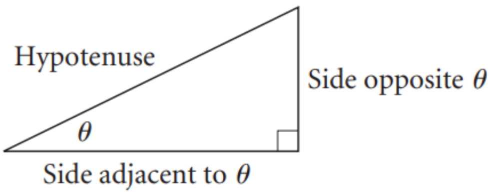
Chapter 1 The Trigonometry of Right Triangle

The trigonometric functions, including ***sine***, ***cosine***, and ***tangent*** were first defined as ***ratios of sides in a right triangle***. Many applications involve ***an angle of elevation*** or ***an angle of depression***. Suppose that the pilot of the plane is looking forward and then looks up, the pilot's eyes pass through an angle of elevation. If he is looking straight ahead and then he moves his eye down for landing, the pilot's eyes pass through an angle of depression.

The Trigonometric Ratios

Consider a right triangle with one of its acute angles labeled θ. The side opposite the right angle is called the ***hypotenuse***. The other sides of the right triangle are referenced by their position relative to the acute angle θ. One side is called ***side opposite*** $\boldsymbol{\theta}$ and one is called ***side adjacent to*** $\boldsymbol{\theta}$.



Trigonometric ratios are the ratios of the length of sides of a right triangle. In trigonometry, there are six trigonometric ratios, namely, ***sine(sin), cosine(cos), tangent(tan), secant(sec), cosecant(csc), and cotangent(cot)***.

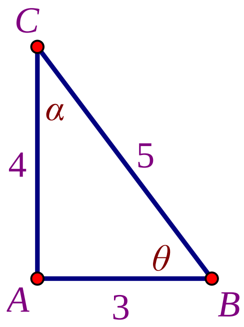
Trigonometric Ratio Values of an Acute Angle $\boldsymbol{\theta}$

Let θ be an acute angle of a right triangle. Then the six trigonometric ratios of θ are as follows:

$\sin\theta = \dfrac{side\ opposite\ \theta}{hypotenuse} = \dfrac{opp}{hyp}$	$\csc\theta = \dfrac{hypotenuse}{side\ opposite\ \theta} = \dfrac{hyp}{opp}$
$\cos\theta = \dfrac{side\ adjacent\ to\ \theta}{hypotenuse} = \dfrac{adj}{hyp}$	$\sec\theta = \dfrac{hypotenuse}{side\ adjacent\ to\ \theta} = \dfrac{hyp}{adj}$
$\tan\theta = \dfrac{side\ opposite\ \theta}{side\ adjacent\ to\ \theta} = \dfrac{opp}{adj}$	$\cot\theta = \dfrac{side\ adjacent\ to\ \theta}{side\ opposite\ \theta} = \dfrac{adj}{opp}$

See the example 1.1 illustrated how to find the value of trigonometric ratios.

Example 1.1 In the right triangle shown below, find the values of six trigonometric ratios values of (a) $\boldsymbol{\theta}$ and (b) $\boldsymbol{\alpha}$.



Solution:

a) $\sin\theta = \frac{opp}{hyp} = \frac{4}{5}$, $\qquad \csc\theta = \frac{hyp}{opp} = \frac{5}{4}$

$\cos\theta = \frac{adj}{hyp} = \frac{3}{5}$, $\qquad \sec\theta = \frac{hyp}{adj} = \frac{5}{3}$

$\tan\theta = \frac{opp}{adj} = \frac{4}{3}$, $\qquad \cot\theta = \frac{adj}{opp} = \frac{3}{4}$

b) $\sin\alpha = \frac{opp}{hyp} = \frac{3}{5}$, $\qquad \csc\boldsymbol{\alpha} = \frac{hyp}{opp} = \frac{5}{3}$

$\cos\boldsymbol{\alpha} = \frac{adj}{hyp} = \frac{4}{5}$, $\qquad \sec\boldsymbol{\alpha} = \frac{hyp}{adj} = \frac{5}{4}$

$\tan\boldsymbol{\alpha} = \frac{opp}{adj} = \frac{3}{4}$, $\qquad \cot\boldsymbol{\alpha} = \frac{adj}{opp} = \frac{4}{3}$ #

Reciprocal Trigonometric Ratios

In Example 1.1(a), there are the reciprocal relationship between the values of $\sin\theta$, $\frac{4}{5}$ and $\csc\theta$, $\frac{5}{4}$, the values of $\cos\theta$,$\frac{3}{5}$ and $\sec\theta$, $\frac{5}{3}$, and the values of $\tan\theta$, $\frac{4}{3}$ and $\cot\theta$, $\frac{3}{4}$. Likewise, for any angle, the cosecant (csc), secant (sec), and cotangent (cot) are the reciprocals of the sine, cosine, and tangent ratio values, respectively. There are the reciprocal identities of the trigonometric ratios as follows:

- The ***reciprocal sine*** is ***cosecan***t:

$$\sin\theta = \frac{1}{\csc\theta} \quad and \quad \csc\theta = \frac{1}{\sin\theta}$$

- The ***reciprocal cosine*** is ***secant***:

$$\cos\theta = \frac{1}{\sec\theta} \ and \ \sec\theta = \frac{1}{\cos\theta}$$

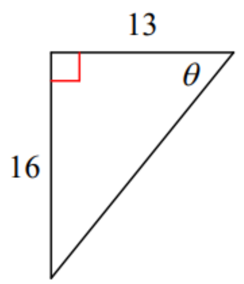
- The ***reciprocal tangent*** is ***cotangent***:

$$\tan\theta = \frac{1}{\cot\theta} \ and \ \cot\theta = \frac{1}{\tan\theta}$$

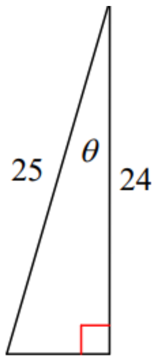
When solving right triangles, the reciprocal identities can be helpful in solving the value of the trigonometric ratios as illustrated in Example 1.2 and Example 1.3.

Example 1.2 Find the values of the trigonometric ratios indicated. (You will need to use Pythagorean theorem to find the missing side length)

a) $\csc\theta$



b) $\sec\theta$



Solution:

a) Using Pythagorean theorem:
Hypotenuse $= \sqrt{16^2 + 13^2} = \sqrt{256 + 169} = 5\sqrt{17}$
Since $\csc\theta$ is the reciprocal sine, and $\sin\theta = \frac{16}{5\sqrt{17}}$
Therefore, $\csc\theta = \frac{5\sqrt{17}}{16}$ #

b) Since $\sec\theta$ is the reciprocal cosine, and $\cos\theta = \frac{24}{25}$
Therefore, $\sec\theta = \frac{25}{24}$ #

Example 1.3 Find the values of trigonometric ratios indicated. (You will need to draw a right triangle)

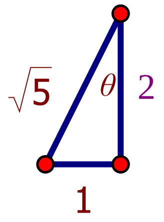
a) Find $\csc\theta$ if $\sec\theta = \frac{\sqrt{5}}{2}$

b) Find $\cot\theta$ if $\sec\theta = \frac{5}{3}$

Solution:

a) Since $\sec\theta$ is the reciprocal cosine, and $\sec\theta = \frac{\sqrt{5}}{2}$

Then $\cos\theta = \frac{2}{\sqrt{5}}$



Using Pythagorean theorem:

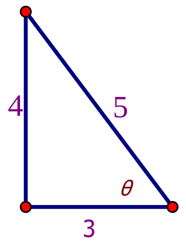
$$\text{Opposite }\theta = \sqrt{(\sqrt{5})^2 - 2^2} = \sqrt{5-4} = 1$$

Since $\csc\theta$ is the reciprocal sine, and $\sin\theta = \frac{1}{\sqrt{5}}$

Therefore $\csc\theta = \sqrt{5}$ #

b) Since $\sec\theta$ is the reciprocal cosine, and $\sec\theta = \frac{5}{3}$

Then $\cos\theta = \frac{3}{5}$



Using Pythagorean theorem:

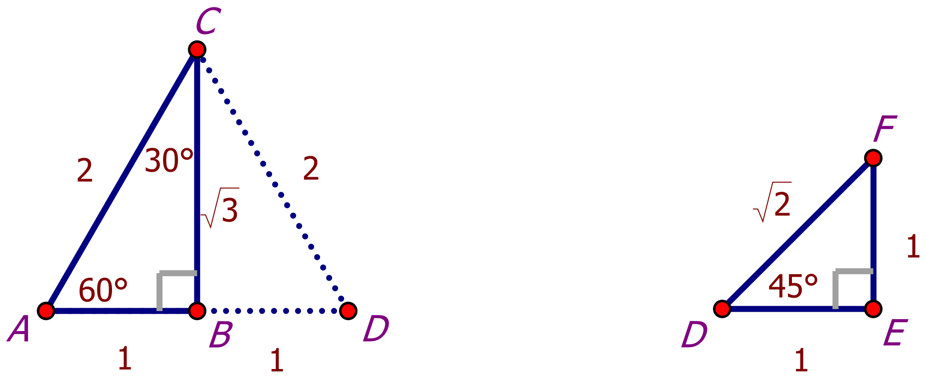
$$\text{Opposite }\theta = \sqrt{5^2 - 3^2} = \sqrt{25-9} = \sqrt{16} = 4$$

Since $\cot\theta$ is the reciprocal tangent, and $\tan\theta = \frac{4}{3}$

Therefore $\cot\theta = \frac{3}{4}$ #

The Trigonometric Ratio for $30°, 45°, 60°$

The trigonometric ratios for $30°, 45°$, and $60°$ are based on some standard triangles as follows:



The left figure shows 30° and 60° based on an equilateral triangle *ADC* with sides of length 2 and with one of the angles bisected perpendicular to side $\overline{AD}$. It follows that $\angle ACB = 30°$ and $AB = 1$. By using Pythagorean theorem, $BC = \sqrt{3}$.

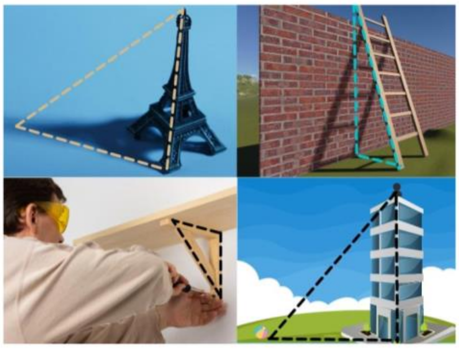
The right figure shows 45° angle based on an isosceles with the equal sides ($DE = EF$) having a length of 1. By using Pythagorean theorem, $DF = \sqrt{2}$.

Then, the trigonometric ratios for $30°, 45°$, and $60°$ are listed in the following table:

θ	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\csc\theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

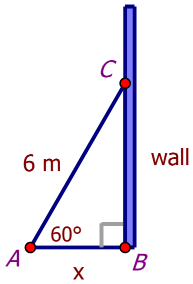
Applications of Right Triangle Trigonometry

Many real situations involve right triangles. In geometry, solving the problems involving distances can be used the Pythagorean theorem. In this topic, solving the problems involving measure of all three angles and the lengths of all three sides of a right triangle can be used the knowledge of trigonometric ratios.



Source: https://numberdyslexia.com/real-life-applications-of-right-angle-triangle/

Example 1.4 A 6-meter ladder leans against a brick wall forming angle θ of $60°$ with the ground. How far is the base of the ladder from the wall?



Solution: Let x be the distance along the ground to the wall, as in the above picture.

Since a 6-meter ladder forms an angle of $60°$ with the ground against a brick wall.

Then, it forms a right triangle ABC with $\angle ABC$ is a right angle, $\overline{AB}$ is the adjacent side with the length x and $\overline{AC}$ is the hypotenuse with the length 6 meters.

Since $\cos\theta = \frac{adj}{hyp}$

Then, $\cos 60° = \frac{AB}{AC} = \frac{x}{6}$ (equation 1)

Since $\cos 60° = \frac{1}{2}$

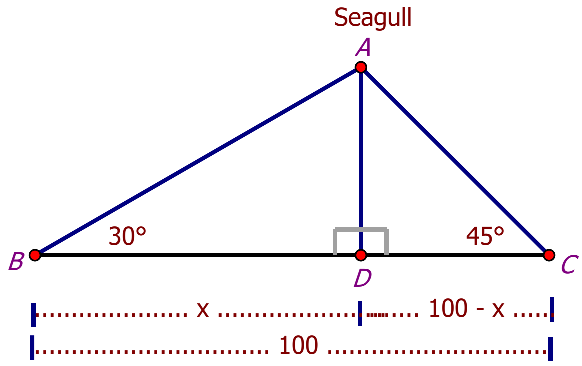
Substitute $\cos 60°$ in equation 1 and solving for x:

$$\frac{1}{2} = \frac{x}{6}$$

$$x = 3$$

Therefore, the base of the ladder 3 meters far from the wall. #

Example 1.5 Two girls have the same height are standing 100 meters apart. They both see a beautiful seagull in the air between them. The angles of elevation from their eyes to the bird are $30°$ and $45°$, respectively. How high up is the seagull?



Solution: In ΔABC, let $\overline{AD}$ perpendicular to $\overline{BC}$ at point D.

Divide up the 100 meters into x and $100 - x$

In ΔABD, $\tan 30° = \frac{AD}{x}$ ....(eq. 1)

In ΔACD, $\tan 45° = \frac{AD}{100-x}$ ....(eq. 2)

Since, $\tan 30° = \frac{1}{\sqrt{3}}$ and $\tan 45° = 1$

Substitute $\tan 30° = \frac{1}{\sqrt{3}}$ in eq. 1 and $\tan 45° = 1$ in eq. 2

From eq. 1, $\frac{1}{\sqrt{3}} = \frac{AD}{x}$ or $AD = \frac{x}{\sqrt{3}}$ ....(eq. 3)

From eq. 2, $1 = \frac{AD}{100-x}$ or $AD = 100 - x$ ....(eq. 4)

Since eq.3 = eq.4, then $\frac{x}{\sqrt{3}} = 100 - x$ ....(eq. 5)

Solve eq. 5 for x, $x + \sqrt{3}x = 100\sqrt{3}$

$$(1 + \sqrt{3})x = 100\sqrt{3}$$

$$x = \frac{100\sqrt{3}}{1+\sqrt{3}}$$

Thus, $AD = 100 - x = 100 - \frac{100\sqrt{3}}{1+\sqrt{3}}$

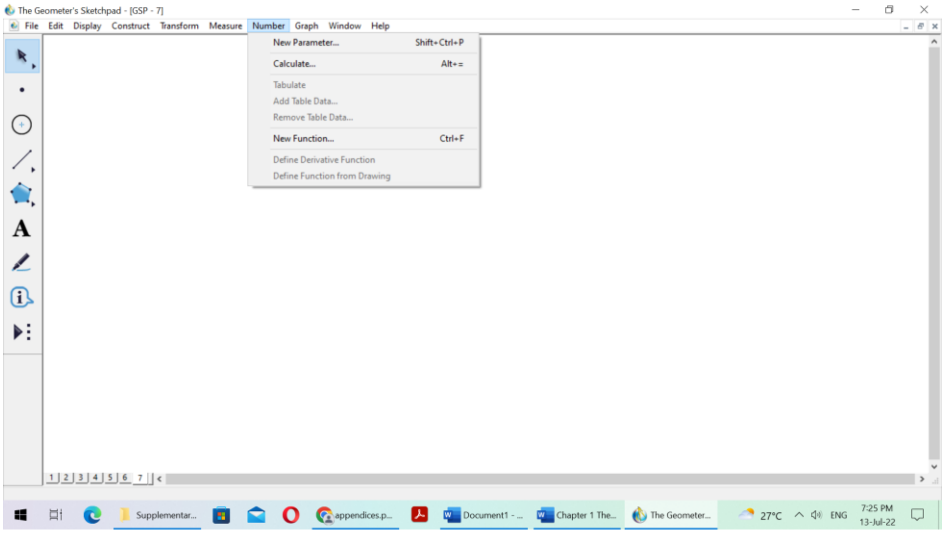
$$AD \approx 36.60$$

Therefore, the seagull is about 37 meters high. #

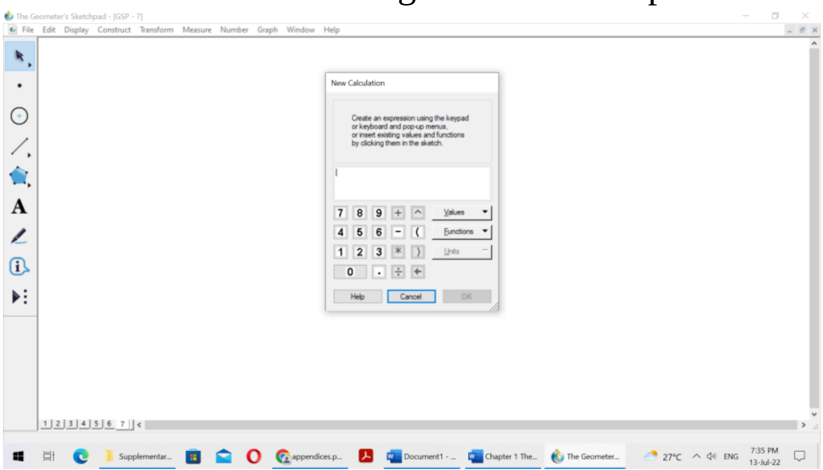
Finding Trigonometric Ratios by Using Calculators

The values of trigonometric ratios of any angles can be found by using a calculator. For this topic, the use of Geometer's Sketchpad (V. 5) to calculate the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for any angle θ measured in degree is shown in the following steps.

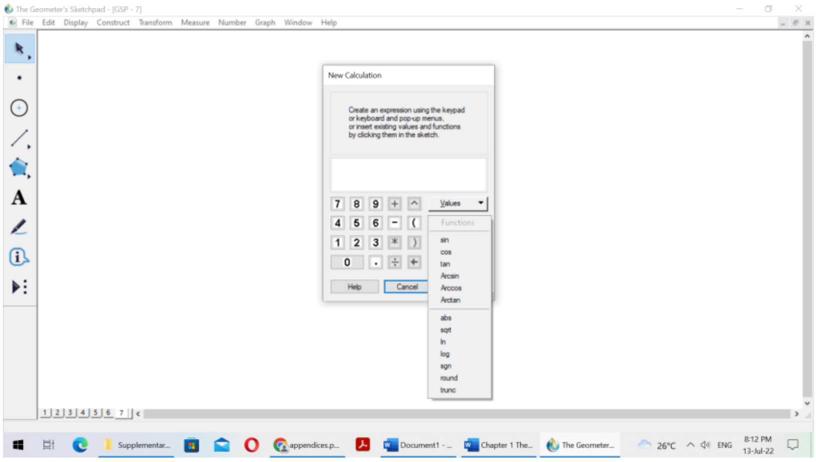
1. To invoke the calculator, choose "Calculate" from under the Number menu.



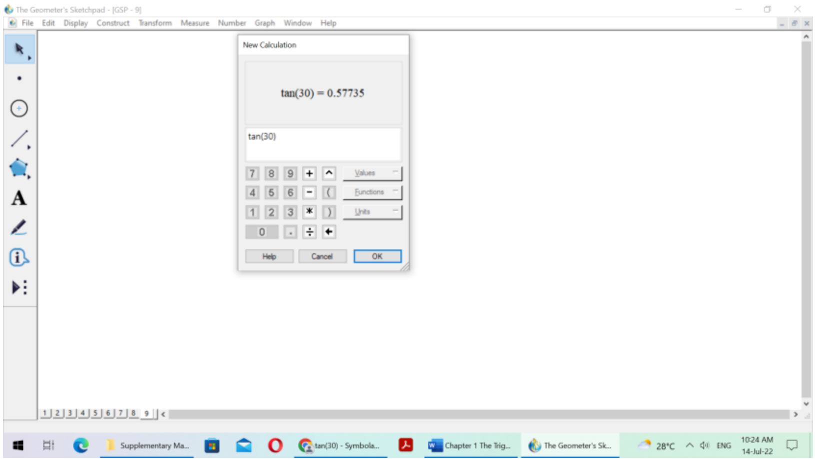
2. Click on "Calculate", the display area shows the expression being built. Click a value on the keypad or type from keyboard to enter it into the calculator. The keypad has operators for addition (+), subtraction (-), multiplication (*), and division (/). The next column contains parentheses, carat symbol (^) for raising to a power, and arrow to delete or change the value or operator.



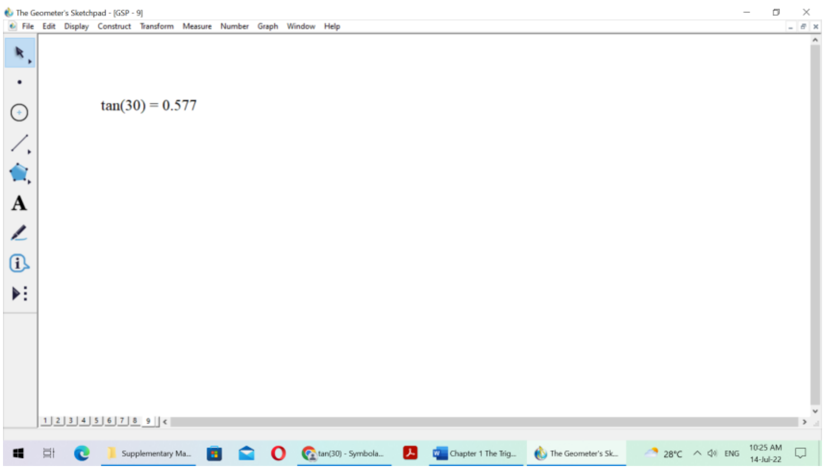
3. Click on "Function" button and enter "sin" or "cos" or "tan" into the calculator.



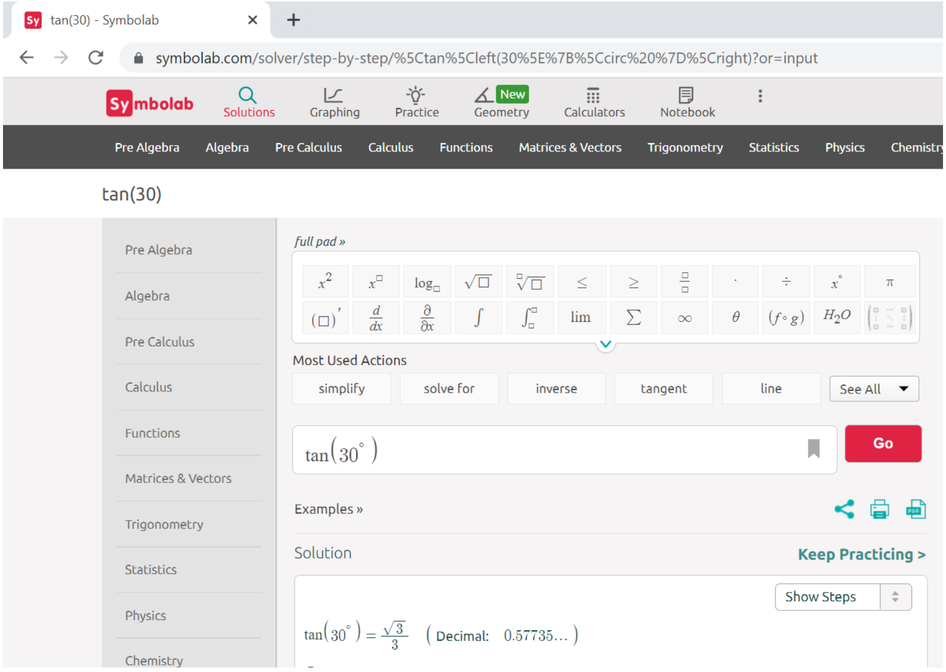
4. Enter the measure of angle to find the value of trigonometric ratios. The angles can be measured in different units, for measuring in "degree" click the value on keypad.



5. Click on "OK", then the solution will show on GSP screen.

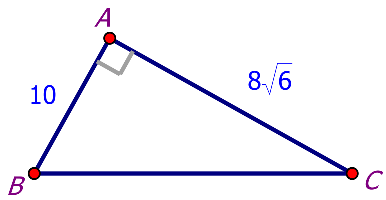


Different calculators may use different keying as shown one online calculator (https://www.symbolab.com/solver/step-by-step/).



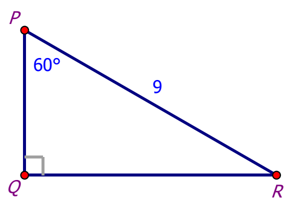
Practice 1.1

1. In $\Delta\, ABC$, $AB = 10$ cm, $AC = 8\sqrt{6}$, and $\angle BAC = 90°$.

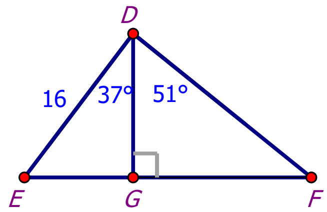


   (a) Find the length of $\overline{BC}$.
   (b) Find the values of $\sin B, \cos B$, $\sin C$, *and* $\cos C$, giving the answers as fractions in lowest terms.
   (c) What is the relation between (i) $\sin B$ and $\cos C$, (ii) $\cos B$ and $\sin C$.

2. In the diagram, $PR = 9$ cm, $\angle\, QPR = 60°$, and $\angle\, PQR = 90°$. Find the length of $\overline{QR}$ and $\overline{PQ}$, correct to one decimal place (round to the nearest 10^{th}).



3. In $\Delta\, ABC, \angle\, BAC = 90°, \angle\, BCA = 60°$ and $AC = 12$ cm. Find the length of $\overline{AB}$ and $\overline{BC}$, correct to 1 decimal point.
4. In $\Delta\, DEF, \angle\, EDF = 90°, \angle\, DFE = 28°,$ and $DF = 16\ cm.$ Find the lengths of $\overline{DE}$ and $\overline{EF}$, correct to 2 decimal places (round to the nearest 100^{th}).
5. In the diagram, $\overline{DG} \perp \overline{EF}, \angle\, EDG = 37°, \angle\, GDF = 51°$and $DE = 16$ cm. Find the lengths of $\overline{DG}$ and $\overline{FG}$.



6. The angle of elevation of the top of the building at distance of 50 meters from its fool on a horizontal plane is found to be $60°$. Find the height of the building.
7. A ladder placed against a wall such that it reaches the top of the wall of the height 6 meters and the ladder is inclined at an angle of $60°$. Find how far the ladder is from the foot of the wall.
8. From the top of the tower 30 meters height, a man is observing the base of the tree at an angle of depression measuring $30°$. Find the distance between the tree and the tower.
9. A man wants to determine the height of the light house. He measured the angle at A and found that $\tan A = \frac{3}{4}$. What is the height of the light house if A is 40 meters from the base?
10. A kite is flying at the height of 65 meters attached to a string. If the inclination of the string with the ground is $31°$. Find the length of the string.

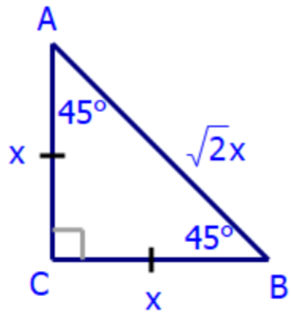
Special Right Triangles

There are two special right triangles appear throughout the study of trigonometry:

- 45-45-90 Triangle
- 30-60-90 Triangle

The 45-45-90 Triangles

A 45-45-90 triangle has two equal angles (45°) and one right angle (90°). It is called an isosceles right triangle because it has the characteristic of both the isosceles and the right triangles. Since the sum of the interior angles is $180°$, then the isosceles triangle has one 90 degree angle and two 45 degree angles as shown in the following figures.



In the above figure, ΔABC is an isosceles triangle with 45, 45, 90 degrees.

Let x units be the length of equal legs which opposite to the angle of $45°$ and the hypothenuse is opposite to the angle of $90°$.

By the Pythagorean Theorem,

$$AB^2 = CA^2 + CB^2 = x^2 + x^2 = 2x^2$$

Therefore, $AB = \sqrt{2}x$.

So, the ratio of three sides of 45-45-90 triangle is always $x: x: \sqrt{2}x$ or $1: 1: \sqrt{2}$.

Calculating the side of 45-45-90 triangle

If one of the sides of an isosceles right triangle is known, we can apply the ratio of three sides of the 45-45-90 triangle to calculate the other sides as the following examples.

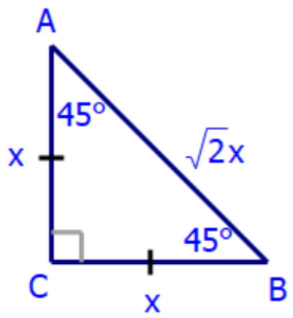
Example 1.6 If one of the equal sides of an isosceles right triangle is 4, what are the measures of the other two sides?

Solution:

Method 1: Using the ratio $x: x: \sqrt{2}x$ for isosceles right triangle.

Since $x = 4$, then the other sides must be 4 and $4\sqrt{2}$ #

Method 2: Using the Pythagorean Theorem and the fact that the legs of this right triangle are equal ($x = 4$).



By the Pythagorean Theorem:

$AB^2 = CA^2 + CB^2 = x^2 + x^2 = 4^2 + 4^2 = 16 + 16 = 32$

$AB = 4\sqrt{2}$

Therefore, the other sides must be 4 and $4\sqrt{2}$ #

Method 3: Let $CA = 4$

Using the trigonometric ratios:

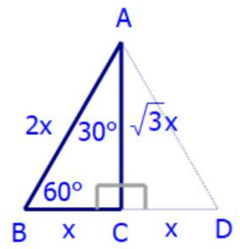
Since $\sin 45° = \frac{CA}{AB}$, $\sin 45° = \frac{1}{\sqrt{2}}$, and $CB = CA$

It follows that, $\frac{1}{\sqrt{2}} = \frac{4}{AB}$

Therefore, $AB = 4\sqrt{2}$ and $CB = 4$ #

The 30-60-90 Triangles

A 30-60-90 triangle is a scalene right triangle where one angle is 90° and the other two angles are different measurement (30° and 60°) and add up to 90°. The ratio of three sides of 30-60-90 triangle is always $x:\sqrt{3}x:2x$ or $1:\sqrt{3}:2$ as shown in the following figure.



In the above figure, ΔABD is an equilateral triangle with three equal angles (60°) and three equal sides ($2x$). The perpendicular $\overline{AC}$ traced from the vertex ($\angle BAD$) to the opposite side ($\overline{BD}$) at point C ($m\angle ACB = 90°$) and divides it into equal halves ($BC = CD = x$) and splits the vertex into equal angles of 30 degrees each ($m\angle BAC = m\angle DAC = 30°$).

Since ΔABC is a right triangle with $m\angle ACB = 90°$.

By the Pythagorean Theorem,

$$AB^2 = CA^2 + CB^2$$

$$(2x)^2 = CA^2 + x^2$$

$$CA^2 = 4x^2 - x^2 = 3x^2$$

$$CA = \sqrt{3}x$$

Therefore, the ratio of three sides of 30-60-90 triangle is always $x: \sqrt{3}x: 2x$ or $1: \sqrt{3}: 2$.

Calculating the side of 30-60-90 triangle

If one of the sides of a 30-60-90 triangle is known, we can apply the ratio of three sides of the 30-60-90 triangle to calculate the other sides as the following examples.

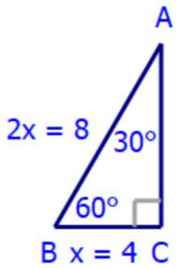
Example 1.7: If the side opposite to angle measure of 30° (the shortest side) is 4, what is the measure of the other two sides?

Solution:

Method 1: Using the ratio $x: \sqrt{3}x: 2x$ for 30-60-90 triangle.

Since $x = 4$, then the other sides must be 2(4) = 8 and $4\sqrt{3}$ #

Method 2: Using the Pythagorean Theorem and the fact that one leg of this right triangle (shortest side) is $x = 4$.



Thus, the hypothenuse is $2x = 2(4) = 8$.

By the Pythagorean Theorem,

$$AB^2 = CA^2 + CB^2$$

$$(8)^2 = CA^2 + 4^2$$

$$CA^2 = 64 - 16 = 48$$

$$CA = 4\sqrt{3}$$

Therefore, the other sides must be 8 and $4\sqrt{3}$ #

Method 3: Let $BC = 4$

Using the trigonometric ratios:

Since, $\sin 30° = \frac{BC}{AB}$, $\sin 30° = \frac{1}{2}$

It follows that, $\frac{1}{2} = \frac{4}{AB}$

Therefore, $AB = 4(2) = 8$.

Since, $\cos 30° = \frac{AC}{AB}$ and $\cos 30° = \frac{\sqrt{3}}{2}$

It follows that, $\frac{\sqrt{3}}{2} = \frac{AC}{8}$

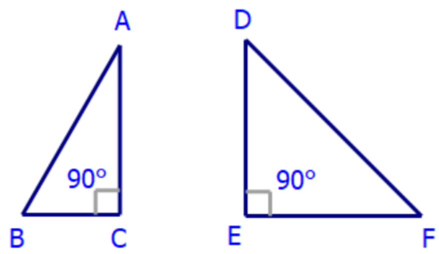
Therefore, $AC = \frac{8\sqrt{3}}{2} = 4\sqrt{3}$

Therefore, the other sides must be 8 and $4\sqrt{3}$ #

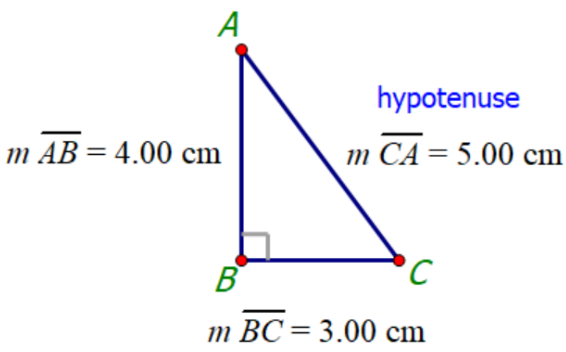
Investigation Properties of Right Triangles

There are some important concepts of right triangles that require for investigation by using dynamic geometry software such as the Geometer's Sketchpad or GeoGebra. The following concepts are helpful in learning trigonometry.

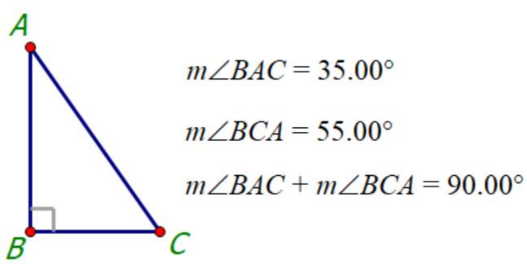
- A ***right triangle*** is a triangle with one angle of 90°.



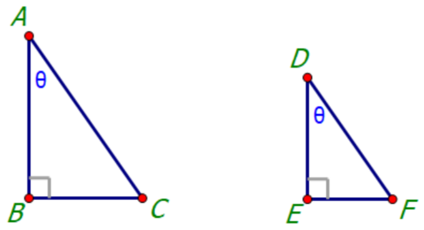
- The ***hypotenuse*** is the longest side in a right triangle. This is always the side opposite the right-angle.



- In a right triangle, the two acute angles are ***complementary***.



- Right triangles will be ***similar*** if an acute angle of one is equal to an angle of the other. Therefore, the sides that make the equal angles will be ***proportional***.



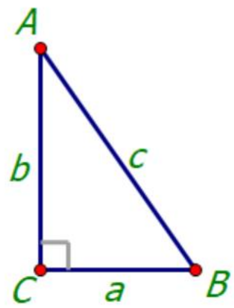
In the right triangles *ABC* and *DEF*, if the acute angle at A is equal to the acute angle at D, then ΔABC and ΔDEF will be similar denoted by $\Delta ABC \sim \Delta DEF$. Therefore, the ratios of the corresponding sides will be in the same ratio:

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

- ***The Pythagorean Theorem***: In a right triangle with sides

$a, b,$ and c, where c is the hypothesis, then:

$$a^2 + b^2 = c^2$$



- There are fundamental relations between sides of a right triangle

when its angle equals 60 degrees.

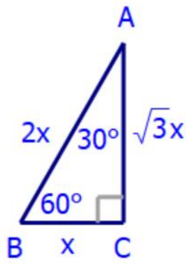
1) The length of adjacent side is equal to half of the length of

hypothesis.

2) The length of opposite side is equal to $\frac{\sqrt{3}}{2}$ times the length of

hypothesis.

3) The third angle of right triangle is 30 degrees



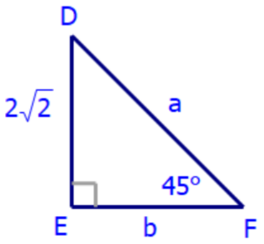
In above figure, $AB = 2x$, $BC = \frac{1}{2}AB = \frac{1}{2}(2x) = x$, $AC = \frac{\sqrt{3}}{2}(2x) = \sqrt{3}x$.

Note that the ratio of three sides of 30-60-90 triangle is always $x{:}\sqrt{3}x{:}\,2x$ or $1{:}\sqrt{3}{:}\,2$.

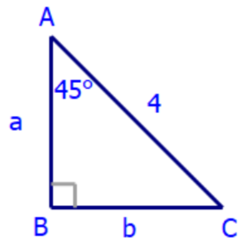
Practice 1.2

1. Find the missing side lengths of 45-45-90 triangle. Leave the answer as radicals in simplest form.

1)

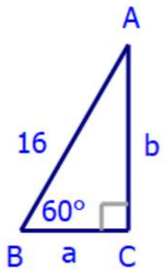


2)

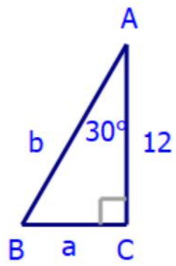


2. Find the missing side lengths of 30-60-90 triangle. Leave the answer as radicals in simplest form.

1)



2)



3. Investigate the following properties by using dynamic geometry software.

1) Right triangles will be ***similar*** if an acute angle of one is equal to an angle of the other. Therefore, the sides that make the equal angles will be ***proportional***.

2) A ***Pythagorean triple*** consists of three positive integers $a, b,$ and c such that $a^2 + b^2 = c^2$. Such a triple is commonly written (a, b, c) and a well-known example is (3, 4, 5). If (a, b, c) is a Pythagorean triple, then so is (ka, kb, kc) for any positive integer k.

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