which a dependent variable can hold a value of ‘x’ changes all through the domain.

Function	Domain	Range
$\sin x$	$(-\infty, +\infty)$	$[-1, 1]$
$\cos x$	$(-\infty, +\infty)$	$[-1, 1]$
$\tan x$	$R - \left\{x \middle\vert x = \frac{(2n+1)\pi}{2}\right\}$	$(-\infty, +\infty)$
$\csc x$	$R - \{x \vert x = n\pi\}$	$(-\infty, -1] \cup [1, +\infty)$
$\sec x$	$R - \left\{x \middle\vert x = \frac{(2n+1)\pi}{2}\right\}$	$(-\infty, -1] \cup [1, +\infty)$
$\cot x$	$R - \{x \vert x = n\pi\}$	$(-\infty, +\infty)$

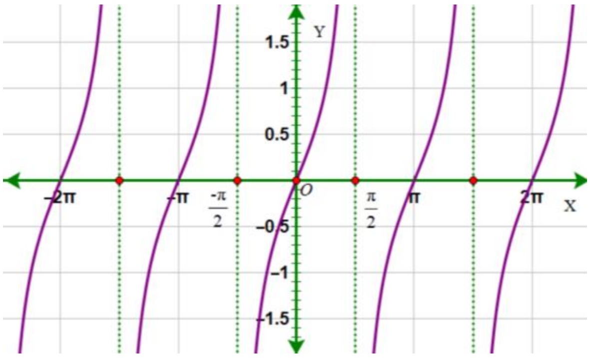
The graphs of trigonometric functions have the domain value θ represented on the horizontal x-axis and the range value represented along the vertical y-axis. The graph of $\sin\theta$ and $\tan\theta$ passes through the origin and the graph of other trigonometric functions do not pass through the origin. Since the angle θ and $\theta + 2\pi$ correspond to the same point and same value. Consequently, the trigonometry are ***periodic functions***.



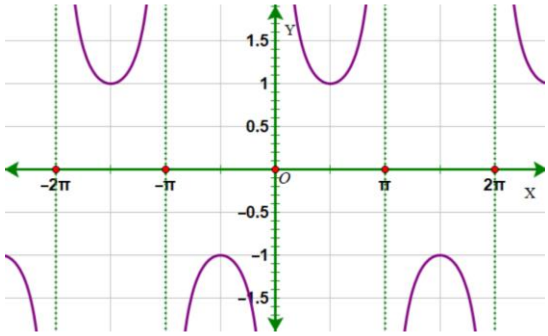
Graph of sine function (period = 2π)



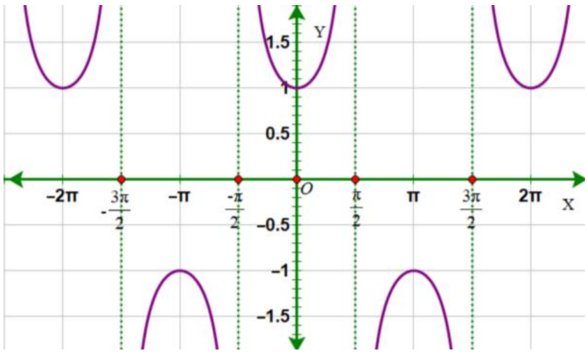
Graph of cosine function (period = 2π)



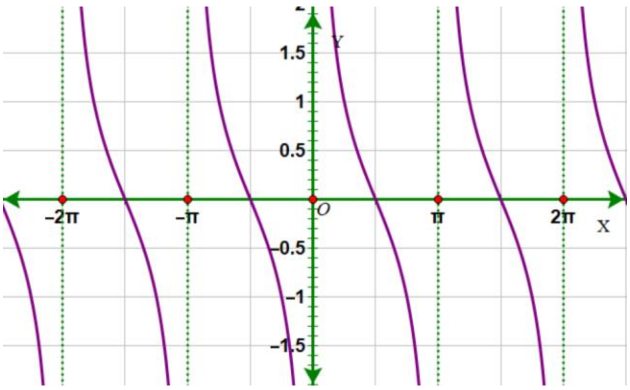
Graph of tangent function (period = π)



Graph of cosec function (period = 2π)



Graph of sec function (period = 2π)



Graph of cotangent function (period = π)

Trigonometric Identities

A trigonometric identity is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. The following identities can be used to simplify equations for solving the trigonometric functions.

- **Reciprocal identities**:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

- **Pythagorean identities**:

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

- **Addition and subtraction formulas**:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

- **Double-angle formulas**:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

Explicit Functions and Implicit Functions

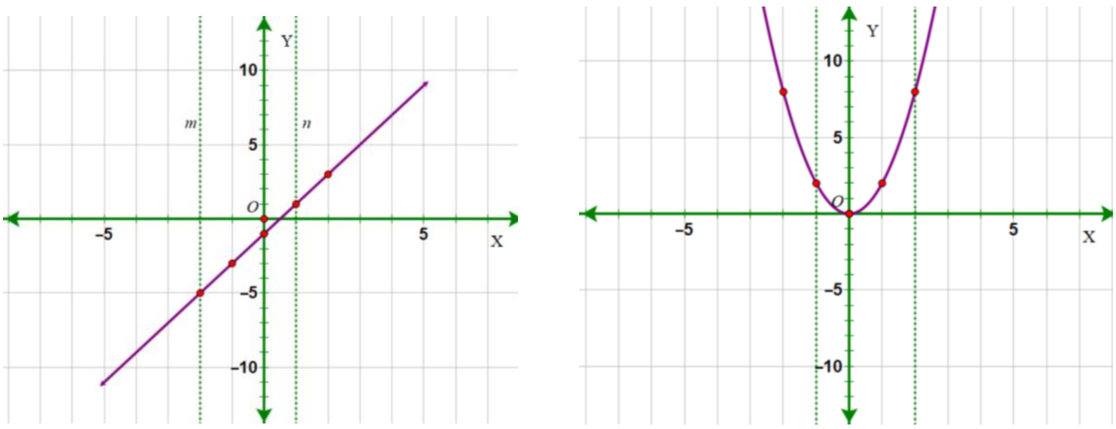
In mathematics, an equation does not necessarily represent a function. For example, the equation $2x - 1 = 11$, the value of 'x' is 6 which is derived through the transposition of the coefficient, On the other hand, the function $y = f(x) = 2x - 1$ can have varied solutions depending on the assigned value for 'x'.

Explicit Functions

An ***explicit function*** is a function that can be written a dependent variable in terms of the independent variable. From the above function $y = f(x) = 2x - 1$, the variable y can be isolated on the left side of the equation. The values of the explicit functions vary from assigned value of x as listed in the following table.

x	-2	-1	0	1	2
$y = f(x) = 2x - 1$	-5	-3	-1	1	3
$y = g(x) = 2x^2$	8	2	0	2	8

Graphs of explicit functions satisfy any vertical lines which only intersect at most one point (see Figure 1.16).

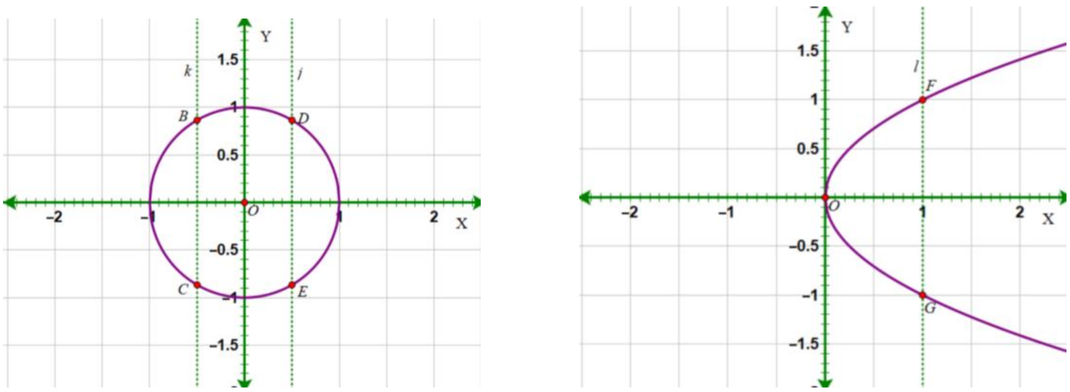


$y = 2x - 1$ $y = 2x^2$

Figure 1.16

Implicit Functions

An ***implicit function*** is a function that cannot find a single value of *y* for a given value of *x*. The graph of implicit function does not satisfy any vertical line tests which intersect graph more than one point such as $x^2 + y^2 = 1$ (circle) and $y^2 = x$ (parabola opens right) as shown in Figure 1.17.



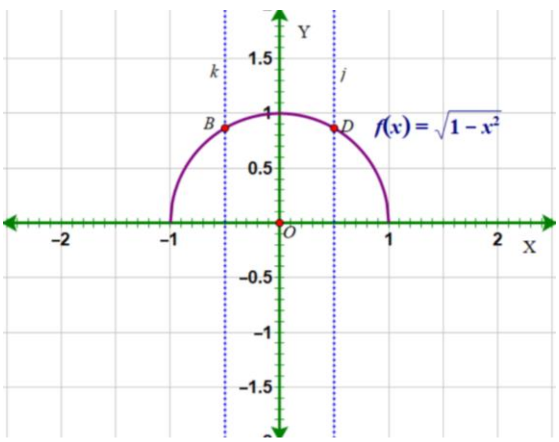
$x^2 + y^2 = 1$ $\qquad$ $y^2 = x$

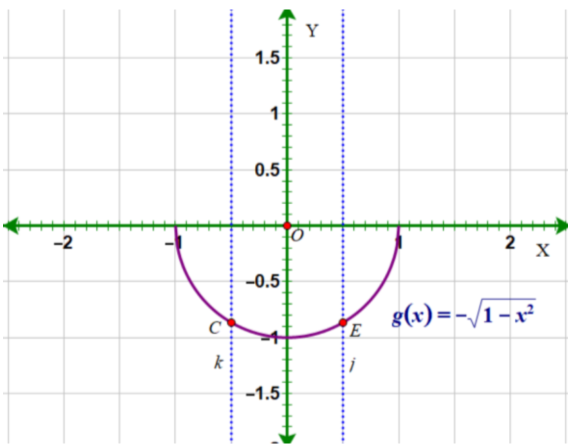
Figure 1.17

Some implicit functions can be rewritten as the explicit solutions. Consider the equation $x^2 + y^2 = 1$, there are two explicit solutions such as

1) $y = f_1(x) = +\sqrt{1 - x^2}, x \in [-1, 1]$
2) $y = f_2(x) = -\sqrt{1 - x^2}, \in [-1, 1]$

In this context the equation $x^2 + y^2 = 1$ is an ***implicit function***, and two explicit solutions can be plotted as the following graphs.





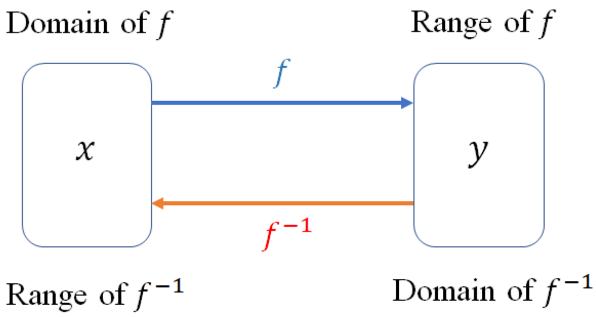
Inverse Functions

In mathematics, a function has exactly one output for each input. The inverse function is defined as follows:

Definition: Let *f* be a function with domain D and range R. A function g with domain R and range D is an ***inverse function*** for *f* if for all *x* in D,

$$y = f(x) \text{ if and only if } x = g(y).$$

We can write $g(y)$ in terms of the composition of f and g as $g(f(x)) = x$. Denoting inverse function g as f^{-1} is read as "f inverse", then $g(f(x)) = f^{-1}(f(x)) = x$. The relationship between the domain and range of f and the domain and range of f^{-1} is shown in the following figure.



The inverse function maps each element from the range of f back to its corresponding element from the domain of f. From the definition of inverse function, we can notice that:

Let $\boldsymbol{f}$ be a function with domain $\boldsymbol{D}$ and range $\boldsymbol{R}$. The following are equivalent:

- $f(x)$ has an inverse $f^{-1}(x)$.
- There is a function f^{-1} such that $f^{-1}(f(x)) = x$ for all $x \in D$ and $f(f^{-1}(y)) = y$ for all $y \in R$.
- $f(x)$ is one to one function.
- Horizontal Line Test meets the graph of $f(x)$ in at most one point.

Graphs of Inverse Functions

Inverse functions have graphs that are reflections over the line $y = x$ and thus reversed ordered pairs, that means if $(x, y) \in f(x)$ then $(y, x) \in f^{-1}(x)$.

Guidelines for identifying an inverse function $f^{-1}(x)$ by its graph $f(x)$:

Step 1: Sketch both graphs on the same coordinate grid.

Step 2: Draw the line $y = x$ and look for the symmetry by reflection.

a. If no symmetry by reflection is apparent, the function is not the inverse function.

b. If symmetry by reflection is apparent, go to **Step 3** to verify.

Step 3: Compare the coordinates of $(x, y) \in f(x)$ to determine if they are reversed $(y, x) \in f^{-1}(x)$, then $f(x)$ and $f^{-1}(x)$ are reverses.

Example: Sketch the graph of $f(x) = 2x^2$ for $x \geq 0$ and identify $f^{-1}(x)$ by graphing.

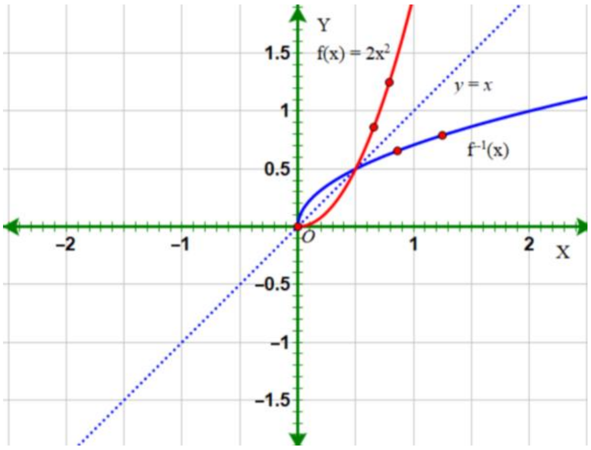


Figure 1.18

Finding the Inverse of a Function

Given the function $f(x)$, the way to find the inverse of a function $f^{-1}(x)$ as the following steps:

Step 1: Replace $f(x)$ in terms of y for given equation.

Step 2: Switch the variables x and y in Step 1.

Step 3: Solve the equation from Step 2 for y.

Step 4: Replace y with the inverse notation $f^{-1}(x)$.

Note that $f(x)$ and $f^{-1}(x)$ are one-to-one functions, then $f^{-1}(x)$ will be an ***inverse function*** by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true and satisfy the horizontal line test.

Example 1: Given the function $f(x) = 2x - 1$. Find the inverse of $f(x)$.

Solution:

Step 1: Replace $f(x)$ in terms of y: $y = 2x - 1$.

Step 2: Switch the variables x and y in Step 1, then $x = 2y - 1$.

Step 3: Solve for y:

$$x = 2y - 1$$

Then $x + 1 = 2y$

Thus, $y = \frac{x+1}{2}$ is the inverse of $f(x)$.

Therefore, $f^{-1}(x) = \frac{x+1}{2}$. #

The graphs of $f(x) = 2x - 1$ and $f^{-1}(x) = \frac{x+1}{2}$ are illustrated in Figure 1.18.:

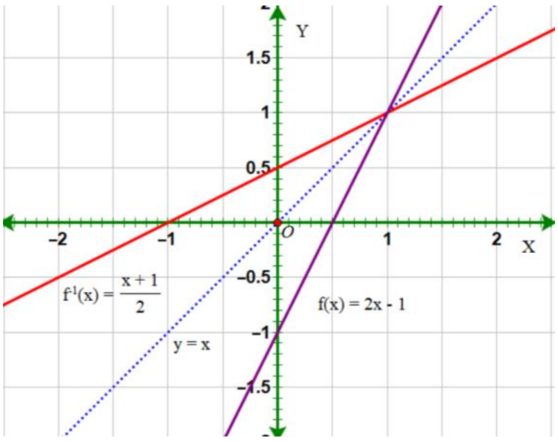


Figure 1.19

Verify to check whether $f^{-1}(x)$ is an inverse function:

$$(f \circ f^{-1})(x) = f\left(f^{-1}(x)\right) = f\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = x$$

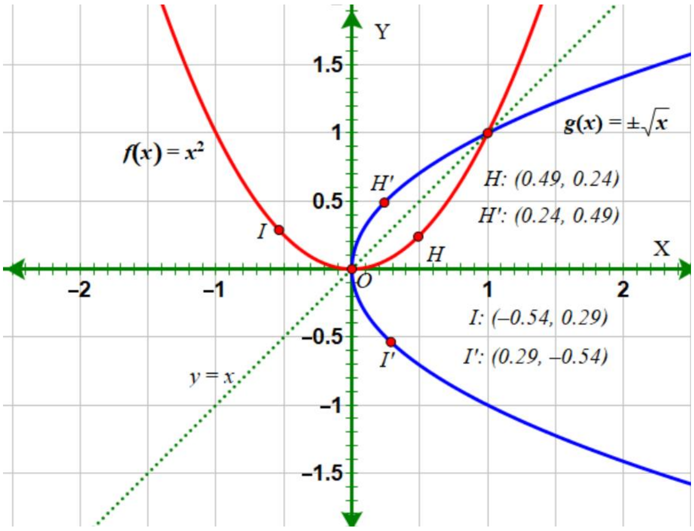
and $(f^{-1} \circ f)(x) = f^{-1}\left(f(x)\right) = f^{-1}(2x-1) = \frac{(2x-1)+1}{2} = x$.

Thus, $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true.

Therefore, $f^{-1}(x)$ is an inverse function.

In Figure 1.19, the symmetric line of $f(x)$ *and* $f^{-1}(x)$ is $y = x$ and any horizontal line intersect the inverse of a function at most one point.

Example 2: Given the function $f(x) = x^2$. Find the inverse of $f(x)$.



Solution: Graph of a parabola with the equation $y = f(x) = x^2$, the U-Shaped curve opening up. This function $f(x)$ fails the horizontal line test, and therefore the inverse of the function $g(x)$ is not the inverse function. #

If we use the steps in Example 1, we will find that the inverse of a parabola is not the inverse function:

Step 1: Replace $f(x)$ in terms of *y*: $y = x^2$.

Step 2: Switch the variables x and y in Step 1, then $x = y^2$.

Step 3: Solve for *y*:

$$x = y^2$$

Then $y = \pm\sqrt{x}$.

Let $g(x) = y = \pm\sqrt{x}$ *is the inverse of function* $f(x)$.

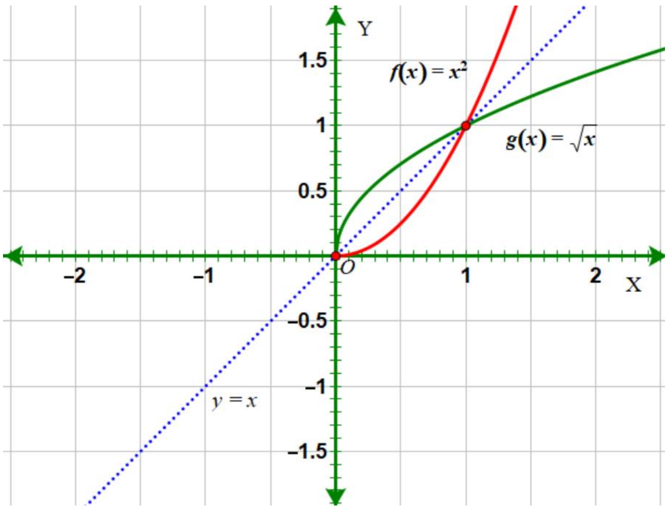
But any given *x*-value will correspond to two different y-values, one from $+\sqrt{x}$ and other from $-\sqrt{x}$.

Since $g(x)$ fails the vertical line test, then $g(x)$ *is not the inverse function* $f(x)$. #

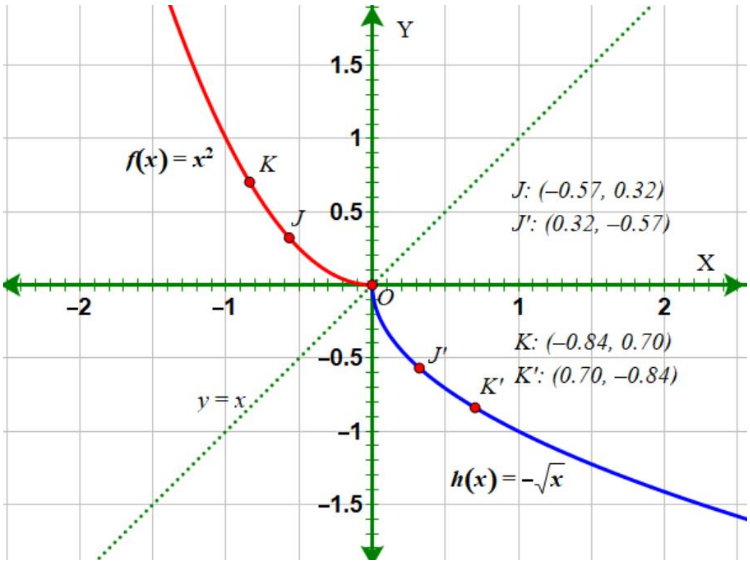
Restricted Domain

Since $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of *f* such that the function is one-to-one. This subset is called a ***restricted domain***.

For the function $f(x) = x^2$, if we restrict the domain to be $[0, \infty)$, then we find that $f(x)$ passes the horizontal line test and $g(x)$ meets the vertical line test will be the inverse function as follows:



On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could define a new function h such that the domain of h is $(-\infty, 0]$ and inverse is given by the formula $h(x) = -\sqrt{x}$ as shown in the following graph.



Example 3: Given the function $f(x) = x^3 + 4$. Find the inverse of $f(x)$.

Solution:

Step 1: Replace $f(x)$ in terms of y: $y = x^3 + 4$.

Step 2: Switch the variables x and y in Step 1, then $x = y^3 + 4$.

Step 3: Solve for y:

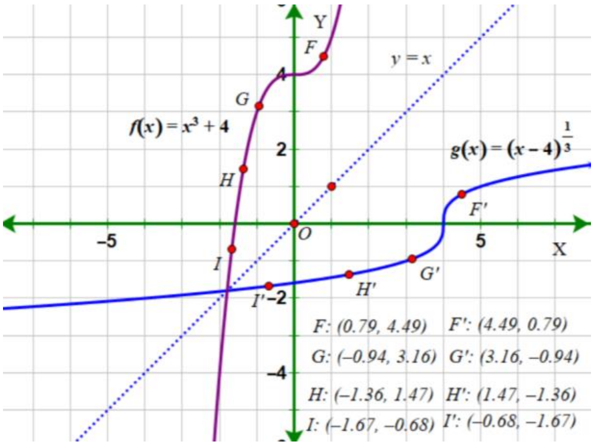
$x = y^3 + 4.$

Then $y^3 = x - 4$

Thus, $y = \sqrt[3]{x-4}$ is the inverse of $f(x)$.

Therefore, $f^{-1}(x) = \sqrt[3]{x-4}$ #

Verify by graphing as follows:



Notice that graph of $g(x) = \sqrt[3]{x-4} = (x-4)^{\frac{1}{3}}$ meets the vertical line test, it is the inverse function of $f(x) = x^3 + 4$ and written as $f^{-1}(x) = \sqrt[3]{x-4}$.

Verify by finding the composition of $f(x)$ and $g(x)$ as follows:

$$(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x-4}) = (\sqrt[3]{x-4})^3 + 4 = x + 4 - 4 = x.$$

and $(g \circ f)(x) = g(f(x)) = g(x^3 + 4) = \sqrt[3]{(x^3+4) - 4} = \sqrt[3]{x^3 + 4 - 4} = \sqrt[3]{x^3} = x$.

Thus, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ are both true.

Therefore, $f(x)$ and $g(x)$ are inverses and written $g(x)$ as $f^{-1}(x)$.

Practice 1.7

From given function $f(x)$ and $g(x)$, verify whether both functions are inverses by (a) graphing, and (b) finding the composition.

(i) $f(x) = (x+3)^2 - 7$, $g(x) = \sqrt{x+7} - 3$. (Assume the domain of $f(x)$ is $[-3, \infty)$).

(ii) $f(x) = 3x + 7$, $g(x) = \frac{1}{3}x - 7$.

(iii) $f(x) = 2\sqrt[3]{x-5} + 7$, $g(x) = \left(\frac{x-7}{2}\right)^3 + 5$.

(iv) $f(x) = \frac{(x-7)^3}{2} + 5$, $g(x) = 2\sqrt[3]{x-5} + 7$.

References

Ayres, F. & Mendelson, E. (2013). Calculus Sixth Edition. USA: McGraw-Hill Companies, Inc.

Herman, E.J. & Strang, G. (2018). Calculus Volume 1. Texas: Rice University.